Symmetry analysis and conservation laws for higher order Camassa-Holm equation

Vahid Shirvani
Department of Mathematics, Eslamabad-E-Gharb Branch, Islamic Azad University, Eslamabad-E-Gharb, Iran.
E-mail: v.shirvani@kiau.ac.ir

Mehdi Nadjafikhah
School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16846-13114, Iran.
E-mail: n.nadjafikhah@iust.ac.ir

Abstract
In this paper, Lie symmetry group method is applied to study for the higher order Camassa-Holm equation. Complete analysis of symmetries and nonclassical symmetries is discussed. Furthermore, optimal system, preliminary classification of its group invariant solutions and symmetry reduction are investigated. Finally conservation laws for the higher order Camassa-Holm equation which conserved quantities arise from multipliers by using homotopy operator are presented.

Keywords. Lie symmetry, Higher order Camassa-Holm equation, Optimal system, Similarity solutions, Conservation laws.

2010 Mathematics Subject Classification. 58J70, 35L65, 22E70.

1. INTRODUCTION

In recent years, many researchers have been researched on the Camassa-Holm equation. They extend the studies to the generalized CH equation, higher order CH equations and so on. Lixin Tian, Chunyu Shen and Danping Ding gave the optimal control of the viscous CH equation under the boundary condition and proved the existence and uniqueness of optimal solution to the viscous CH equation in a short interval [14]. Using geometrical methods, higher order CH equations have been treated in [7]. The well-posedness of higher order CH equations were considered in [6]. Conservation laws and soliton solutions for modified Camassa-Holm equation were studied in [8]. Several new types of bounded wave solutions for the generalized two-component Camassa-Holm equation obtained in [15].

The formulation of the higher order Camassa-Holm equation which was recently derived by Coclite, Holden and Karlsen in [6] is

\[ \partial_t u = B_k(u, u), \quad t > 0, \quad k \in \mathbb{N} \cup \{0\}, \quad x \in \mathbb{R}, \]

Received: 28 April 2018; Accepted: 01 January 2019.
* Corresponding author.
where $u = u(x,t) : \mathbb{R} \times [0,\infty) \rightarrow \mathbb{R}$ is the unknown function and

$$
B_k(u,u) := A_k^{-1}C_k(u) - u\partial_x u, \\
A_k(u) := \sum_{j=0}^{k} (-1)^j \partial_x^{2j} u, \\
C_k(u) := -uA_k(\partial_x u) + A_k(u\partial_x u) - 2\partial_x uA_k(u).
$$

In cases $k = 0$ and $k = 1$, equation (1.1) becomes the inviscid Burgers equation and the Camassa-Holm equation respectively.

In this paper we consider the case $k = 2$ of equation (1.1). It also can be rewritten as

$$
\Delta := u_t - u_{x^2} + u_{x^4} + 3uu_x - 2u_xu_{x^2} - uu_{x^3} + 2u_xu_{x^3} + uu_{x^4} = 0. \tag{1.2}
$$

2. Lie symmetries for the higher order CH equation

In this section, we give the general form of an infinitesimal generator admitted by equation (1.2) and find transformed solutions. For the general procedure to determining symmetries for any system of partial differential equations and more details see [4, 5, 9, 11].

We consider the one parameter Lie group of infinitesimal transformations on $(x,t,u)$, 

$$
\tilde{x} = x + s\xi(x,t,u) + O(s^2), \\
\tilde{t} = t + s\eta(x,t,u) + O(s^2), \\
\tilde{u} = u + s\varphi(x,t,u) + O(s^2), \tag{2.1}
$$

where $s$ is the group parameter and $\xi$, $\eta$ and $\varphi$ are the infinitesimals of the transformations for the independent and dependent variables, respectively. The associated vector field is of the form:

$$
v = \xi(x,t,u)\partial_x + \eta(x,t,u)\partial_t + \varphi(x,t,u)\partial_u. \tag{2.2}
$$

and, its fifth prolongation is

$$
\text{Pr}^{(5)}v = v + \varphi_x \partial_{u_x} + \varphi_t \partial_{u_t} + \varphi_{x^2} \partial_{u_{x^2}} + \varphi_{x^3} \partial_{u_{x^3}} + \varphi_{x^4} \partial_{u_{x^4}} + \varphi_{x^5} \partial_{u_{x^5}} + \varphi_{x^6} \partial_{u_{x^6}} + \cdots + \varphi_{x^7} \partial_{u_{x^7}}, \tag{2.3}
$$

where

$$
\varphi_x = D_x(\varphi - \xi u_x - \eta u_t), \\
\varphi_t = D_t(\varphi - \xi u_x - \eta u_t), \\
\vdots \\
\varphi_{x^5} = D_x^5(\varphi - \xi u_x - \eta u_t). \tag{2.4}
$$
Table 1. The commutator table

<table>
<thead>
<tr>
<th>[v_i, v_j]</th>
<th>v_1</th>
<th>v_2</th>
<th>v_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>v_1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>v_2</td>
<td>0</td>
<td>0</td>
<td>v_2</td>
</tr>
<tr>
<td>v_3</td>
<td>0</td>
<td>−v_2</td>
<td>0</td>
</tr>
</tbody>
</table>

where $D_x$ and $D_t$ are the total derivatives with respect to $x$ and $t$ respectively. The vector field $v$ generates a one parameter symmetry group of the equation (1.2) if and only if

$$P_v^{(5)}v[\Delta] |_{\Delta=0} = 0,$$

(2.5)

The condition (2.5) is equivalent to

$$\begin{align*}
(3u_x - u_x^3 + u_x^5) \phi + \phi^t + (3u - 2u_x^2 + 2u_x^4) \phi x - u\phi x^3 \\
- 2u_x \phi x^2 - \phi x^2 t + 2u_x \phi x^4 + u\phi x^5 + \phi x^4 t |_{\Delta=0} &= 0,
\end{align*}$$

(2.6)

Substituting (2.4) into (2.6), and equating the coefficients of the various monomials in partial derivatives with respect to $x$ and various power of $u$, we can find the determining equations for the symmetry group of the equation (1.2). Solving these equations, we get the following forms of the coefficient functions

$$\begin{align*}
\xi &= c_3, \\
\eta &= c_1 t + c_2, \\
\phi &= -c_1 u.
\end{align*}$$

(2.7)

where $c_1$, $c_2$ and $c_3$ are arbitrary constant. Thus, the Lie algebra $g$ of infinitesimal symmetry of the equation (1.2) is spanned by the three vector fields

$$v_1 = \partial_x, \quad v_2 = \partial_t, \quad v_3 = t\partial_t - u\partial_u.$$

(2.8)

The commutation relations between these vector fields are given in the Table 1. The Lie algebra $g$ is solvable, because if $g^{(1)} = \langle v_i, [v_i, v_j] \rangle = [g, g]$, we have $g^{(1)} = \langle v_1, v_2, v_3 \rangle$, and $g^{(2)} = [g^{(1)}, g^{(1)}] = \langle v_2 \rangle$, so, we have a chain of ideals $g^{(1)} \supset g^{(2)} \supset \{0\}$.

To obtain the group transformation which is generated by the infinitesimal generators $v_i$ for $i = 1, 2, 3$ we need to solve the three systems of first order ordinary differential equations

$$\begin{align*}
\frac{d\tilde{x}(s)}{ds} &= \xi_i(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)), & \tilde{x}(0) &= x, \\
\frac{d\tilde{t}(s)}{ds} &= \eta_i(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)), & \tilde{t}(0) &= t, & i &= 1, 2, 3 \\
\frac{d\tilde{u}(s)}{ds} &= \phi_i(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)), & \tilde{u}(0) &= u.
\end{align*}$$

We get the one-parameter groups $G_i(s)$ generated by $v_i$ for $i = 1, 2, 3$

$$\begin{align*}
G_1 : (t, x, u) \mapsto (x + s, t, u), \\
G_2 : (t, x, u) \mapsto (x, t + s, u), \\
G_3 : (t, x, u) \mapsto (x, e^{st}, e^{-s}u).
\end{align*}$$

(2.9)
Consequently,

**Theorem 2.1.** If $u = f(x,t)$ is a solution of higher order CH equation, so are the functions

\[
\begin{align*}
G_1(s) \cdot f(x,t) &= f(x-s,t), \\
G_2(s) \cdot f(x,t) &= f(x,t-s), \\
G_3(s) \cdot f(x,t) &= f(x,te^{-s})e^{-s}.
\end{align*}
\]  

\quad (2.10)

3. Nonclassical symmetries for the higher order CH equation

In order to obtain nonclassical symmetries for the equation (1.2), the n-th prolongation of the vector field $v$ must be tangent to the intersection $E \cap E_Q^{(n)}$

\[\mathcal{P}_n^{(n)}v(\Delta)|_{E_Q \cap E_Q^{(n)}} = 0, \quad (3.1)\]

where $E_Q$ is system of partial differential equations:

\[Q^\alpha (x,u,u^{(1)}) = \varphi^{(\alpha)}(x,u) - \sum_{i=1}^{p} \xi^i(x,u)u^{\alpha}_i = 0, \quad (3.2)\]

known as the invariant surface conditions and the n-th prolongation of the invariant surface conditions (3.2) will be denoted by $E_Q^{(n)}$, which is a n-th order system of partial differential equations obtained by appending to (3.2) its partial derivatives with respect to the independent variables of orders $j \leq n - 1$. For more theoretical background see [1, 11].

If we assume that the coefficient of $\partial_t$ of the vector field (2.2) does not identically equal zero, then for the vector field

\[v = \xi(x,t,u)\partial_x + \partial_t + \varphi(x,t,u)\partial_u \quad (3.3)\]

the invariant surface condition is

\[u_t + \xi u_x = \varphi. \quad (3.4)\]

Calculating equations (3.1) and inserting $\varphi$ from (3.4) into it, we can find the determining equations by equating the coefficients of the various monomials in partial derivatives with respect to $x$ and various power of $u$. Solving these equations, we get $\xi = c$ and $\varphi = 0$, where $c$ is arbitrary constant.

Now assume that the coefficient of $\partial_t$ in (3.3) equals zero and try to find the infinitesimal nonclassical symmetries of the form

\[v = \partial_x + \varphi(x,t,u)\partial_u \quad (3.5)\]

for which the invariant surface conditions is $u_x = \varphi$. Similar the previous case, we can find determining equations. Solving this equations, we get $\varphi = 0$. This means that no supplementary symmetries, of non-classical type, are specific for our models.
Table 2. Adjoint representation table

<table>
<thead>
<tr>
<th>$Ad(\exp(\varepsilon v_i)v_j)$</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>$v_1$</td>
<td>$v_2$</td>
<td>$v_3$</td>
</tr>
<tr>
<td>$v_2$</td>
<td>$v_1$</td>
<td>$v_2$</td>
<td>$v_3 - \varepsilon v_2$</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$v_1$</td>
<td>$v_2 + \varepsilon v_3$</td>
<td>$v_3$</td>
</tr>
</tbody>
</table>

4. Optimal system for the higher order CH equation

In this section, we obtained the complete group classifications of the equation (1.2). We use the Lie series method, to compute the adjoint representation

$$Ad(\exp(\varepsilon v_i)v_j) = v_j - \varepsilon [v_i, v_j] + \frac{\varepsilon^2}{2} [v_i, [v_i, v_j]] - \cdots,$$

where $[v_i, v_j]$ is the commutator for the Lie algebra, $\varepsilon$ is a parameter, and $i, j = 1, 2, 3$. Then we have the Table 2.

**Theorem 4.1.** An optimal system of the higher order CH equation is

(1) $\alpha v_1 + v_3$, (2) $\beta v_1 + v_2$

**Proof.** Consider the symmetry algebra $g$ of the equation (1.2) whose adjoint representation was determined in table 2 and let $F^\varepsilon_i: g \to g$ defined by $v \mapsto Ad(\exp(\varepsilon v_i)v$ is a linear map, for $i = 1, 2, 3$. The matrices $M^\varepsilon_i$ of $F^\varepsilon_i$, $i = 1, 2, 3$, with respect to basis $\{v_1, v_2, v_3\}$ are

$$M^\varepsilon_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M^\varepsilon_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \varepsilon \\ 0 & 0 & 1 \end{pmatrix}, \quad M^\varepsilon_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\varepsilon} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $X = \sum_{i=1}^{3} a_i v_i$ is a nonzero vector field in $g$. We will simplify as many of the coefficients $a_i$ as possible by acting these matrices on a vector field $X$ alternatively.

Suppose first that $a_3 \neq 0$, scaling $X$ if necessary we can assume that $a_3 = 1$, then we can make the coefficients of $v_2$ vanish by $M^\varepsilon_2$, and $X$ reduced to case 1. If $a_3 = 0$ and $a_2 \neq 0$, then we cannot make vanish the coefficients of $v_1$ and $v_2$ by acting any matrices $M^\varepsilon_i$. Scaling $X$ if necessary, we can assume that $a_2 = 1$ and $X$ reduced to case 2. □

5. Symmetry reduction and differential invariants for the higher order CH equation

In this section, we will reduce equation (1.2) to ordinary differential equations and obtain some invariant solutions with respect to symmetries.

We can now compute the invariants associated with the symmetry operators, they can be obtained by integrating the characteristic equations. For example for the operator $\alpha v_1 + v_2 = \alpha \partial_x + \partial_t$ characteristic equation is

$$\frac{dx}{\alpha} = \frac{dt}{1} = \frac{du}{0}.$$

(5.1)
**Table 3. Invariants**

<table>
<thead>
<tr>
<th>operator</th>
<th>y</th>
<th>v</th>
<th>u</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v_1)</td>
<td>(t)</td>
<td>u</td>
<td>(v(y))</td>
</tr>
<tr>
<td>(v_2)</td>
<td>(x)</td>
<td>u</td>
<td>(v(y))</td>
</tr>
<tr>
<td>(v_3)</td>
<td>(x)</td>
<td>(t u)</td>
<td>(\frac{1}{y} v(y))</td>
</tr>
<tr>
<td>(\alpha v_1 + v_3)</td>
<td>(x - \log(t))</td>
<td>(t u)</td>
<td>(\frac{1}{7} v(y))</td>
</tr>
<tr>
<td>(\alpha v_1 + v_2)</td>
<td>(x - \alpha t)</td>
<td>u</td>
<td>(v(y))</td>
</tr>
</tbody>
</table>

**Table 4. Reduced equations**

<table>
<thead>
<tr>
<th>operator</th>
<th>similarity reduced equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v_1)</td>
<td>(v_y = 0)</td>
</tr>
<tr>
<td>(v_2)</td>
<td>(3 v v_y - 2 v_y v_y^2 - v v_y^3)</td>
</tr>
<tr>
<td>(v_3)</td>
<td>(-v + v_y^2 - v y^4 + 3 v v_y - 2 v_y v_y^2)</td>
</tr>
<tr>
<td>(\alpha v_1 + v_2)</td>
<td>(-\alpha v_y + \alpha v_y^3 - \alpha v_y^5 + 3 v v_y)</td>
</tr>
<tr>
<td>(\alpha v_1 + v_3)</td>
<td>(-v - \alpha v_y + v_y^2 + \alpha v_y^3 - v y^4 - \alpha v_y^5 + 3 v v_y - 2 v_y v_y^2 - v v_y^3 + 2 v_y v_y^4 + v v_y^5 = 0)</td>
</tr>
</tbody>
</table>

The corresponding invariants are \(y = x - \alpha t\), \(v = u\) therefore, a solution of our equation in this case is \(u = v(y)\). By substituting derivatives of \(u\) are given in terms of \(v\) and \(y\), we obtain the ordinary differential equation

\[-\alpha v_y + \alpha v_y^3 - \alpha v_y^5 + 3 v v_y - 2 v_y v_y^2 - v v_y^3 + 2 v_y v_y^4 + v v_y^5 = 0. \quad (5.2)\]

All results are coming in the tables 3 and 4.

For finding the differential invariants of the equation (1.2) up to order two, we should solve the following systems of PDEs:

\[
\frac{\partial I}{\partial x}, \quad \frac{\partial I}{\partial t}, \quad t \frac{\partial I}{\partial t} - u \frac{\partial I}{\partial u}, \quad (5.3)
\]

where \(I\) is a smooth function of \((x, t, u)\),

\[
\frac{\partial I_1}{\partial x}, \quad \frac{\partial I_1}{\partial t}, \quad t \frac{\partial I_1}{\partial t} - u \frac{\partial I_1}{\partial u} - u_x \frac{\partial I_1}{\partial u_x} - 2 u_t \frac{\partial I_1}{\partial u_t}, \quad (5.4)
\]

where \(I_1\) is a smooth function of \((x, t, u, u_x, u_t)\).
\[
\frac{\partial I_2}{\partial x} - u \frac{\partial I_2}{\partial t} - u_x \frac{\partial I_2}{\partial u_x} - 2 u_t \frac{\partial I_2}{\partial u_t} - 2 u_x \frac{\partial I_2}{\partial u_x} - 3 u_x \frac{\partial I_2}{\partial u_{xx}}.
\]

\(I_2\) is a smooth function of \((x, t, u, u_x, u_{xx}, u_{xt}, u_{tt})\). The solutions of PDEs systems (5.3), (5.4) and (5.5) coming in table 5.

**Table 5. differential invariants**

<table>
<thead>
<tr>
<th>vector field</th>
<th>invariant</th>
<th>1st order</th>
<th>2nd order</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v_1)</td>
<td>(t, u)</td>
<td>(u_x, u_t)</td>
<td>(u_{xx}, u_{xt}, u_{tt})</td>
</tr>
<tr>
<td>(v_2)</td>
<td>(x, u)</td>
<td>(u_x, u_t)</td>
<td>(u_{xx}, u_{xt}, u_{tt})</td>
</tr>
<tr>
<td>(v_3)</td>
<td>(x, t u)</td>
<td>(t u_x, t^2 u_t)</td>
<td>(t u_{xx}, t^2 u_{xt}, t^3 u_{tt})</td>
</tr>
</tbody>
</table>

6. Conservation laws for the higher order CH equation

There are several methods for computing conservation laws as the method based on the Noether’s theorem, the multiplier method, Lie-Bäcklund symmetry generators of the PDE, the direct method, etc. [2, 3, 11, 12].

Now, we use the multiplier method and apply the homotopy operator [13] to construct conservation laws for equation (1.2).

By Theorem 1.3.3 in [3], a set of non-singular local multipliers

\[\{\Lambda_\nu(x, U, \partial U, \cdots, \partial^r U)\}_{\nu=1}^l\]

yields a local conservation law for the system \(\Delta_\nu(x, u^{(n)})\) if and only if the set of identities

\[E_{U_j}(\Lambda_\nu(x, U, \partial U, \cdots, \partial^r U)\Delta_\nu(x, u^{(n)})) \equiv 0,\]

for \(j = 1, \cdots, q\) holds for arbitrary functions \(U(x)\), where \(E_{U_j}\) is the Euler operator with respect to \(U^j\).

Now, we seek all local conservation law multipliers of the form \(\Lambda = \xi(x, t, u)\) of the equation (1.2). The determining equations (6.1) become

\[E_U[\xi(x, t, U)(U_{x_t} - U_{x^2 t} + U_{x^3 t} + 3UU_x - 2U_x U_{x^3} - 2U_x U_{x^4} + UU_{x^5}]) \equiv 0,\]

where \(U(x, t)\) are arbitrary function. The solution of the determining system (6.2) given by

\[\xi = c_1 U + c_2,\]

where \(c_1\) and \(c_2\) are arbitrary constants. So local multipliers given by

1) \(\xi = 1,\) \hspace{1cm} 2) \(\xi = U,\)
Each of the local multipliers $\xi$ determines a nontrivial local conservation law $D_t\Psi + D_x\Phi = 0$ with the characteristic form
\[
D_t\Psi + D_x\Phi \equiv \xi(U_t - U_{xt} + U_{x^3} + 3UU_x - 2U_xU_{x^2} - UU_{x^3} + 2UU_{x^4} + UU_{x^5}),
\]
(6.5)

To calculate the conserved quantities $\Psi$ and $\Phi$, we apply the homotopy operator (see [10]) which yield of multiplier $\xi = 1$, therefore
\[
\Phi = u - \frac{1}{3} u_{x^2} + \frac{1}{5} u_{x^4},
\]
\[
\Psi = \frac{2}{3} u^2 - \frac{1}{2} u_x^2 - \frac{2}{3} u_{xt} - uu_{x^2} + u_x u_{x^3} - \frac{1}{2} u_{x^2}^2 + \frac{4}{5} u_{x^4} + uu_{x^4},
\]
(6.6)

so, we have the first conservation law of the higher order CH equation respect to multiplier $\xi = 1$
\[
D_t\left(u - \frac{1}{3} u_{x^2} + \frac{1}{5} u_{x^4}\right) + D_x\left(\frac{2}{3} u^2 - \frac{1}{2} u_x^2 - \frac{2}{3} u_{xt} - uu_{x^2} + u_x u_{x^3} - \frac{1}{2} u_{x^2}^2 + \frac{4}{5} u_{x^4} + uu_{x^4}\right) = 0.
\]
(6.7)

Similarly for conservation law respect to multiplier $\xi = u$, by applying 2-dimensional homotopy operator, we have
\[
\Phi = \frac{1}{2} u^2 - \frac{1}{4} uu_{x^2} + \frac{1}{6} u_x^2 + \frac{1}{5} uu_{x^4} - \frac{1}{3} u_{x^2} u_{x^3} + \frac{1}{10} u_{x^4}^2,
\]
\[
\Psi = u^3 + u^2 u_{x^2} - \frac{1}{3} uu_{x^4} + \frac{1}{3} u_x u_x - u^2 u_{x^2} + \frac{4}{3} uu_{x^3} - \frac{1}{2} u_{x^2} u_{x^4} - \frac{3}{5} u_{x^2} u_{x^4} + \frac{2}{5} u_{x^2} u_x - \frac{2}{3} u_{x^2} u_{x^4} + \frac{2}{3} uu_{x^4},
\]
(6.8)

so, the second conservation low of the higher order CH equation is
\[
D_t\left(\frac{1}{2} u^2 - \frac{1}{3} uu_{x^2} + \frac{1}{6} u_x^2 + \frac{1}{5} uu_{x^4} - \frac{1}{3} u_{x^2} u_{x^3} + \frac{1}{10} u_{x^4}^2\right) + D_x\left(u^3 + u^2 u_{x^2} - \frac{1}{3} uu_{x^4} + \frac{1}{3} u_x u_x - u^2 u_{x^2} + \frac{4}{3} uu_{x^3} - \frac{1}{2} u_{x^2} u_{x^4} - \frac{3}{5} u_{x^2} u_{x^4} + \frac{2}{5} u_{x^2} u_x - \frac{2}{3} u_{x^2} u_{x^4} + \frac{2}{3} uu_{x^4}\right) = 0.
\]
(6.9)

REFERENCES


