Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 9, No. 1, 2021, pp. 211-224
DOI:10.22034/cmde.2019.31983.1492

Quintic Spline functions and Fredholm integral equation

Khosrow Maleknejad*
School of Mathematics,
Iran University of Science and Technology, Tehran, Iran.
E-mail: maleknejad@iust.ac.ir

Jalil Rashidinia
School of Mathematics,
Iran University of Science and Technology, Tehran, Iran.
E-mail: rashidinia@iust.ac.ir

Hamed Jalilian
School of Mathematics,
Iran University of Science and Technology, Tehran, Iran.
E-mail: hjalilian@mathdep.iust.ac.ir

Abstract A new six order method developed for the approximation Fredholm integral equation of the second kind. This method is based on the quintic spline functions (QSF). In our approach, we first formulate the Quintic polynomial spline then the solution of integral equation approximated by this spline. But we need to develop the end conditions which can be associated with the quntic spline. To avoid the reduction accuracy, we formulate the end condition in such a way to obtain the band matrix and also to obtain the same order of accuracy. The convergence of the method is discussed by using matrix algebra. Finally, four test problems have been used for numerical illustration to demonstrate the practical ability of the new method.

Keywords. Fredholm integral equation, Quintic spline function (QSF).

2010 Mathematics Subject Classification. 65R20.

1. Introduction

We consider the following Fredholm integral equation of the second kind

$$y(t) = f(t) + \int_a^b k(t,x)y(x)dx, \tag{1.1}$$

where the functions f and k are known and sufficiently smooth while y is the unknown function (for problems that are not perfectly smooth, other models of splines can be used in parallel to our proposed method, for example, trigonometric spline [31]). The existence and uniqueness of the smooth solution are given by many authors, (see, for example, [15, 34]). This study is an effort to propose a new method to obtain the numerical solution of linear Fredholm integral equations. Many methods have been developed for solving this equations such as ([5, 10, 19, 21, 32]).

Received: 9 February 2019 ; Accepted: 31 December 2019.
* corresponding.

A modified approach to the numerical solution of Fredholm integral equations of the second kind had been developed by Panda et al. [26]. Maleknejad and Mahmoudi [21] proposed hybrid Taylor and block pulse functions for solving these equations. A Sinc-collocation scheme for Fredholm integral equations of the second kind was proposed by Rashidinia and Zarebnia [28] and Tomoaki Okayama et al. [32] improved this version. Moreover, in [9] discrete collocation iterative method has been proposed to obtain the numerical solution of nonlinear Fredholm integral equations.

In [1] convergence of integro quartic and sextic B-spline interpolation is presented. Haghighi, A.R and Roohi, M. proposed the fractional Cubic Spline Interpolation without using the derivative values for studying the curves and surfaces [13]. Various methods have been used by many authors to solve Optimal Control Problems such as, [14, 24, 25, 27]. The method involving expansion of the unknown function of a Fredholm integral equation of the second kind in terms of polynomials has been obtained by A. Chakrabarti and S.C. Martha [6].

Also, the fractional differential equations of various types, play important roles not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena, can be converted to Volterra integral equation. A.Akgul et al. [2, 3, 4] solved many important models of fractional differential equations by reproducing kernel method.

The high accuracy cubic spline finite difference approximation for the solution of one-space dimensional non-linear wave equations have been developed by Mahanty et al. [22] the quintic polynomial and Non-polynomial Spline functions are used on differential equations repeatedly (see, [16, 17, 29, 30, 31, 33]) but have not been directly applied for the approximation of the integral equations so far. This work is an effort to this end.

The important relations in our method are equations (2.8) and (2.9). By using the Taylor series we obtained the truncation error of order $O(h^6)$. The better accuracy of our method compared to the existing methods such as [26] with the same number of grid points is indicated in tables.

The structure of this paper is organized as follows. Quintic polynomial spline formulation is derived in section 2. In section 3, the methodology and formulation QSF method for Fredholm integral equations are brought. Section 4, is devoted to convergence analysis. Finally, in section 5, Numerical examples are given to clarify the applications and efficiency of the method. At the end, we have a conclusion of our study.

2. Quintic spline

The numerical scheme has been developed on the domain of integration $\Omega = [a, b]$ with partitions $\Delta : \ a = x_0 < x_1 < ... < x_n = b$, where $h = \frac{b-a}{n}$. Let $S_j(x)$ be the interpolating quintic spline function which interpolates y at x_j, by following previous work in quintic spline such as ([8, 29, 33]), defined on $[x_j, x_{j+1}]$, $j = 0, ..., n-1$ as

$$S_j(x) = a_j(x-x_j)^5+b_j(x-x_j)^4+c_j(x-x_j)^3+d_j(x-x_j)^2+e_j(x-x_j)+f_j, \quad (2.1)$$

where a_j, b_j, c_j, d_j, e_j and f_j are real numbers. To derive the coefficients a_j, b_j, c_j, d_j, e_j and f_j, we define:

$$\begin{aligned} &S_j(x_i) = y_i, S_j(x_{i+1}) = y_{i+1}, S_j''(x_i) = M_i \\ &S_j''(x_{i+1}) = M_{i+1}, S_j''''(x_i) = F_i, S_j''''(x_{i+1}) = F_{i+1}, \end{aligned} \tag{2.2}$$

By algebraic manipulation, we get the following expression :

$$a_j = \frac{F_{j+1} - F_j}{120h},\ b_j = \frac{F_j}{24},\ c_j = \frac{M_{j+1} - M_j}{6h} - \frac{h(F_{j+1} + F_j)}{36},\ d_j = \frac{M_j}{2}, \tag{2.3}$$

$$e_j = \frac{y_{j+1} - y_j}{h} - \frac{h(M_{j+1} + 2M_j)}{6} + \frac{h^3(7F_{j+1} + 8F_j)}{360},\ f_j = y_j. \tag{2.4}$$

The continuity of the first derivative at $x = x_j$ implies

$$M_{j+1} + 4M_j + M_{j-1} = \frac{6}{h^2}(y_{j+1} - 2y_j + y_{j-1}) + \frac{h^2}{60}(7F_{j+1} + 16F_j + 7F_{j-1}), \tag{2.5}$$

and the continuity of the third derivative at $x = x_j$ implies

$$M_{j+1} - 2M_j + M_{j-1} = \frac{h^2}{6}(F_{j+1} + 4F_j + F_{j-1}), \quad j = 1(1)N-1, \tag{2.6}$$

subtracting (2.6) from (2.5) and dividing by 6 we obtain

$$M_j = \frac{1}{h^2}(y_{j+1} - 2y_j + y_{j-1}) - \frac{h^2}{120}(F_{j+1} + 8F_j + F_{j-1}), \quad j = 1(1)N-1, \tag{2.7}$$

Eliminating of M_j's between (2.5) and (2.6) leads to the following useful relation for $j = 2(1)N-2$

$$\begin{aligned} &F_{j+2} + 26F_{j+1} + 66F_j + 26F_{j-1} + F_{j-2} \\ &\qquad = \frac{120}{h^4}(y_{j+2} - 4y_{j+1} + 6y_j - 4y_{j-1} + y_{j-2}), \end{aligned} \tag{2.8}$$

Using the similar manner the following spline relation may be obtained

$$\begin{aligned} &M_{j+2} + 26M_{j+1} + 66M_j + 26M_{j-1} + M_{j-2} \\ &\qquad = \frac{20}{h^2}(y_{j+2} + 2y_{j+1} - 6y_j + 2y_{j-1} + y_{j-2}), \end{aligned} \tag{2.9}$$

3. The end condition

To obtain the unique solution of the linear systems (2.8) and (2.9) we need eight more equations. By using Taylor series and method of undetermined coefficients, the boundary formulas associated with boundary conditions for the fourth-order method can be determined as follows.

To obtain the fourth-order boundary formula, we define the following identities

$$\begin{aligned}
&\sum_{k=0}^{2}\gamma_k y_k'' = \frac{20}{h^2}(\sum_{k=0}^{5}\eta_k y_k) + O(h^4), j = 1,\\
&\sum_{k=0}^{3}\nu_k y_k'' = \frac{20}{h^2}(\sum_{k=0}^{5}\sigma_k y_k) + O(h^4), j = 2,\\
&\sum_{k=0}^{3}\nu_k y_{N-k}'' = \frac{20}{h^2}(\sum_{k=0}^{5}\sigma_k y_{N-k}) + O(h^4), j = N-1,\\
&\sum_{k=0}^{2}\gamma_k y_{N-k}'' = \frac{20}{h^2}(\sum_{k=0}^{5}\eta_k y_{N-k}) + O(h^4), j = N,\\
&\sum_{k=0}^{2}\gamma_k y_k'''' = \frac{120}{h^4}(\sum_{k=0}^{7}\alpha_k y_k) + O(h^4), j = 1,\\
&\sum_{k=0}^{3}\nu_k y_k'''' = \frac{120}{h^4}(\sum_{k=0}^{7}\beta_k y_k) + O(h^4), j = 2,\\
&\sum_{k=0}^{3}\nu_k y_{N-k}'''' = \frac{120}{h^4}(\sum_{k=0}^{7}\beta_k y_{N-k}) + O(h^4), j = N-1,\\
&\sum_{k=0}^{2}\gamma_k y_{N-k}'''' = \frac{120}{h^4}(\sum_{k=0}^{7}\alpha_k y_{N-k}) + O(h^4), j = N,
\end{aligned} \tag{3.1}$$

By using Taylor's expansion, we obtain the unknown coefficients in (3.1) as follows:

$$\begin{aligned}
&(\gamma_0,\gamma_1,\gamma_2,\nu_0,\nu_1,\nu_2,\nu_3) = (65, 26, 1, 26, 66, 26, 1),\\
&(\eta_0,\eta_1,\eta_2,\eta_3,\eta_4,\eta_5) = (\frac{199}{15}, \frac{-649}{15}, \frac{287}{5}, \frac{-122}{3}, \frac{238}{15}, \frac{-13}{5})\\
&(\sigma_0,\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5) = (\frac{451}{60}, \frac{-4579}{240}, \frac{189}{10}, \frac{-1373}{120}, \frac{59}{12}, \frac{-13}{16}),\\
&(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7) = (\frac{419}{12}, \frac{-3071}{90}, \frac{689}{8}, \frac{-21907}{180}, \frac{37561}{360}, \frac{-272}{5}, \frac{1151}{72}, \frac{-367}{180})\\
&(\beta_0,\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6,\beta_7) = (\frac{589}{144}, \frac{-4081}{180}, \frac{4327}{80}, \frac{-6527}{90}, \frac{42839}{720}, \frac{-601}{20}, \frac{6233}{720}, \frac{-49}{45}),
\end{aligned}$$

It can be seen that systems (2.8) and (2.9) with the above boundary condition are strictly diagonally dominant that have a unique solution to obtain $M_0, M_1, ..., M_N$ and $F_0, F_1, ..., F_N$. In the matrix notation, the relations (2.8) and (2.9) can be expressed as follows:

$$PM = \frac{20}{h^2}TY,\ PF = \tfrac{120}{h^4}UY,$$

where T and U are five-diagonal matrices,

$Y = (y_0, y_1, y_2, ..., y_N)^T$, $M = (M_0, M_1, M_2, ..., M_N)^T$, $F = (F_0, F_1, F_2, ..., F_N)^T$ and

$$P = \begin{bmatrix} 65 & 26 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 26 & 66 & 26 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 1 & 26 & 66 & 26 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 26 & 66 & 26 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 26 & 66 & 26 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 26 & 65 \end{bmatrix} \tag{3.2}$$

In (3.2) the matrix P is given by

$$P = Q^2 + 18Q - 24I,$$

where $Q = (q_{ij})$ is three diagonal matrix defined by

$$q_{ij} = \begin{cases} 4 & i = j, \\ 1 & |i - j| = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

4. Methodology

Consider the following Fredholm integral equation of the second kind

$$y(t) = f(t) + \int_a^b k(t, x)y(x)dx \tag{4.1}$$

from (2.1), (2.3) and (2.4) we have

$$\begin{aligned} S_j(x) &= \frac{F_{j+1} - F_j}{120h}(x - x_j)^5 \\ &+ \frac{F_j}{24}(x - x_j)^4 + \left[\frac{M_{j+1} - M_j}{6h} - h\frac{F_{j+1} + F_j}{36}\right](x - x_j)^3 + \frac{M_j}{2}(x - x_j)^2 \\ &+ \left[\frac{y_{j+1} - y_j}{h} - h\frac{M_{j+1} + 2M_j}{6} + h^3\frac{7F_{j+1} + 8F_j}{360}\right](x - x_j) + y_j, \end{aligned} \tag{4.2}$$

As we will show in Theorem (5.2), that for y, solution of the Eq. (1.1), smooth enough and for all $t \in [a, b]$ we have $\|y - S\| = O(h^6)$.

Hence, from (1.1), we obtain for all $i = 0, ..., n$

$$
\begin{aligned}
y_i &= f_i + \int_a^b k(t_i, x) y(x) dx \\
&= f_i + \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} k(t_i, x) y(x) dx \\
&\approx f_i + \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} k(t_i, x) S_j(x) dx + O(h^6)
\end{aligned} \tag{4.3}
$$

The above relation is equivalent to

$$
\begin{aligned}
y_i &= f_i + \sum_{j=0}^{n-1} \frac{F_{j+1}}{120h} \int_{x_j}^{x_{j+1}} k(t_i, x)(x - x_j)^5 \\
&- \sum_{j=0}^{n-1} \frac{F_j}{120h} \int_{x_j}^{x_{j+1}} k(t_i, x)(x - x_j)^5 + \sum_{j=0}^{n-1} \frac{F_j}{24} \int_{x_j}^{x_{j+1}} k(t_i, x)(x - x_j)^4 \\
&+ \sum_{j=0}^{n-1} \left[\frac{M_{j+1}}{6h} - h\frac{F_{j+1}}{36} \right] \int_{x_j}^{x_{j+1}} k(t_i, x)(x - x_j)^3 \\
&- \sum_{j=0}^{n-1} \left[\frac{M_j}{6h} + h\frac{F_j}{36} \right] \int_{x_j}^{x_{j+1}} k(t_i, x)(x - x_j)^3 \\
&+ \sum_{j=0}^{n-1} \frac{M_j}{2} \int_{x_j}^{x_{j+1}} k(t_i, x)(x - x_j)^2 + \sum_{j=0}^{n-1} \left[\frac{y_{j+1}}{h} \right. \\
&- \left. \sum_{j=0}^{n-1} h\frac{M_{j+1}}{6} + \sum_{j=0}^{n-1} 7h^3 \frac{F_{j+1}}{360} \right] \int_{x_j}^{x_{j+1}} k(t_i, x)(x - x_j) \\
&+ \sum_{j=0}^{n-1} \left[\frac{-y_j}{h} - h\frac{M_j}{3} + 8h^3 \frac{F_j}{360} \right] \int_{x_j}^{x_{j+1}} k(t_i, x)(x - x_j) \\
&+ \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} k(t_i, x) y_j dx
\end{aligned}
$$

$$
\begin{aligned}
&= f_i + \frac{1}{120h}\sum_{j=0}^{n} F_j a_{i,j} - \frac{1}{120h}\sum_{j=0}^{n} F_j b_{i,j} + \frac{1}{24}\sum_{j=0}^{n} F_j c_{i,j} + \frac{1}{6h}\sum_{j=0}^{n} M_j d_{i,j} \\
&- \frac{h}{36}\sum_{j=0}^{n} F_j d_{i,j} - \frac{1}{6h}\sum_{j=0}^{n} M_j e_{i,j} - \frac{h}{36}\sum_{j=0}^{n} F_j e_{i,j} + \frac{1}{2}\sum_{j=0}^{n} M_j g_{i,j} \\
&+ \frac{1}{h}\sum_{j=0}^{n} y_j l_{i,j} - \frac{h}{6}\sum_{j=0}^{n} M_j l_{i,j} + \frac{7h^3}{360}\sum_{j=0}^{n} F_j l_{i,j} + \frac{8h^3}{360}\sum_{j=0}^{n} F_j o_{i,j} \\
&- \frac{1}{h}\sum_{j=0}^{n} y_j o_{i,j} - \frac{h}{3}\sum_{j=0}^{n} M_j o_{i,j} + \sum_{j=0}^{n} y_j v_{i,j}, \quad i = 0(1)n \qquad (4.4)
\end{aligned}
$$

where, $a_{i0} = b_{in} = c_{in} = d_{i0} = e_{in} = g_{in} = l_{i0} = o_{in} = v_{in} = 0$.
It is supposed that y_i, M_i and F_i is approximated by $\hat{y}_i, \hat{M}_i$ and $\hat{F}_i$ such that satisfy in (2.8) and (2.9) for $i = 0, 1, ..., n$ then we have

$$
\begin{aligned}
\hat{y} &= f_i + \frac{1}{120h}\sum_{j=0}^{n} \hat{F}_j a_{i,j} - \frac{1}{120h}\sum_{j=0}^{n} \hat{F}_j b_{i,j} + \frac{1}{24}\sum_{j=0}^{n} \hat{F}_j c_{i,j} \\
&+ \frac{1}{6h}\sum_{j=0}^{n} \hat{M}_j d_{i,j} - \frac{h}{36}\sum_{j=0}^{n} \hat{F}_j d_{i,j} - \frac{1}{6h}\sum_{j=0}^{n} \hat{M}_j e_{i,j} - \frac{h}{36}\sum_{j=0}^{n} \hat{F}_j e_{i,j} \\
&+ \frac{1}{2}\sum_{j=0}^{n} \hat{M}_j g_{i,j} + \frac{1}{h}\sum_{j=0}^{n} \hat{y}_j l_{i,j} - \frac{h}{6}\sum_{j=0}^{n} \hat{M}_j l_{i,j} + \frac{7h^3}{360}\sum_{j=0}^{n} \hat{F}_j l_{i,j} \\
&+ \frac{8h^3}{360}\sum_{j=0}^{n} \hat{F}_j o_{i,j} - \frac{1}{h}\sum_{j=0}^{n} \hat{y}_j o_{i,j} - \frac{h}{3}\sum_{j=0}^{n} \hat{M}_j o_{i,j} + \sum_{j=0}^{n} \hat{y}_j v_{i,j}, i = 0(1)n \qquad (4.5)
\end{aligned}
$$

The matrix form of the relation (4.4) can be expressed as follows:

$$
\begin{aligned}
\hat{Y} &= f + \underbrace{(\frac{1}{120h}A - \frac{1}{120h}B + \frac{1}{24}C - \frac{h}{36}D - \frac{h}{36}E + \frac{7h^3}{360}L + \frac{8h^3}{360}O)}_{W}\hat{F} \\
&+ \underbrace{(\frac{1}{6h}D - \frac{1}{6h}E + \frac{1}{2}G - \frac{h}{6}L - \frac{h}{3}O)}_{Z}\hat{M} + \underbrace{(\frac{1}{h}L - \frac{1}{h}O + V)}_{X}\hat{Y} \qquad (4.6)
\end{aligned}
$$

where, $A = (a_{ij}), B = (b_{ij}), C = (c_{ij}), D = (d_{i,j}), E = (e_{i,j}), L = (l_{i,j}), O = (o_{ij}), G = (g_{ij}), V = (v_{i,j})$ and $f = (f_0, f_1, f_2, \ldots, f_{n-1}, f_n)^T$. Finally,

$$
\left[I - (X + \frac{120}{h^4}WP^{-1}U + \frac{20}{h^2}ZP^{-1}T)\right]\hat{Y} = f, \qquad (4.7)
$$

Hence, we can approximate the exact solution y using quintic spline $\hat{S} = \hat{S}_i$ on $\Omega_i = [x_i, x_{i+1}],\ for\ i = 0, 1, ..., n-1$ where,

$$\begin{aligned}\hat{S}_j(x) &= \frac{\hat{F}_{j+1} - \hat{F}_j}{120h}(x - x_j)^5 \\ &+ \frac{\hat{F}_j}{24}(x - x_j)^4 + \left[\frac{\hat{M}_{j+1} - \hat{M}_j}{6h} - h\frac{\hat{F}_{j+1} + \hat{F}_j}{36}\right](x - x_j)^3 + \frac{\hat{M}_j}{2}(x - x_j)^2 \\ &+ \left[\frac{\hat{y}_{j+1} - \hat{y}_j}{h} - h\frac{\hat{M}_{j+1} + 2\hat{M}_j}{6} + h^3\frac{7\hat{F}_{j+1} + 8\hat{F}_j}{360}\right](x - x_j) + \hat{y}_j,\end{aligned} \tag{4.8}$$

5. Analysis of convergence

Lemma 5.1. *Assume Ω be a $n \times n$ matrix with $\|\Omega\|_\infty < 1$, then, the matrix $(I - \Omega)$ is invertible. In addition to $\|(I - \Omega)^{-1}\|_\infty \leq \frac{1}{1 - \|\Omega\|_\infty}$.*

Theorem 5.2. *Let $S(x)$ be the unique quintic spline satisfying in (2.5)-(2.8), for a given function $y \in C^{10}[a, b]$. Then the following error estimates hold :*

$$\|S - y\| = O(h^6),$$

Proof. Assuming the continuity of derivatives of y of sufficiently high order, by using (2.5)-(2.8), we obtain

$$\begin{aligned}&S'(x_j) = y'_j + \tfrac{1}{5040}h^6 y_j^7 - \tfrac{1}{21600}h^8 y_j^9 + O(h^{10}) \\ &S''(x_j) = y''_j + \tfrac{1}{720}h^4 y_j^6 - \tfrac{1}{3360}h^6 y_j^8 + \tfrac{1}{86400}h^8 y_j^{10} + O(h^{10}) \\ &S'''(x_j) = y'''_j - \tfrac{1}{240}h^4 y_j^7 + \tfrac{11}{30240}h^6 y_j^9 + O(h^8) \\ &S^{(4)}(x_j) = y_j^{(4)} - \tfrac{1}{12}h^2 y_j^6 + \tfrac{1}{240}h^4 y_j^8 - \tfrac{1}{7560}h^6 y_j^{10} + O(h^8) \\ &\tfrac{1}{2}\left[S^{(5)}(x_j^+) + S^{(5)}(x_j^-)\right] = y_j^5 + \tfrac{1}{12}h^2 y_j^7 - \tfrac{h^4}{720}y_j^9 + O(h^6) \\ &\tfrac{1}{h}\left[S^{(5)}(x_j^+) - S^{(5)}(x_j^-)\right] = y_j^6 + \tfrac{h^7}{6\times 7!}h^2 y_j^{12} + O(h^9)\end{aligned} \tag{5.1}$$

Now, let $e(x) = y(x) - S(x)$, then for $0 \leq \theta \leq 1$,

$$\begin{aligned}e(x_j + \theta h) = e(x_j) + \theta h e'(x_j) + \frac{\theta^2 h^2}{2!}e''(x_j) + \frac{\theta^3 h^3}{3!}e'''(x_j) \\ + \frac{\theta^4 h^4}{4!}e^{(4)}(x_j) + \frac{\theta^5 h^5}{5!}e^{(5)}(x_j) + \ldots\end{aligned} \tag{5.2}$$

Subsitituting from (5.1) in (5.2), we obtain

$$e(x_j + \theta h) = \frac{\theta^2(\theta - 1)^2(2\theta^2 - 2\theta - 1)}{1440}h^6 y_j^6 + O(h^7)\ \square. \tag{5.3}$$

□

Lemma 5.3. *Suppose $Q_{N\times N}$ is the matrix defined in equation (3.3), then, $\|Q^{-1}\| \leq \frac{1 - sech\frac{N}{2}\theta}{2}$, $cosh\theta = 2$.*

Proof. Following [33]. □

Lemma 5.4. $\|P^{-1}\| \leq \frac{1-sech\frac{N}{2}\theta}{24sech\frac{N}{2}\theta}$

Proof. Based on the definition of the matrix Q and also using Lemma (5.1) and (5.3), we have:
$\|P^{-1}\|= \|(Q^2+18Q-24I)^{-1}\|$
$=\|(18Q(I-\left[\frac{24}{18}Q^{-1}-\frac{1}{18}Q\right]))^{-1}\|=\frac{1}{18}\|(I-\left[\frac{24}{18}Q^{-1}-\frac{1}{18}Q\right])^{-1}Q^{-1}\|$. Hence,
$\|P^{-1}\| \leq \frac{1}{18}\|(I-\left[\frac{24}{18}Q^{-1}-\frac{1}{18}Q\right])^{-1}\| \; \|Q^{-1}\| \leq \frac{1-sech\frac{N}{2}\theta}{24sech\frac{N}{2}\theta}$ □. □

Theorem 5.5. *Let $y \in C^{10}(I)$ and $k \in C^{10}(I\times I)$ such that,*
$\|k\|_\infty < \frac{\left[2-\frac{2834777(sech\frac{N}{2}\theta-1)}{17280sech\frac{N}{2}\theta}\right]^{-1}}{(b-a)}$
where $\theta = cosh^{-1}(2)$, then (4.8) define a unique approximation and the resulting error $\hat{e} := y-\hat{s}$ satisfies

$$\|\hat{e}\|_{\infty\psi} \leq \alpha h^6, \forall\psi \subset I,$$

where α is a constant.

Proof. It is easy to verify that $\|A\|_\infty, \|B\|_\infty \leq \frac{\|k\|_\infty(b-a)h^5}{6}$, $\|C\|_\infty \leq \frac{\|k\|_\infty(b-a)h^4}{5}$
$\|D\|_\infty, \|E\|_\infty \leq \frac{\|k\|_\infty(b-a)h^3}{4}$,$\|G\|_\infty \leq \frac{\|k\|_\infty(b-a)h^2}{3}$, $\|L\|_\infty, \|O\|_\infty \leq \frac{\|k\|_\infty(b-a)h^2}{2}$
and also $\|V\|_\infty \leq \|k\|_\infty(b-a)$. Hence, $\|X\|_\infty \leq 2\|k\|_\infty(b-a)$,$\|W\|_\infty \leq \frac{33}{720}\|k\|_\infty(b-a)h^4$ and $\|Z\|_\infty \leq \frac{10}{24}\|k\|_\infty(b-a)h^2$. Then we have:

$$\begin{aligned}&\|X+\frac{120}{h^4}WP^{-1}U+\frac{20}{h^2}ZP^{-1}T\|_\infty\\&\leq \|k\|_\infty(b-a)\left[2+\frac{2834777(1-sech\frac{N}{2}\theta)}{17280sech\frac{N}{2}\theta}\right] < 1.\end{aligned}$$

Using **lemma** (5.1), the system (4.7) has a unique solution $\hat{y}$. It follows that the Eq(4.8) defines a unique solution $\hat{S}$. Now, let $\hat{e} = y-\hat{y} = (y_0-\hat{y}_0, ..., y_n-\hat{y}_n)^T$. Then from (4.3) we get $\left[I-(X+\frac{120}{h^4}WP^{-1}U+\frac{20}{h^2}ZP^{-1}T)\right]\hat{e} = O(h^6)$. Therefore, $\hat{e} = \left[I-(X+\frac{120}{h^4}WP^{-1}U+\frac{20}{h^2}ZP^{-1}T)\right]^{-1}O(h^6)$, which implies by **lemma** (5.1), that there exist $\alpha_1 > 0$ such that

$$\|\hat{e}\|_\infty \leq \underbrace{\frac{\alpha_1}{1-\|k\|_\infty(b-a)\left[2+\frac{2834777(1-sech\frac{N}{2}\theta)}{17280sech\frac{N}{2}\theta}\right]}}_{\alpha_2}h^6.$$

On the other hand, from (2.8), we have $(M-\hat{M}) = \frac{20}{h^2}P^{-1}T\hat{e}$ and from (2.9),we have $(F-\hat{F}) = \frac{120}{h^4}P^{-1}U\hat{e}$. Therefore, there exist $\alpha_3, \alpha_4 > 0$ such that $\|M-\hat{M}\|_\infty \leq \alpha_3 h^4$ and $\|F-\hat{F}\|_\infty \leq \alpha_4 h^2$.

Table 1. Absolute errors for Example 1.

M, n	$M = 20$	$n = 10$
t	Method in [28]	QSF method
0	1.5×10^{-5}	2.2×10^{-8}
0.1	1.9×10^{-5}	1.1×10^{-8}
0.2	2.3×10^{-5}	5.4×10^{-9}
0.3	2.9×10^{-5}	2.3×10^{-9}
0.4	3.5×10^{-5}	1.0×10^{-9}
0.5	4.3×10^{-5}	5.8×10^{-10}
0.6	5.2×10^{-5}	7.2×10^{-10}
0.7	6.4×10^{-5}	1.2×10^{-9}
0.8	7.8×10^{-5}	2.1×10^{-9}
0.9	9.6×10^{-5}	3.4×10^{-9}
1	1.1×10^{-4}	5.2×10^{-9}

In consequence, for all $i = 0, 1..., n-1$ and $x \in \Omega_i$, we have

$$
\begin{aligned}
&|S_i(x) - \hat{S}_i(x)| \\
&\leq \frac{|F_{j+1} - \hat{F}_{j+1}|h^4}{18} + \frac{|F_j - \hat{F}_j|h^4}{10} + \frac{|M_{j+1} - \hat{M}_{j+1}|h^2}{3} + |M_j - \hat{M}_j|h^2 \\
&+ |y_j - \hat{y}_j| + 2|y_j - \hat{y}_j| \leq \frac{28}{180}\alpha_4 h^6 + \frac{4}{3}\alpha_3 h^6 + 3\alpha_2 h^6
\end{aligned}
$$

$$\Longrightarrow |S_i(x) - \hat{S}_i(x)| \leq \alpha_5 h^6,$$

where $\alpha_5 = \frac{28}{180}\alpha_4 + \frac{4}{3}\alpha_3 + 3\alpha_2$. It follows that

$$\|y - \hat{S}\|_\infty \leq \|y - S\|_\infty + \|S - \hat{S}\|_\infty \leq \alpha_1 h^6 + \alpha_5 h^6.$$

Thus, the proof is completed by taking $\alpha = \alpha_1 + \alpha_5$. □ □

6. Computational illustrations

In this section, we have implemented our method (QSF) for solving examples of the Fredholm integral equations. To compare the presented method with the other methods, some examples are selected from other papers. The absolute error in the solution is compared with the methods in [23, 26, 28]. Comparison between results confirms the efficiency and simplicity of the presented method is considerable. It should be noted that the associated computations with the examples were performed using **MATLAB R2013a**.

Example 1. Consider the following integral equation
$y(t) = f(t) + \int_0^1 k(t,x)y(x)dx, \ t \in [0,1]$,
where, $k(t,x) = e^{x-t-12}$, $f(t) = cos(t) - \frac{1}{2}e^{-t-12}(e\sin 1 + e\cos 1 - 1)$ and the exact solution is given by the relation $y(t) = \cos t$. The absolute errors in the solution presented in **Table 1**.

TABLE 2. Absolute errors for Example 2.

n	10	10	10
t	Method in [23]	Method in [26]	QSF method
0	1.5×10^{-4}	3.9×10^{-5}	8.2×10^{-15}
0.1	1.9×10^{-4}	4.8×10^{-5}	3.4×10^{-16}
0.2	2.3×10^{-4}	5.9×10^{-5}	1.2×10^{-16}
0.3	2.9×10^{-4}	7.2×10^{-5}	4.2×10^{-16}
0.4	3.5×10^{-4}	8.8×10^{-5}	2.1×10^{-16}
0.5	4.3×10^{-4}	1.0×10^{-4}	3.5×10^{-16}
0.6	5.2×10^{-4}	1.3×10^{-4}	5.6×10^{-16}
0.7	6.4×10^{-4}	1.6×10^{-4}	7.2×10^{-16}
0.8	7.8×10^{-4}	1.9×10^{-4}	1.1×10^{-15}
0.9	9.6×10^{-4}	2.4×10^{-4}	3.6×10^{-15}
1	1.1×10^{-3}	2.9×10^{-4}	8.6×10^{-15}

TABLE 3. Absolute errors for Example 3.

M, n	$n = 10$	$M = 20$	$n = 10$
t	Method in [23]	Method in [28]	QSF
0	1.36×10^{-4}	1.4×10^{-5}	$3..3\times10^{-8}$
0.1	2.3×10^{-4}	2.6×10^{-5}	1.6×10^{-8}
0.2	3.0×10^{-4}	3.3×10^{-6}	9.6×10^{-9}
0.3	3.7×10^{-4}	5.5×10^{-6}	4.6×10^{-9}
0.4	4.2×10^{-4}	6.3×10^{-6}	2.1×10^{-9}
0.5	4.4×10^{-4}	7.7×10^{-6}	1.0×10^{-8}
0.6	4.2×10^{-4}	1.5×10^{-5}	8.7×10^{-8}
0.7	3.7×10^{-4}	1.9×10^{-5}	1.2×10^{-7}
0.8	3.0×10^{-4}	2.4×10^{-5}	2.1×10^{-7}
0.9	2.3×10^{-4}	3.5×10^{-5}	3.4×10^{-7}
1	1.6×10^{-4}	5.9×10^{-5}	5.1×10^{-7}

Example 2. Consider the following integral equation
$y(t) = f(t) + \int_{-1}^{1} k(t,x)y(x)dx,\ t \in [-1,1],$
where, $k(t,x) = e^{t-x-3}$, $f = 1 - e^{t-2} + e^{t-4}$ and the exact solution is given by the relation $y(t) = 1$. The absolute errors in the solution presented in **Table 2**.

Example 3. Consider the following integral equation
$y(t) = e^{-t^2}(1 - e^{-2} + e^{4}) + \int_{-1}^{1} e^{x^2-x-t^2-3}\ y(x)dx,$
The unique solution is given by $y(t) = e^{-t^2}$. The absolute error in the solution is presented in **Table 3**.

Example 4. Consider the following Lichtenstein-Gershgorin integral equation for $\nu = 1.2$ and $-1 \le t \le 1$,

TABLE 4. Absolute errors for Example 4.

M, n	$n = 10$	$n = 10$	$n = 10$
t	Method in [12]	Method in [26]	QSF
0.1	0.215	0.2158	0.2158
0.2	0.461	0.4625	0.4625
0.3	0.766	0.7673	0.7679
0.4	1.142	1.1434	1.1434
0.5	1.570	1.5721	1.5721
0.6	1.999	2.0007	2.0007
0.7	2.375	2.3765	2.3767
0.8	2.679	2.6808	2.6808
0.9	2.926	2.9266	2.9265
1	3.141	3.1416	3.1416

$$y(t) = 2\tan^{-1}\left[\frac{\nu \sin \pi t}{\nu^2(\cos \pi t + \cos^2 \pi t) + \sin^2 \pi t}\right] + \int_{-1}^{1} \frac{\nu y(x)}{(\nu^2+1) - (\nu^2-1)\cos \pi(x+t)} dx.$$

This equation has been used for the determination of conformal mapping of an ellipse onto a circle as mentioned in [12]. Since the exact solution of this integral equation is unknown, the approximate solution compared with the available result of [26]. The absolute error in the solution is presented in **Table 4**. From this table, we find that the computed solution of this integral equation is in a good agreement with the available result.

7. Conclusion

This work is an effort to obtain the numerical solution of the integral equation of the second kind. Analysis of convergence is investigated. Four test examples are considered from previous work in [23, 26, 28]. The computational solutions are compared with the exact solution. The absolute errors in the solutions by our QSF method are considerably accurate compared to [23, 26, 28].

References

[1] J. Ahmadi, A. Haghighi, and E. Soleymani, *Convergence of integro quartic and sextic B-spline interpolation*, Sahand Communications in Mathematical Analysis, *10* (2018), 97–108.

[2] A. Akgul, E. Karatas, and D. Baleanu, *Numerical solutions of fractional differential equations of Lane-Emden type by an accurate technique*, Advances in Difference Equations, *1* (2015), 220–232.

[3] A. Akgul, Y. Khan, and EK. Akgul, *Solutions of nonlinear systems by reproducing kernel method*, The Journal of Nonlinear Sciences and Applications, *10* (2017), 4408–4417.

[4] A. AkguL, A. Klman, and M. Inc, *Improved Expansion Method for the Space and Time Fractional Foam Drainage and KdV Equations*, Hindawi Publishing Corporation, *2013* (2013), 1–7.

[5] C. Allouch and P. Sablonnire, *Iteration methods for Fredholm integral equations of the second kind based on spline quasi-interpolants*, Math. Comput. Simul, *99* (2014), 19–27.

[6] C. Aloknath and S. C. Martha, *Approximate solutions of Fredholm integral equations of the second kind*, Applied Mathematics and Computation, *211* (2009), 459–466.

[7] G. Akram and S. S. Siddiqi, *Solution of sixth order boundary value problems using non-polynomial spline technique*, Appl. Math. Comput, *181* (2006), 708–720.

[8] T. Aziz, A. Khan, and J. Rashidinia, *Spline methods for the solution of fourth-order parabolic partial differential equations*, Applied mathematics and computation, *167* (2005), 153–166.

[9] A. H. Borzabadi and O. S. Fard, *A numerical scheme for a class of nonlinear Fredholm integral equations of the second kind*, J. Comput. Appl. Math, *232* (2009), 449–454.

[10] L. M. Delves and J. L. Mohamed, *Computational Methods for Integral Equations*, CUP Archive, (1988).

[11] M. El-Mikkawy and A. Karawia, *Inversion of general tridiagonal matrices*, Appl. Math. Lett, *19* (2006), 712–720.

[12] D. Elliott, *A Chebyshev series method for the numerical solution of Fredholm integral equations*, Comput. J, *6* (1963), 102–112.

[13] A. R. Haghighi and M. Roohi, *The fractional Cubic Spline Interpolation without using the derivative values*, Indian Journal of Science and Technology, *5* (2012), 3433–3470.

[14] A. R. Haghighi and N. Pirhadi, *A Numerical Study of Heat Transfer and Flow Characteristics of Pulsatile Blood Flow in a Tapered Artery with a Combination of Stenosis and Aneurysm*, International Journal of Heat and Technology, *37* (2019), 11–21.

[15] H. Hochstadt, *Integral Equations*, John Wiley and Sons., *91* (2011) , (2011).

[16] A. Khan, I. Khan, and T. Aziz, *Sextic spline solution for solving a fourth-order parabolic partial differential equation*, International Journal of Computer Mathematics, *82* (2005), 871–879.

[17] A. Khan, A. Khan, and T. Sultana, *Parametric quintic spline solution of third-order boundary value problems*, International Journal of Computer Mathematics, *89* (2012), 1663–1677.

[18] M. A. Khan, I. A. Tirmizi, and E. H. Twizell, *Non polynomial spline approach to the solution of a system of third-order boundary-value problems*, Appl. Math. Comput, *168* (2005), 152–163.

[19] G. Long and G. Nelakanti, *Iteration methods for Fredholm integral equations of the second kind*, Comput. Math. with Appl, *53* (2007), 886–894.

[20] R. R. Love, *The electrostatic field of two equal circular co-axial conducting disks*, Q. J. Mech. Appl. Math, *2* (1949), 428–451.

[21] K. Maleknejad and Y. Mahmoudi, *Numerical solution of linear Fredholm integral equation by using hybrid Taylor and block-pulse functions*, Appl. Math. Comput, *149* (2004), 799–806.

[22] R. K. Mohanty, P. L. Sachdev, and N. Jha, *An $O(h^4)$ accurate cubic spline TAGE method for nonlinear singular two point boundary value problems*, Applied Mathematics and Computation, *158* (2004), 853–868.

[23] F. Muller and W. Varnhorn, *On approximation and numerical solution of Fredholm integral equations of second kind using quasi-interpolation*, Appl. Math. Comput, *217* (2011), 6409–6416.

[24] A. Nemati S. Yousefi, F. Soltanian, and J. S. Ardabili, *An efficient numerical solution of fractional optimal control problems by using the Ritz method and Bernstein operational matrix*, Asian Journal of Control, *18* (2016), 2272–2282.

[25] A. Nemati and K. Mamehrashi, *The Use of the Ritz Method and Laplace Transform for Solving 2D Fractional Order Optimal Control Problems Described by the Roesser Model*, Asian Journal of Control, *21* (2019), 1189–1201.

[26] S. Panda, S. C. Martha, and A. Chakrabarti, *A modified approach to numerical solution of Fredholm integral equations of the second kind*, Appl. Math. Comput, *271* (2015), 102–112.

[27] R. Pradip, K. Thula, and R. Agarwal. *Non-optimal fourth-order and optimal sixth-order B-spline collocation methods for Lane-Emden boundary value problems*, App. Num. Mathematics, *145* (2019), 342–360.

[28] J. Rashidinia and M. Zarebnia, *Numerical solution of linear integral equations by using Sinc-collocation method*, Applied mathematics and computation, *168* (2005), 806–822.

[29] J. Rashidinia and A. Golbabaee, *Convergence of numerical solution of a fourth-order boundary value problem*, Applied mathematics and computation, *171* (2005), 1296–1305.

[30] J. Rashidinia, R. Mohammadi, and R. Jalilian, *Quintic spline solution of boundary value problems in the plate deflection theory*, Scientia Iranica. Transaction D, Computer Science and Engineering, Electrical, *16* (2009), 53–59.

[31] J. Rashidinia and R. Jalilian, *Non-polynomial spline for solution of boundary-value problems in plate deflection theory*, International Journal of Computer Mathematics, *84* (2007), 1483–1494.

[32] O. Tomoaki, T. Matsuo, and M. Sugihara, *Improvement of a Sinc-collocation method for Fredholm integral equations of the second kind*, BIT Numerical Mathematics, *51* (2011), 339–366.

[33] R. A. Usmani and S. A. Warsi, *Quintic spline solutions of boundary value problems*, Computers and Mathematics with Applications, *6* (1980), 197–203.

[34] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, Springer Science and Business Media, (2013).