Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 8, No. 2, 2020, pp. 364-372
DOI:10.22034/cmde.2020.26920.1356

Symmetry analysis and conservation laws for higher order Camassa-Holm equation

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Abstract In this paper, Lie symmetry group method is applied to study for the higher order Camassa-Holm equation. Complete analysis of symmetries and nonclassical symmetries is discussed. Furthermore, optimal system, preliminary classification of its group invariant solutions and symmetry reduction are investigated. Finally conservation laws for the higher order Camassa-Holm equation which conserved quantities arise from multipliers by using homotopy operator are presented.

Keywords. Lie symmetry, Higher order Camassa-Holm equation, Optimal system, Similarity solutions, Conservation laws.

2010 Mathematics Subject Classification. 58J70, 35L65, 22E70.

1. INTRODUCTION

In recent years, many researchers have been researched on the Camassa-Holm equation. They extend the studies to the generalized CH equation, higher order CH equations and so on. Lixin Tian, Chunyu Shen and Danping Ding gave the optimal control of the viscous CH equation under the boundary condition and proved the existence and uniqueness of optimal solution to the viscous CH equation in a short interval [14]. Using geometrical methods, higher order CH equations have been treated in [7]. The well-posedness of higher order CH equations were considered in [6]. Conservation laws and soliton solutions for modified Camassa-Holm equation were studied in [8]. Several new types of bounded wave solutions for the generalized two-component Camassa-Holm equation obtained in [15].

The formulation of the higher order Camassa-Holm equation which was recently derived by Coclite, Holden and Karlsen in [6] is

$$\partial_t u = B_k(u, u), \qquad t > 0,\ k \in \mathbb{N} \cup 0,\ x \in \mathbb{R}, \tag{1.1}$$

Received: 28 April 2018 ; Accepted: 01 January 2019.

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where $u = u(x,t) : \mathbb{R} \times [0,\infty) \longrightarrow \mathbb{R}$ is the unknown function and

$$\begin{aligned} B_k(u,u) &:= A_k^{-1} C_k(u) - u\partial_x u, \\ A_k(u) &:= \sum_{j=0}^{k} (-1)^j \partial_x^{2j} u, \\ C_k(u) &:= -uA_k(\partial_x u) + A_k(u\partial_x u) - 2\partial_x u A_k(u). \end{aligned}$$

In cases $k = 0$ and $k = 1$, equation (1.1) becomes the inviscid Burgers equation and the Camassa-Holm equation respectively.

In this paper we consider the case $k = 2$ of equation (1.1). It also can be rewritten as

$$\Delta := u_t - u_{x^2 t} + u_{x^4 t} + 3uu_x - 2u_x u_{x^2} - uu_{x^3} + 2u_x u_{x^4} + uu_{x^5} = 0. \tag{1.2}$$

2. Lie symmetries for the higher order CH equation

In this section, we give the general form of a infinitesimal generator admitted by equation (1.2) and find transformed solutions. For the general procedure to determining symmetries for any system of partial differential equations and more details see [4, 5, 9, 11].

We consider the one parameter Lie group of infinitesimal transformations on (x,t,u),

$$\begin{aligned} \tilde{x} &= x + s\xi(x,t,u) + O(s^2), \\ \tilde{t} &= t + s\eta(x,t,u) + O(s^2), \\ \tilde{u} &= u + s\varphi(x,t,u) + O(s^2), \end{aligned} \tag{2.1}$$

where s is the group parameter and ξ, η and φ are the infinitesimals of the transformations for the independent and dependent variables, respectively. The associated vector field is of the form:

$$\mathbf{v} = \xi(x,t,u)\partial_x + \eta(x,t,u)\partial_t + \varphi(x,t,u)\partial_u. \tag{2.2}$$

and, its fifth prolongation is

$$\begin{aligned} \mathrm{Pr}^{(5)}\mathbf{v} = \mathbf{v} &+ \varphi^x \,\partial_{u_x} + \varphi^t \,\partial_{u_t} + \varphi^{x^2} \,\partial_{u_{x^2}} + \varphi^{xt} \,\partial_{u_{xt}} \\ &+ \varphi^{t^2} \,\partial_{u_{t^2}} + \varphi^{x^3} \,\partial_{u_{x^3}} + \varphi^{x^2 t} \,\partial_{u_{x^2 t}} + \varphi^{xt^2} \,\partial_{u_{xt^2}} \\ &+ \varphi^{t^3} \,\partial_{u_{t^3}} + \cdots + \varphi^{xt^4} \,\partial_{u_{xt^4}} + \varphi^{t^5} \,\partial_{u_{t^5}}. \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \varphi^x &= D_x(\varphi - \xi\, u_x - \eta\, u_t) + \xi\, u_{x^2} + \eta\, u_{xt}, \\ \varphi^t &= D_t(\varphi - \xi\, u_x - \eta\, u_t) + \xi\, u_{xt} + \eta\, u_{t^2}, \\ &\ \vdots \\ \varphi^{t^5} &= D_x^5(\varphi - \xi\, u_x - \eta\, u_t) + \xi\, u_{x^5 t} + \eta\, u_{t^5}, \end{aligned} \tag{2.4}$$

TABLE 1. The commutator table

$[\mathbf{v}_i, \mathbf{v}_j]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	0	0	0
$\mathbf{v}_2$	0	0	$\mathbf{v}_2$
$\mathbf{v}_3$	0	$-\mathbf{v}_2$	0

where D_x and D_t are the total derivatives with respect to x and t respectively. The vector field $\mathbf{v}$ generates a one parameter symmetry group of the equation (1.2) if and only if

$$\mathrm{Pr}^{(5)}\mathbf{v}[\Delta] \mid_{\Delta=0}= 0, \tag{2.5}$$

The condition (2.5) is equivalent to

$$\begin{aligned}&(3u_x - u_{x^3} + u_{x^5})\varphi + \varphi^t + (3u - 2u_{x^2} + 2u_{x^4})\varphi^x - u\varphi^{x^3}\\ &- 2u_x\varphi^{x^2} - \varphi^{x^2 t} + 2u_x\varphi^{x^4} + u\varphi^{x^5} + \varphi^{x^4 t} \mid_{\Delta=0}= 0,\end{aligned} \tag{2.6}$$

Substituting (2.4) into (2.6), and equating the coefficients of the various monomials in partial derivatives with respect to x and various power of u, we can find the determining equations for the symmetry group of the equation (1.2). Solving these equations, we get the following forms of the coefficient functions

$$\xi = c_3, \qquad \eta = c_1 t + c_2, \qquad \varphi = -c_1 u. \tag{2.7}$$

where c_1, c_2 and c_3 are arbitrary constant. Thus, the Lie algebra $\mathbf{g}$ of infinitesimal symmetry of the equation (1.2) is spanned by the three vector fields

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t, \qquad \mathbf{v}_3 = t\partial_t - u\partial_u. \tag{2.8}$$

The commutation relations between these vector fields are given in the Table 1. The Lie algebra $\mathbf{g}$ is solvable, because if $\mathbf{g}^{(1)} = \langle \mathbf{v}_i, [\mathbf{v}_i, \mathbf{v}_j] \rangle = [\mathbf{g}, \mathbf{g}]$, we have $\mathbf{g}^{(1)} = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$, and $\mathbf{g}^{(2)} = [\mathbf{g}^{(1)}, \mathbf{g}^{(1)}] = \langle \mathbf{v}_2 \rangle$, so, we have a chain of ideals $\mathbf{g}^{(1)} \supset \mathbf{g}^{(2)} \supset \{0\}$.

To obtain the group transformation which is generated by the infinitesimal generators $\mathbf{v}_i$ for $i = 1, 2, 3$ we need to solve the three systems of first order ordinary differential equations

$$\begin{aligned}\frac{d\tilde{x}(s)}{ds} &= \xi_i(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)), \quad \tilde{x}(0) = x,\\ \frac{d\tilde{t}(s)}{ds} &= \eta_i(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)), \quad \tilde{t}(0) = t, \qquad i = 1, 2, 3\\ \frac{d\tilde{u}(s)}{ds} &= \varphi_i(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)), \quad \tilde{u}(0) = u.\end{aligned}$$

We get the one-parameter groups $G_i(s)$ generated by $\mathbf{v}_i$ for $i = 1, 2, 3$

$$\begin{aligned}G_1 : (t, x, u) &\longmapsto (x + s, t, u),\\ G_2 : (t, x, u) &\longmapsto (x, t + s, u),\\ G_3 : (t, x, u) &\longmapsto (x, \mathrm{e}^s t, \mathrm{e}^{-s} u).\end{aligned} \tag{2.9}$$

Consequently,

Theorem 2.1. *If $u = f(x,t)$ is a solution of higher order CH equation, so are the functions*

$$\begin{aligned} G_1(s)\cdot f(x,t) &= f(x-s,t), \\ G_2(s)\cdot f(x,t) &= f(x,t-s), \\ G_3(s)\cdot f(x,t) &= f(x,te^{-s})e^{-s}. \end{aligned} \tag{2.10}$$

3. NONCLASSICAL SYMMETRIES FOR THE HIGHER ORDER CH EQUATION

In order to obtain nonclassical symmetries for the equation (1.2), the n-th prolongation of the vector field $\mathbf{v}$ must be tangent to the intersection $E \cap E_Q^{(n)}$

$$\mathrm{Pr}^{(n)}\mathbf{v}(\Delta)|_{E_Q \cap E_Q^{(n)}} = 0, \tag{3.1}$$

where E_Q is system of partial differential equations:

$$Q^\alpha(x,u,u^{(1)}) = \varphi^{(\alpha)}(x,u) - \sum_{i=1}^{p} \xi^i(x,u)u_i^\alpha = 0, \tag{3.2}$$

known as the invariant surface conditions and the n-th prolongation of the invariant surface conditions (3.2) will be denoted by $E_Q^{(n)}$, which is a n-th order system of partial differential equations obtained by appending to (3.2) its partial derivatives with respect to the independent variables of orders $j \leq n-1$. For more theoretical background see [1, 11].

If we assume that the coefficient of ∂_t of the vector field (2.2) does not identically equal zero, then for the vector field

$$\mathbf{v} = \xi(x,t,u)\partial_x + \partial_t + \varphi(x,t,u)\partial_u \tag{3.3}$$

the invariant surface condition is

$$u_t + \xi u_x = \varphi. \tag{3.4}$$

Calculating equations (3.1) and inserting φ from (3.4) in to it, we can find the determining equations by equating the coefficients of the various monomials in partial derivatives with respect to x and various power of u. Solving these equations, we get $\xi = c$ and $\varphi = 0$, where c is arbitrary constant.

Now assume that the coefficient of ∂_t in (3.3) equals zero and try to find the infinitesimal nonclassical symmetries of the form

$$\mathbf{v} = \partial_x + \varphi(x,t,u)\partial_u \tag{3.5}$$

for which the invariant surface conditions is $u_x = \varphi$. Similar the previous case, we can find determining equations. Solving this equations, we get $\varphi = 0$. This means that no supplementary symmetries, of non-classical type, are specific for our models.

TABLE 2. Adjoint representation table

$Ad(\exp(\varepsilon\mathbf{v}_i)\mathbf{v}_j)$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3-\varepsilon\mathbf{v}_2$
$\mathbf{v}_3$	$\mathbf{v}_1$	$\mathbf{v}_2+\varepsilon\mathbf{v}_3$	$\mathbf{v}_3$

4. OPTIMAL SYSTEM FOR THE HIGHER ORDER CH EQUATION

In this section, we obtained the complete group classifications of the equation (1.2). We use the Lie series method, to compute the adjoint representation

$$\mathrm{A}d(\exp(\varepsilon\mathbf{v}_i)\mathbf{v}_j)=\mathbf{v}_j-\varepsilon[\mathbf{v}_i,\mathbf{v}_j]+\frac{\varepsilon^2}{2}[\mathbf{v}_i,[\mathbf{v}_i,\mathbf{v}_j]]-\cdots,$$

where $[\mathbf{v}_i,\mathbf{v}_j]$ is the commutator for the Lie algebra, ε is a parameter, and $i,j=1,2,3$. Then we have the Table 2.

Theorem 4.1. *An optimal system of the higher order CH equation is*
(1) $\alpha\boldsymbol{v}_1+\boldsymbol{v}_3$, *(2)* $\beta\boldsymbol{v}_1+\boldsymbol{v}_2$

Proof. Consider the symmetry algebra $\mathbf{g}$ of the equation (1.2) whose adjoint representation was determined in table 2 and let $F_i^\varepsilon:\mathbf{g}\to\mathbf{g}$ defined by $\mathbf{v}\mapsto\mathrm{Ad}(\exp(\varepsilon\mathbf{v}_i)\mathbf{v})$ is a linear map, for $i=1,2,3$. The matrices M_i^ε of F_i^ε, $i=1,2,3$, with respect to basis $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ are

$$M_1^\varepsilon=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ M_2^\varepsilon=\begin{pmatrix}1&0&0\\0&1&\varepsilon\\0&0&1\end{pmatrix},\ M_3^\varepsilon=\begin{pmatrix}1&0&0\\0&\mathrm{e}^{-\varepsilon}&0\\0&0&1\end{pmatrix}.$$

Let $\mathbf{X}=\sum_{i=1}^3 a_i\mathbf{v}_i$ is a nonzero vector field in $\mathbf{g}$. We will simplify as many of the coefficients a_i as possible by acting these matrices on a vector field $\mathbf{X}$ alternatively.

Suppose first that $a_3\neq 0$, scaling $\mathbf{X}$ if necessary we can assume that $a_3=1$, then we can make the coefficients of $\mathbf{v}_2$ vanish by M_2^ε, and $\mathbf{X}$ reduced to case 1. If $a_3=0$ and $a_2\neq 0$, then we cannot make vanish the coefficients of $\mathbf{v}_1$ and $\mathbf{v}_2$ by acting any matrices M_i^ε. Scaling $\mathbf{X}$ if necessary, we can assume that $a_2=1$ and $\mathbf{X}$ reduced to case 2. □

5. SYMMETRY REDUCTION AND DIFFERENTIAL INVARIANTS FOR THE HIGHER ORDER CH EQUATION

In this section, we will reduce equation (1.2) to ordinary differential equations and obtain some invariant solutions with respect to symmetries.

We can now compute the invariants associated with the symmetry operators, they can be obtained by integrating the characteristic equations. For example for the operator $\alpha\mathbf{v}_1+\mathbf{v}_2=\alpha\partial_x+\partial_t$ characteristic equation is

$$\frac{dx}{\alpha}=\frac{dt}{1}=\frac{du}{0}. \tag{5.1}$$

TABLE 3. Invariants

operator	y	v	u
$\mathbf{v}_1$	t	u	$v(y)$
$\mathbf{v}_2$	x	u	$v(y)$
$\mathbf{v}_3$	x	$t\,u$	$\frac{1}{t}v(y)$
$\alpha\,\mathbf{v}_1+\mathbf{v}_3$	$x-\log(t)$	$t\,u$	$\frac{1}{t}v(y)$
$\alpha\,\mathbf{v}_1+\mathbf{v}_2$	$x-\alpha\,t$	u	$v(y)$

TABLE 4. Reduced equations

operator	similarity reduced equations
$\mathbf{v}_1$	$v_y=0$
$\mathbf{v}_2$	$3vv_y-2v_yv_{y^2}-vv_{y^3}$ $+2v_yv_{y^4}+vv_{y^5}=0$
$\mathbf{v}_3$	$-v+v_{y^2}-v_{y^4}+3vv_y-2v_yv_{y^2}$ $-vv_{y^3}+2v_yv_{y^4}+vv_{y^5}=0$
$\alpha\,\mathbf{v}_1+\mathbf{v}_2$	$-\alpha\,v_y+\alpha\,v_{y^3}-\alpha\,v_{y^5}+3\,v\,v_y$ $-2\,v_yv_{y^2}-v\,v_{y^3}+2\,v_yv_{y^4}+v\,v_{y^5}=0$
$\alpha\,\mathbf{v}_1+\mathbf{v}_3$	$-v-\alpha\,v_y+v_{y^2}+\alpha\,v_{y^3}-v_{y^4}-\alpha\,v_{y^5}+$ $3vv_y-2v_yv_{y^2}-vv_{y^3}+2v_yv_{y^4}+vv_{y^5}=0$

The corresponding invariants are $y=x-\alpha\,t$, $v=u$ therefore, a solution of our equation in this case is $u=v(y)$. By substituting derivatives of u are given in terms of v and y, we obtain the ordinary differential equation

$$-\alpha v_y+\alpha v_{y^3}-\alpha v_{y^5}+3vv_y-2v_yv_{y^2}-vv_{y^3}+2v_yv_{y^4}+vv_{y^5}=0. \tag{5.2}$$

All results are coming in the tables 3 and 4.

For finding the differential invariants of the equation (1.2) up to order two, we should solve the following systems of PDEs:

$$\frac{\partial I}{\partial x},\qquad \frac{\partial I}{\partial t},\qquad t\,\frac{\partial I}{\partial t}-u\,\frac{\partial I}{\partial u}, \tag{5.3}$$

where I is a smooth function of (x,t,u),

$$\frac{\partial I_1}{\partial x},\qquad \frac{\partial I_1}{\partial t},\qquad t\,\frac{\partial I_1}{\partial t}-u\,\frac{\partial I_1}{\partial u}-u_x\,\frac{\partial I_1}{\partial u_x}-2u_t\,\frac{\partial I_1}{\partial u_t}, \tag{5.4}$$

where I_1 is a smooth function of (x,t,u,u_x,u_t),

$$
\begin{aligned}
&\frac{\partial I_2}{\partial x}, \qquad \frac{\partial I_2}{\partial t}, \\
&t\,\frac{\partial I_2}{\partial t} - u\,\frac{\partial I_2}{\partial u} - u_x\,\frac{\partial I_1}{\partial u_x} - 2u_t\,\frac{\partial I_2}{\partial u_t} - u_{x^2}\,\frac{\partial I_2}{\partial u_{x^2}} - 2\,u_{xt}\,\frac{\partial I_2}{\partial u_{xt}} - 3\,u_{t^2}\,\frac{\partial I_2}{\partial u_{t^2}},
\end{aligned}
\tag{5.5}
$$

I_2 is a smooth function of $(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt})$. The solutions of PDEs systems (5.3), (5.4) and (5.5) coming in table 5.

TABLE 5. differential invariants

vector field	invariant	1st order	2nd order
$\mathbf{v}_1$	t, u	u_x, u_t	u_{xx}, u_{xt}, u_{tt}
$\mathbf{v}_2$	x, u	u_x, u_t	u_{xx}, u_{xt}, u_{tt}
$\mathbf{v}_3$	$x, t\,u$	$t\,u_x, t^2\,u_t$	$t\,u_{xx}, t^2\,u_{xt}, t^3\,u_{tt}$

6. CONSERVATION LAWS FOR THE HIGHER ORDER CH EQUATION

There are several methods for computing conservation laws as the method based on the Noether's theorem, the multiplier method, Lie-Bcklund symmetry generators of the PDE, the direct method, etc.[2, 3, 11, 12].

Now, we use the multiplier method and apply the homotopy operator [13] to construct conservation laws for equation (1.2).

By Theorem 1.3.3 in [3], a set of non-singular local multipliers

$$\{\Lambda_\nu(x, U, \partial_U, \cdots, \partial_U^r)\}_{\nu=1}^{l}$$

yields a local conservation law for the system $\Delta_\nu(x, u^{(n)})$ if and only if the set of identities

$$E_{U^j}(\Lambda_\nu(x, U, \partial_U, \cdots, \partial_U^r)\Delta_\nu(x, u^{(n)})) \equiv 0, \tag{6.1}$$

for $j = 1, \cdots q$ holds for arbitrary functions $U(x)$, where E_{U^j} is the Euler operator with respect to U^j.

Now, we seek all local conservation law multipliers of the form $\Lambda = \xi(x, t, u)$ of the equation (1.2). The determining equations (6.1) become

$$
\begin{aligned}
&E_U[\xi(x, t, U)(U_t - U_{x^2t} + U_{x^4t} + 3UU_x - \\
&\quad 2U_xU_{x^2} - UU_{x^3} + 2U_xU_{x^4} + UU_{x^5})] \equiv 0,
\end{aligned}
\tag{6.2}
$$

where $U(x, t)$ are arbitrary function. The solution of the determining system (6.2) given by

$$\xi = c_1\,U + c_2, \tag{6.3}$$

where c_1 and c_2 are arbitrary constants. So local multipliers given by

$$1)\ \xi = 1, \qquad 2)\ \xi = U, \tag{6.4}$$

Each of the local multipliers ξ determines a nontrivial local conservation law $D_t\Psi + D_x\Phi = 0$ with the characteristic form

$$\begin{aligned} D_t\Psi + D_x\Phi \equiv \xi(&U_t - U_{x^2t} + U_{x^4t} + 3UU_x - \\ &2U_xU_{x^2} - UU_{x^3} + 2U_xU_{x^4} + UU_{x^5}), \end{aligned} \tag{6.5}$$

To calculate the conserved quantities Ψ and Φ, we apply the homotopy operator (see [10]) which yield of multiplier $\xi = 1$, therefore

$$\begin{aligned} \Phi &= u - \tfrac{1}{3}\,u_{x^2} + \tfrac{1}{5}\,u_{x^4}, \\ \Psi &= \tfrac{3}{2}\,u^2 - \tfrac{1}{2}\,u_x^2 - \tfrac{2}{3}\,u_{xt} - uu_{x^2} + u_xu_{x^3} - \\ &\quad \tfrac{1}{2}\,u_{x^2}^2 + \tfrac{4}{5}\,u_{x^3t} + uu_{x^4}, \end{aligned} \tag{6.6}$$

so, we have the first conservation low of the higher order CH equation respect to multiplier $\xi = 1$

$$\begin{aligned} &D_t\big(u - \tfrac{1}{3}\,u_{x^2} + \tfrac{1}{5}\,u_{x^4}\big) + D_x\big(\tfrac{3}{2}\,u^2 - \tfrac{1}{2}\,u_x^2 - \tfrac{2}{3}\,u_{xt} \\ &\quad -uu_{x^2} + u_xu_{x^3} - \tfrac{1}{2}\,u_{x^2}^2 + \tfrac{4}{5}\,u_{x^3t} + uu_{x^4}\big) = 0. \end{aligned} \tag{6.7}$$

similarly for conservation law respect to multiplier $\xi = u$, by applying 2-dimensional homotopy operator, we have

$$\begin{aligned} \Phi &= \tfrac{1}{2}\,u^2 - \tfrac{1}{3}\,uu_{x^2} + \tfrac{1}{6}\,u_x^2 + \tfrac{1}{5}\,uu_{x^4} - \tfrac{1}{5}\,u_xu_{x^3} + \tfrac{1}{10}\,u_{x^2}^2, \\ \Psi &= u^3 + u^2u_{x^4} - \tfrac{2}{3}\,uu_{xt} + \tfrac{1}{3}\,u_xu_t - u^2u_{x^2} + \tfrac{4}{5}uu_{x^3t} \\ &\quad -\tfrac{1}{5}\,u_{x^3}u_t - \tfrac{3}{5}\,u_xu_{x^2t} + \tfrac{2}{5}\,u_{x^2}u_{xt}, \end{aligned} \tag{6.8}$$

so, the second conservation low of the higher order CH equation is

$$\begin{aligned} &D_t\big(\tfrac{1}{2}\,u^2 - \tfrac{1}{3}\,uu_{x^2} + \tfrac{1}{6}\,u_x^2 + \tfrac{1}{5}\,uu_{x^4} - \tfrac{1}{5}\,u_xu_{x^3} + \tfrac{1}{10}\,u_{x^2}^2\big) \\ &+D_x\big(u^3 + u^2u_{x^4} - \tfrac{2}{3}\,uu_{xt} + \tfrac{1}{3}\,u_xu_t - u^2u_{x^2} + \tfrac{4}{5}uu_{x^3t} \\ &\quad -\tfrac{1}{5}\,u_{x^3}u_t - \tfrac{3}{5}\,u_xu_{x^2t} + \tfrac{2}{5}\,u_{x^2}u_{xt}\big) = 0. \end{aligned} \tag{6.9}$$

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