Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 9, No. 1, 2021, pp. 187-210
DOI:10.22034/cmde.2019.33771.1551

Numerical solution of fractional Riesz space telegraph equation: stability and error

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Abstract In this paper, a numerical method based on polynomial approximation is presented for the Riesz fractional telegraph equation. First, a system of fractional differential equations are obtained from the telegraph equation with respect to the time variable by using the method of lines. Then a new numerical algorithm, as well as its modification for solving fractional differential equations (FDEs) based on the polynomial interpolation, is proposed. The algorithms are designed to estimate to linear fractional systems. The convergence order and stability of the fractional order algorithms are proved. At the end three examples with known exact solutions are chosen. Numerical results show that accuracy of present scheme is of order $O(\Delta t^2)$.

Keywords. Fractional telegraph equation, Polynomial approximation, Riemann-Liouville fractional derivative, Riesz fractional equation, Discretization.

2010 Mathematics Subject Classification. 34A08, 33F05.

1. Introduction

During the last three decades, fractional calculus has been recently applied to physics and other natural sciences [29]. Because of recent considerations in science and engineering have proved that their equations may be explained more exactly by using differential equations of non-integer order [23]. The use of differential equations of fractional order appears more and more frequently in several research areas [18]. Kilbas et al. [19] developed differentiation of fractional order and some of their applications to differential equations. Another distinguishing of their book is that most of the theory of such operations is concerned with functions of

Received: 29 May 2019 ; Accepted: 24 December 2019.
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one variable. Also, they showed different forms of fractional integro-differentiation of functions of multi variables. We refer the interested reader to [25] to experience more application of the fractional differential equations.

Some numerical schemes have been developed for fractional differential equations and for an approximation of fractional derivatives by generalized finite differences [10]. Irandoust-Pakchin et al. [14], proposed a new homotopy perturbation method for solving fractional order nonlinear cable equation. Also, Javidi and Bashir [16], proposed a numerical method for solving fractional partial differential equations based on Laplace transforms.

The fractional telegraph equation, as a typical fractional diffusion wave equation, is used into signal analysis for transmission and modeling of the reaction diffusion. The telegraph equation is achieved by the variational relationship between the voltage wave and the current wave on the well-proportional transmission line. So it is also called the transmission line equation. However, the telegraph equation can not well explain the oddness diffusion event during the finite long transmits progress, where the voltage wave or the current wave possibly exists [18]. Therefore it is necessary to inquire the fractional telegraph equation, including the time and (or) space fractional derivatives.

Asgari et al. maintain telegraph equations are hyperbolic partial differential equations that are applicable in various branches of engineering and biological sciences in [2]. The authors of [12], implement the meshless method for solving the time-fractional telegraph equation by using a radial basis function. In [9], a numerical method introduced to solve fractional telegraph equation and stability conditions. Also, Povstenko [26], used the fundamental solutions to the nonhomogeneous space-time- fractional telegraph equations, as well as the associated thermal stresses, are studied in the axisymmetric case. Several articles that their authors showed analytical solution and numerical analysis of fractional telegraph equation [4, 13].

In [6], the powerful and effective approximate analytical mathematical tool like homotopy analysis method is used to solve the telegraph equation with fractional time derivative. some schemes proposed for solving telegraph equations with Dirichlet boundary conditions in [28]. An unconditionally stable fourth-order method for telegraph equation based on hermite interpolation introduced in [21]. This method is presented for the numerical solutions of one-dimensional telegraph equations. Orsinger and Zhao [24], discussed the space fractional telegraph equation by using the Fourier transform technique. Momani [22], gave the analytic and approximate solutions of space and time-fractional telegraph equation by using Adomian decomposition. The authors of [31], presented and discussed the semi-discrete and fully discrete numerical approximations for the time-space fractional telegraph equations.

As a new approach, Celik and Duman [3], used the fractional centered derivative approach to approximate the Riesz fractional derivative. They applied the Crank-Nicolson method for the fractional diffusion equation by using fractional centered difference approach. Chen et al. [5], presented a class of unconditionally stable difference schemes based on the Pade approximation for the Riesz space fractional telegraph equation. Zhang and Lio [30], used the fundamental solutions of the space-time Riesz fractional partial differential equations with periodic conditions.

In this paper, linear interpolation for the time variable and mesh schemes for the space variable is considered with error analysis and stability. In the same line of thoughts, we intend to solve Riesz space- fractional telegraph equation by polynomial approximation.

The structure of the paper is as follows. In sections 3 and 4, we present the Riesz space fractional telegraph equation and revise the previously published theory. In sections 5 and 6, a new numerical solution for solving Riesz space fractional telegraph equation and error analysis are outlined. Some examples in sections 7 show the accuracy of the present scheme. The conclusions are included in the last section.

2. Preliminaries

In this section, we present some definitions, preliminary facts and presentation that will be used further in this study.

Definition 2.1. The Euler's gamma function is defined by the integral [11]

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \mathrm{Re}(z) > 0.$$

$C(J,\mathbb{R})$ denotes the Banach space of all continuous functions from $J = [0,t]$ into $\mathbb{R}$ with the norm

$$\|f\|_\infty = \sup\{|f(t)| : t \in J\}, \quad T > 0.$$

$C^n(J,\mathbb{R})$ denotes the class of all real valued functions defined on $J = [0,t], \quad T > 0$ which have continuous nth order derivatives.

Definition 2.2. The Caputo fractional derivative of order $p > 0$ of the function $f \in C^n(J,\mathbb{R})$ is defined as [19]:

$$\frac{\partial^p}{\partial t^p} f(t) = {}_a^cD_t^p f(t) = \begin{cases} \frac{1}{\Gamma(n-p)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{p-n+1}} ds, & n-1 < p \leq n, \quad n \in N, \\ f^{(n)}(t), & p = n. \end{cases}$$

Definition 2.3. The left and right Riemann-Liouville derivatives with order $p > 0$ of the given function $f(x), x \in (a,b)$ are defined as [25]:

$${}_a^RD_x^p f(x) = \frac{1}{\Gamma(m-p)} \frac{d^m}{dx^m} \int_a^x (x-s)^{m-p-1} f(s) ds,$$

and

$${}_x^RD_b^p f(x) == \frac{(-1)^m}{\Gamma(m-p)} \frac{d^m}{dx^m} \int_x^b (s-x)^{m-p-1} f(s) ds,$$

respectively, where m is a positive integer satisfying $m-1 \leq p < m$.

Definition 2.4. The Riesz derivative with order $\lambda > 0$ of the given function $f(x)$, $x \in (a,b)$ are defined as [25]:

$$ {}_a^{RZ}D_b^{\lambda} f(x) = c_{\lambda}({}_a^{RZ}D_x^{\lambda} f(x) + {}_x^{RZ}D_b^{\lambda} f(x)), $$

where $c_{\lambda} = \frac{-1}{2\cos(\frac{\lambda\pi}{2})}$, $\quad \lambda \neq 2k+1, \quad k = 0,1,2,\ldots$ ${}_a^{RZ}D_b^{\lambda} f(x)$ is sometimes expressed as $\frac{\partial^{\lambda} f(x)}{\partial |x|^{\lambda}}$.

Definition 2.5. The Mittag-Leffler function defined by series when the real part of α is strictly positive [25]

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, $$

where Γ is the Gamma function.

Definition 2.6. The Gershgorin circle [27]: Let A be a $n \times n$ complex matrix, with entries $a_{i,j}$. For $i \in \{1,\ldots,n\}$ let $R_i = \sum_{j \neq i} |a_{ij}|$ be the sum of the absolute values of the non diagonal entries in the i-th row. Let $D(a_{ii}, R_i) \subseteq \mathbb{C}$ be a closed disc centered at a_{ii} with radius R_i.

Theorem 2.7. *Gershgorin circle theorem: Every eigenvalue of A lies within at least one of the Gershgorin discs $D(a_{ii}, R_i)$.*

Proof. (see [27]). □

3. The Model

If n,m are positive integers and $[a,b], [0,T]$ is given, let $h = \frac{b-a}{n}$, $\Delta t = \frac{T}{m}$. The solution domain $[a,b] \times [0,T]$ is covered by a uniform grid of mesh points (x,t). Note that h and Δt are the uniform spatial stepsize and temporal step size.

For every λ, $(1 < \lambda < 2)$ the left and right Riemann-Liouville derivatives exist and match with the left and right Grunwald-Letnikov derivatives under suitable conditions. Then the Riesz derivatie with order λ, $(1 < \lambda < 2)$ can be discretized By the standard, shifted Grunwald-Letnikov formulas, or fractional centered difference method [7].
Recently second-order and fourth-order methods are used for the Riesz space and time fractional diffusion equations. It prognosticates that this methods and techniques are useful for solving some other fractional differential equations with Riesz fractional derivatives.
Now consider the following space-fractional telegraph equation with Riesz operation and frcational derivatives in time over a finite one-dimensional domain

$$ \frac{\partial^p}{\partial t^p}(\frac{\partial^p}{\partial t^p} u(x,t)) + 2\alpha \frac{\partial^p}{\partial t^p} u(x,t) + \beta^2 u(x,t) = \eta \frac{\partial^{\lambda}}{\partial |x|^{\lambda}} u(x,t) + f(x,t), \tag{3.1} $$

subject to the initial conditions

$$u(x,0) = g_1(x), \quad a \le x \le b,$$
$$\frac{\partial^p}{\partial t^p} u(x,0) = g_2(x), \quad a \le x \le b,$$

and Dirichlet boundary conditions

$$u(a,t) = u(b,t) = 0, \quad 0 \le t \le T,$$

where $\alpha > \beta \ge 0$ and $\eta > 0$ are constants, and $1 < \lambda \le 2, 0.5 < p < 1$.

Equations of the form (3.1) arise in the study of propogation of electrical signals in a cable of transmission line and wave phenomena. In fact the telegraph equation is more suitable than ordinary diffusion equation in modeling reaction diffusion [8]. Furthermore, we should mention that with the appropriate coefficient and forcing terms, the one-dimensional telegraph equation describes a diverse array of physical systems; for example, the propogation of voltage and current signals in coaxical transmissions lines of negligible leakage conductance and/or resistance [17]. The Riesz fractional derivative was to read from the kinetics of chaotic dynamics. For the Riesz fractional differential equations, there have remained several analytical and numerical methods.

The Riesz space fractional operator $\frac{\partial^\lambda}{\partial |x|^\lambda}$ over $[a,b]$ is defined by right and left Riemann-Liouville fractional derivation [25] can be writen as (see [5]).

$$\frac{\partial^\lambda}{\partial |x|^\lambda} u(x,t) = -\frac{1}{2\cos\frac{\lambda\pi}{2}} \times \frac{1}{\Gamma(2-\lambda)} \times \frac{\partial^2}{\partial x^2} \int_a^b \frac{u(s,t)}{|x-s|^{\lambda-1}} ds. \tag{3.2}$$

4. Converting to system of FDEs

The authors of [15], proposed the chebyshev spectral collocation for one-dimensional linear hyperbolic telegraph equation. This method is very useful in providing highly accurate solutions to fractional partial differential equations. Other benefit of this method is using of spectral differentiation matrices. Yan et al. [1], used polynomial interpolation to design a novel high-order algorithm for the numerical estimation of fractional differential equations. They utilized Hadamard finite-part integral and the piecewise cubic interpolation polynomial to approximate the integral.

In this section, we present our idea to approximate of Riesz space fractional telegraph equation. We discretize the space-fractional derivative operator through the following fractional central difference [3]:

$$\begin{aligned}\frac{\partial^\lambda}{\partial |x|^\lambda} u(x,t) &= -\frac{1}{h^\lambda} \sum_{p=-\frac{b-x}{h}}^{\frac{x-a}{h}} \frac{(-1)^p \Gamma(\lambda+1)}{\Gamma(\frac{\lambda}{2}-p+1)\Gamma(\frac{\lambda}{2}+p+1)} u(x-ph,t) + O(h^2) \\ &= -\frac{1}{h^\lambda} \sum_{p=-\frac{b-x}{h}}^{\frac{x-a}{h}} w_p u(x-ph,t) + O(h^2),\end{aligned}$$

where $h \to 0, a \le x \le b$ and $1 < \lambda \le 2$.

We introduce a new variable $v(x,t) = \frac{\partial^p}{\partial t^p}u(x,t)$ to transform (3.1) to the following equivalent system

$$\begin{cases} \frac{\partial^p}{\partial t^p}u(x,t) = v(x,t), \\ \frac{\partial^p}{\partial t^p}v(x,t) + 2\alpha v(x,t) + \beta^2 u(x,t) = \eta \frac{\partial^\lambda}{\partial |x|^\lambda}u(x,t) + f(x,t). \end{cases}$$

Now, we define $u(x_i,t) = u_i(t), v(x_i,t) = v_i(t)$.

By approximating $\frac{\partial^\lambda}{\partial |x|^\lambda}u(x_i,t)$ by $\frac{1}{h^\lambda}\sum_{\rho=-\frac{b-x_i}{h}}^{\rho=\frac{x_i-a}{h}} w_\rho u_{i-\rho}(t)$ [3], we have

$$\begin{cases} \frac{d^p}{dt^p}u_i(t) = v_i(t), \quad i = 1,2,3,\dots,n-1, \\ \frac{d^p}{dt^p}v_i(t) + 2\alpha v_i(t) + \beta^2 u_i(t) = -\eta \frac{1}{h^\lambda}\sum_{\rho=-\frac{b-x_i}{h}}^{\rho=\frac{x_i-a}{h}} w_\rho u_{i-\rho}(t) + f_i(t), \end{cases} \tag{4.1}$$

where $w_p = \frac{(-1)^p\Gamma(\lambda+1)}{\Gamma(\frac{\lambda}{2}-p+1)\Gamma(\frac{\lambda}{2}+p+1)}$.

The above discretization techniques is equivalent to the following form

$$\begin{aligned} i=1, \quad & \sum_{\rho=-\frac{b-x_i}{h}}^{\rho=\frac{x_i-a}{h}} w_\rho u_{i-\rho}(t) = \sum_{\rho=-\frac{x_n-x_1}{h}=-(n-1)}^{\rho=\frac{x_1-x_0}{h}} w_\rho u_{1-\rho}(t) \\ = \quad & \sum_{-(n-1)}^{1} w_\rho u_{1-\rho}(t), \\ i=2, \quad & \sum_{\rho=-\frac{b-x_2}{h}}^{\rho=\frac{x_2-a}{h}} w_\rho u_{2-\rho}(t) = \sum_{\rho=-\frac{x_n-x_2}{h}=-(n-2)}^{\rho=\frac{x_2-x_0}{h}} w_\rho u_{2-\rho}(t) \\ = \quad & \sum_{-(n-2)}^{2} w_\rho u_{2-\rho}(t), \\ & . \\ & . \\ & . \\ i=n-1, \quad & \sum_{\rho=-\frac{b-x_{n-1}}{h}}^{\rho=\frac{x_{n-1}-a}{h}} w_\rho u_{n-1-\rho}(t) = \sum_{\rho=-\frac{x_n-x_{n-1}}{h}=-(n-1)}^{\rho=\frac{x_{n-1}-x_0}{h}} w_\rho u_{n-1-\rho}(t) \\ = \quad & \sum_{-1}^{n-1} w_\rho u_{n-1-\rho}(t). \end{aligned}$$

Also from boundary conditions we know that $u_0(t) = u_n(t) = 0$.
By setting

$$u(t) = [u_1(t), u_2(t), \dots, u_{n-1}(t)]^t,$$
$$v(t) = [v_1(t), v_2(t), \dots, v_{n-1}(t)]^t,$$

we can rewrite (4.1), as the following form

$$\begin{cases} \frac{d^p}{dt^p}u(t) = v(t), \\ \frac{d^p}{dt^p}v(t) = -Du(t) - 2\alpha v(t) + f(t), \end{cases} \tag{4.2}$$

where

$$D = \eta C + \beta^2 I_{n-1}.$$

The matrix I_{n-1} is the identity matrix of order n-1, and

$$C=\frac{1}{h^{\lambda}}\begin{bmatrix} w_0 & w_{-1} & . & . & . & w_{-n+2} \\ w_1 & w_0 & . & . & . & w_{-n+3} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ w_{n-2} & w_{n-3} & . & . & . & w_0 \end{bmatrix}_{(n-1)(n-1)}.$$

Let

$$B=\begin{bmatrix} 0 & -I_{n-1} \\ D & 2\alpha I_{n-1} \end{bmatrix}_{(2n-2)(2n-2)}.$$

put

$$X(t)=[u_1(t),u_2(t),...,u_{n-1}(t),v_1(t),v_2(t),...,v_{n-1}(t)]^t.$$

Then, from (4.2), we obtain

$$\begin{cases} \frac{d^p}{dt^p}X(t)=-BX(t)+P(t), \\ X(0)=X_0, \end{cases} \tag{4.3}$$

where

$$P(t)=\begin{bmatrix} 0 \\ f(t) \end{bmatrix}_{(2n-2)\times 1}.$$

Now, we have the following theorems.

Theorem 4.1. *Assume that* $W_k=\frac{(-1)^k\Gamma(\lambda+1)}{\Gamma(\lambda/2+k+1)\Gamma(\lambda/2-k+1)}$, $k=0,\pm1,\pm2,...$ *are the coefficients in the fractional central difference (4.1) for* $1<\lambda\leq 2$*. Then*

1. $w_0>0$.
2. $w_k=w_{-k}<0$, *for all* $|k|\geq 1$.
3. $\sum\limits_{k=-\infty}^{\infty} w_k=0$.
4. $\sum\limits_{\substack{k=-m\\k\neq 0}}^{n} |w_k|\leq w_0$ *for any numbers* $m,n\in\mathbb{N}$.

Proof. (see [19], [5]). □

Theorem 4.2. *If matrix A is of the following form*

$$A=\begin{bmatrix} w_0 & w_{-1} & . & . & . & w_{-n+2} \\ w_1 & w_0 & & & & w_{-n+3} \\ . & & . & & & \\ . & & & . & & \\ . & & & & . & \\ w_{n-2} & w_{n-3} & & & & w_0 \end{bmatrix},$$

and λ *is an eigenvalue of matrix A. Then A is symmetric, strictly diagonally dominant and*

$\lambda \in \mathbb{R}^+$.

Proof. (see [5]). □

Theorem 4.3. *For the matrix B, we have*

$$\|B\|_\infty = Max\left\{1, \frac{2\eta}{h^\lambda}(W_0+\alpha)+\beta^2\right\}.$$

Proof. We have

$$B = \begin{bmatrix} 0 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & -1 \\ \frac{\eta}{h^\lambda}W_0+\beta^2 & \frac{\eta}{h^\lambda}W_{-1} & \dots & \frac{\eta}{h^\lambda}W_{-n+2} & 2\alpha & 0 & \dots & 0 \\ \frac{\eta}{h^\lambda}W_1 & \frac{\eta}{h^\lambda}W_0+\beta^2 & \dots & \frac{\eta}{h^\lambda}W_{-n+3} & 0 & 2\alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\eta}{h^\lambda}W_{n-2} & \frac{\eta}{h^\lambda}W_{n-3} & \dots & \frac{\eta}{h^\lambda}W_0+\beta^2 & 0 & 0 & \dots & 2\alpha \end{bmatrix}_{(2n-2)(2n-2)},$$

where $W_k = \frac{(-1)^k\Gamma(\lambda+1)}{\Gamma(\lambda/2+k+1)\Gamma(\lambda/2-k+1)}$.
It is obvious that
$\forall m,n \in \mathbb{N}: \sum_{\substack{k=-m \\ k\neq 0}}^{n} |W_k| = W_0$.
From definition of $\|B\|_\infty = \underset{i}{Max}\sum_j |b_{ij}|$, we have:

If $\quad i=1, \quad \sum_{j=1}^{2n-2}|b_{1,j}| = 1,$

if $\quad i=2, \quad \sum_{j=1}^{2n-2}|b_{2,j}| = 1,$

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if $\quad i=n-1, \quad \sum_{j=1}^{2n-2}|b_{n-1,j}| = 1,$

if $\quad i=n, \quad \sum_{j=1}^{2n-2}|b_{n,j}| = \left|\frac{\eta}{h^\lambda}W_0+\beta^2\right| + \left|\frac{\eta}{h^\lambda}W_{-1}\right| + \dots + \left|\frac{\eta}{h^\lambda}W_{-n+2}\right| + |2\alpha|$

$$= \frac{\eta}{h^{\lambda}}\{|W_{-1}|+|W_{-2}|+\ldots+|W_{-n+2}|\}+\left|\frac{\eta}{h^{\lambda}}W_0+\beta^2\right|+2\alpha$$

$$= \frac{\eta}{h^{\lambda}}W_0+\left|\frac{\eta}{h^{\lambda}}W_0+\beta^2\right|+2\alpha$$

$$= \frac{2\eta}{h^{\lambda}}(W_0+\alpha)+\beta^2,$$

if $\quad i=n+1, \quad \sum_{j=1}^{2n-2}\left|b_{n+1,j}\right| = \frac{2\eta}{h^{\lambda}}(W_0+\alpha)+\beta^2,$

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if $\quad i=2n-2, \quad \sum_{j=1}^{2n-2}\left|b_{2n-2,j}\right| = \frac{2\eta}{h^{\lambda}}(W_0+\alpha)+\beta^2.$

Therefore, we have

$$\|B\|_{\infty} = Max\left\{1, \frac{2\eta}{h^{\lambda}}(W_0+\alpha)+\beta^2\right\}$$

□

5. Numerical method

It is well known that the initial value problem (4.3), is equivalent to the following Volterra integral equation

$$X(t) = X(t_0)+\frac{1}{\Gamma(p)}\int_0^t (t-\tau)^{p-1}(-BX(\tau)+P(\tau))d\tau. \tag{5.1}$$

Let $Q(t) = -BX(t)+P(t)$. Now we consider (5.1) at $t = t_k$ and rewrite it as the following form:

$$X(t_k) = X(t_0)+\frac{1}{\Gamma(p)}\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(t_k-\tau)^{p-1}Q(\tau)d\tau. \tag{5.2}$$

Now we approximate $Q(t)$ by its piecewise linear interpolation $\bar{Q}(t) = -B\bar{X}(t)+P(t)$ at the nodes t_j and t_{j+1} as the following form

$$\bar{Q}(t) \simeq \frac{t-t_{j+1}}{t_j-t_{j+1}}\bar{Q}(t_j)+\frac{t-t_j}{t_{j+1}-t_j}\bar{Q}(t_{j+1}). \tag{5.3}$$

Let $\bar{X}(t_j)$ be the approximate solution of $X(t_j)$, $j = 1,2,3,\cdots,k$, which have been determined. By using relations (5.1) and (5.3), we can obtain the following formula

$$\bar{X}(t_k) = X(t_0)+\frac{\Delta t^p}{\Gamma(p)}\sum_{j=0}^{k-1}(T_j\bar{Q}(t_j)+R_j\bar{Q}(t_{j+1})), \tag{5.4}$$

where

$$T_j = \frac{(k-j)^p(p+1-k+j)+(k-j-1)^{p+1}}{p(p+1)},$$

$$R_j = \frac{(k-j)^{p+1}-(k-j-1)^p(p+k-j)}{p(p+1)}.$$

Therefore, we can rewrite (5.1) as the following form:

$$\bar{X}(t_k) = X(t_0) + \frac{\Delta t^p}{\Gamma(p)} \sum_{j=0}^{k-1} (T_j(-B\bar{X}(t_j)+P(t_j)) + R_j(-B\bar{X}(t_{j+1})+P(t_{j+1}))). \quad (5.5)$$

Therefore, we have

$$\bar{X}(t_k) = \left(I + \frac{\Delta t^p R_{k-1} B}{\Gamma(p)}\right)^{-1} \left\{X(t_0) + \frac{\Delta t^p}{\Gamma(p)} \times \left(\sum_{j=0}^{k-1} T_j(-B\bar{X}(t_j)+P(t_j)) + \sum_{j=0}^{k-2} R_j(-B\bar{X}(t_{j+1})+P(t_{j+1})) + R_{k-1}P(t_k)\right)\right\}. \quad (5.6)$$

6. Error and Stability analysis

In this section, the error analysis for the proposed scheme in the previous section is discussed based on the error estimate of the compound trapezoidal formula. From previous section, we have

$$X(t_k) = X(0) + \frac{1}{\Gamma(p)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k-\tau)^{p-1} Q(\tau) d\tau,$$

and

$$\bar{X}(t_k) = \bar{X}(0) + \frac{1}{\Gamma(p)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k-\tau)^{p-1} \bar{Q}(\tau) d\tau.$$

We can easily get that

$$X(t_k) - \bar{X}(t_k) = \frac{1}{\Gamma(p)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k-\tau)^{p-1} (Q(\tau)-\bar{Q}(\tau)) d\tau, \quad (6.1)$$

where on each subinterval $[t_j, t_{j+1}]$, $j = 0, 1, ..., n-1$, we have

$$Q(\tau) - \bar{Q}(\tau) = (\tau - t_j)(\tau - t_{j+1}) \frac{Q''(\xi_j)}{2!}, \qquad t_j < \xi_j < t_{j+1}.$$

Let $\|X''(t)\|_\infty = \bar{X}_j$ for $t_j \le t \le t_{j+1}$ and $\|\bar{X}_j\|_\infty = \bar{X}$.
Therefore, we get

$$\|X(t_k) - \bar{X}(t_k)\|_\infty = \left\| \frac{1}{\Gamma(p)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k-\tau)^{p-1}(Q(\tau) - \bar{Q}(\tau)) d\tau \right\|_\infty$$

$$= \frac{1}{2!\Gamma(p)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \left\| (t_k-\tau)^{p-1}(\tau - t_j)(\tau - t_{j+1})(-BX''(\xi_j)) d\tau \right\|_\infty$$

$$\le \frac{1}{2!\Gamma(p)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k-\tau)^{p-1}(\tau - t_j)(t_{j+1} - \tau)(\|B\|_\infty \|X''(\xi_j)\|_\infty)$$

$$\le \frac{1}{2!\Gamma(p)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k-\tau)^{p-1}(\tau - t_j)(t_{j+1} - \tau)(\|B\|_\infty \bar{X}_j) d\tau$$

$$\le \frac{\bar{X}\|B\|_\infty}{2!\Gamma(p)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k-\tau)^{p-1}(\tau - t_j)(t_{j+1} - \tau) d\tau \le \frac{\bar{X}\|B\|_\infty t_k^{1+p} \Delta t^2}{2!\Gamma(p+1)}.$$

Therefore $\|X(t_k) - \bar{X}(t_k)\|_\infty = O(\Delta t^2)$.

Now we give the theoretical stability analysis of our scheme. A numerical initial value problem solver is stable if small perturbations in the initial conditions do not cause the numerical approximation to diverge away from the true solution provided the true solution of the initial value problem is bounded [20].

Theorem 6.1. *Let $X(t_k)$ and $\bar{X}(t_k)$ be numerical solutions in (4.3), with the initial conditions $X(t_0)$ and $\bar{X}(t_0)$, respectively. Then*

$$\|X(t_k) - \bar{X}(t_k)\|_\infty < \|X(t_0) - \bar{X}(t_0)\|_\infty. \tag{6.2}$$

for any k, i.e. the new scheme is numerically stable.

Proof. This proof will be used based on mathematical induction. In view of the given initial condition, suppose that (6.2) is true for (j=1,2,...,k-1). We must prove that this also holds for j=k.
Assume that

$$T_j = \frac{-1}{t} \int_{t_j}^{t_{j+1}} (t_k - \tau)^{p-1}(\tau - t_{j+1}) d\tau,$$

$$R_j = \frac{1}{t}\int\limits_{t_j}^{t_{j+1}} (t_k-\tau)^{p-1}(\tau-t_j)d\tau.$$

From (5.2), we can write

$$X(t_k) = X(t_0) + \frac{1}{\Gamma(p)}\left\{\sum_{j=0}^{k-1}\int\limits_{t_j}^{t_{j+1}} \frac{(t_k-\tau)^{p-1}(\tau-t_{j+1})}{t_j-t_{j+1}}Q(t_j)d\tau\right.$$

$$\left. + \sum_{j=0}^{k-1}\int\limits_{t_j}^{t_{j+1}} \frac{(t_k-\tau)^{p-1}(\tau-t_j)}{t_{j+1}-t_j}Q(t_{j+1})d\tau\right\}.$$

Then, we have

$$\bar{X}(t_k)-X(t_k) = \bar{X}(t_0)-X(t_0)+\frac{B}{\Gamma(p)}\left\{\sum_{j=0}^{k-1}(T_j(\bar{X}(t_j)-X(t_j))+R_j(\bar{X}(t_{j+1})-X(t_{j+1})))\right\}.$$

Also

$$\|\bar{X}(t_k)-X(t_k)\|_\infty \le \|\bar{X}(t_0)-X(t_0)\|_\infty + \frac{\|B\|_\infty}{\Gamma(p)}\left\{\sum_{j=1}^{k-1}\left|T_j\right|\left\|\bar{X}(t_j)-X(t_j)\right\|_\infty\right.$$

$$\left. + |T_0|\,\|\bar{X}(t_0)-X(t_0)\|_\infty + \sum_{j=1}^{k-1}\left|R_{j-1}\right|\left\|\bar{X}(t_j)-X(t_j)\right\|_\infty + |R_{k-1}|\,\|\bar{X}(t_k)-X(t_k)\|_\infty\right\}.$$

Let us

$$\forall j=1,2,...,k-1: \left\|X(t_j)-\bar{X}(t_j)\right\|_\infty \le C_j\|X(t_0)-\bar{X}(t_0)\|_\infty.$$

Since

$$|T_0| \le \frac{1}{p}(t_k^p - t_{k-1}^p) \le \frac{1}{p}t_k^p = \frac{T^p}{p},$$

$$|R_{k-1}| = \frac{1}{p}t^p$$

and there is a $t_j \le \tilde{\tau}_j \le t_{j+1}$ that

$$\left|T_j\right| = \left|\frac{-1}{t}\int\limits_{t_j}^{t_{j+1}} (t_k-\tau)^{p-1}(\tau-t_{j+1})d\tau\right|$$

$$\leq \left|\frac{\tilde{\tau}_j - t_{j+1}}{-t}\right| \int_{t_j}^{t_{j+1}} (t_k-\tau)^{p-1} d\tau \leq \frac{1}{p}\left|\frac{t}{-t}\right| \left(-(t_k-\tau)^p\Big|_{t_j}^{t_{j+1}}\right) = \frac{1}{p}(t_{k-j}^p - t_{k-j-1}^p).$$

Then

$$\sum_{j=1}^{k-1} |T_j| \leq \frac{1}{p} t_{k-1}^p.$$

Similarly we can drive

$$\sum_{j=1}^{k-1} |R_{j-1}| \leq \frac{1}{p}(t_k^p - t_1^p).$$

Combining above results, we can derive

$$\left\{1 - \frac{|R_{k-1}|\,\|B\|_\infty}{\Gamma(p)}\right\} \|\bar{X}(t_k) - X(t_k)\|_\infty \leq \left\{1 + \frac{|T_0|\,\|B\|_\infty}{\Gamma(p)}\right\} \|\bar{X}(t_0) - X(t_0)\|_\infty$$

$$+\frac{\|B\|_\infty}{\Gamma(p)} \left\{\sum_{j=1}^{k-1} (|T_j| + |R_{j-1}|) \|\bar{X}(t_j) - X(t_j)\|_\infty\right\}.$$

If $N = \underset{0\leq j\leq k-1}{Max} \left|\bar{X}(t_j) - X(t_j)\right|, \quad C_1 = 1 + \frac{|T_0|\|B\|_\infty}{\Gamma(p)}$,
since

$$\sum_{j=1}^{k-1} (|T_j| + |R_{j-1}|) \leq \frac{1}{p}(t_{k-1}^p + t_k^p - t_1^p) \leq \frac{2}{p} t_k^p \leq \frac{2}{p} T^p.$$

we have

$$\|\bar{X}(t_k) - X(t_k)\|_\infty \leq \frac{1}{\Gamma(p+1) - t^p \|B\|_\infty} \{C_1 \|\bar{X}(t_0) - X(t_0)\|_\infty$$

$$+2T^p \|B\|_\infty \underset{0\leq j\leq k-1}{Max} \left|\bar{X}(t_j) - X(t_j)\right|\}.$$

Now, applying the mathematical induction and choosing $C_{p,T}$ sufficiently large leads to the end of the proof.

□

7. Numerical examles

In this section, two examples for which the exact solutions are known are solved by the proposed method to illustrate the efficiency and effectiveness of the suggested numerical scheme. We estimate the maximum error and the temporal convergence order of the numerical solution.

Example 7.1. Consider the following Riesz space fractional telegraph equation with constant coefficients

$$\begin{cases} \frac{\partial^p}{\partial t^p}\left(\frac{\partial^p}{\partial t^p}u(x,t)\right)+2\alpha\frac{\partial^p}{\partial t^p}u(x,t)+\beta^2u(x,t)=\mu\frac{\partial^\lambda}{\partial|x|^\lambda}u(x,t)+f(x,t), \\ u(0,t)=u(1,t)=0, \quad 0\le t\le 1, \\ u(x,0)=0, \quad \frac{\partial^p}{\partial t^p}u(x,0)=0. \end{cases}$$

$$f(x,t)=x^2(1-x)^2\Big\{t^{1-2p}E_{2,2-2p}(-t^2)-\tfrac{(2-p)\Gamma(2)}{\Gamma(2-2p)}t^{1-2p}$$
$$+2\alpha(t^{1-p}E_{2,2-p}(-t^2)-\tfrac{(2-p)\Gamma(2)}{\Gamma(2-p)}t^{1-p})+\beta^2\sin t-t\Big\}$$
$$+\tfrac{\mu\sin t-t}{2\cos\frac{\lambda\pi}{2}}\Big\{\tfrac{\Gamma(5)}{\Gamma(5-\lambda)}(x^{4-\lambda}+(1-x)^{4-\lambda})-2\tfrac{\Gamma(4)}{\Gamma(4-\lambda)}(x^{3-\lambda}+(1-x)^{3-\lambda})$$
$$+\tfrac{\Gamma(3)}{\Gamma(3-\lambda)}(x^{2-\lambda}+(1-x)^{2-\lambda})\Big\}.$$

The above problem has the exact solution $u(x,t)=x^2(1-x^2)(\sin t-t)$.
We use the method of (5.6) to solve this problem for $\alpha=10, \quad \beta=5, \quad \mu=1$.
The numerical solution are shown in Table 1. In Table 1 with take $p=.9, \quad \lambda=1.9$ we find that, the numerical results fit well with the theoretical analysis. Figs. 1 and 2 shows the analytical and numerical solution for $u(x,t)=x^2(1-x^2)(\sin t-t)$. Fig. 3 show the comparison between the space and time analytical and the numerical solution for $h={}^1/_{100}$.

TABLE 1. Maximum errors and temporal convergence order of example 1 for $\lambda=1.9$.

h=Δt	Maximum error	Temporal convergence order
.2	1.7087e-4	-
.1	4.3809e-5	1.96360
.05	1.0908e-5	2.00584
.025	2.7224e-6	2.00243
.0125	6.7988e-7	2.00169
.00625	1.6982e-7	2.00127

Example 7.2. Consider the following Riesz space fractional telegraph equation with constant coefficients

$$\begin{cases} \frac{\partial^p}{\partial t^p}\left(\frac{\partial^p}{\partial t^p}u(x,t)\right)+2\alpha\frac{\partial^p}{\partial t^p}u(x,t)+\beta^2u(x,t)=\mu\frac{\partial^\lambda}{\partial|x|^\lambda}u(x,t)+f(x,t), \\ u(0,t)=u(1,t)=0, \quad 0\le t\le 1, \\ u(x,0)=0, \quad \frac{\partial^p}{\partial t^p}u(x,0)=0. \end{cases}$$

FIGURE 1. Analytical solution of example 1.

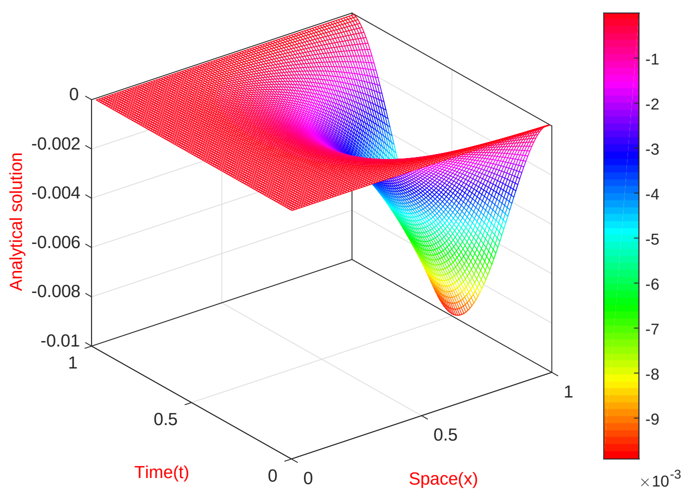
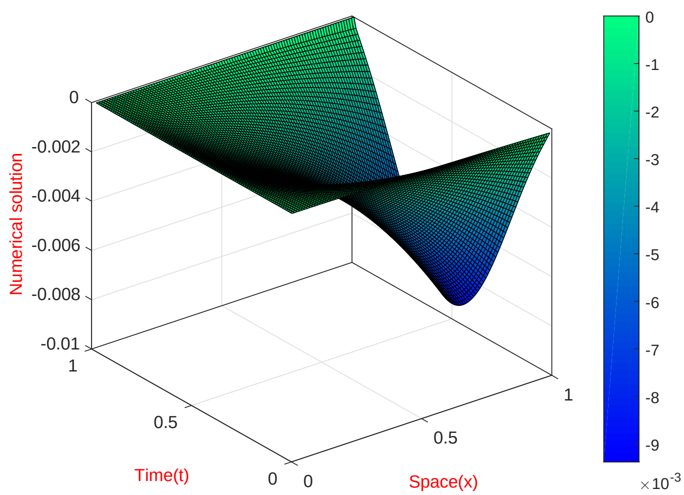


FIGURE 2. Numerical solution of example 1 at $p = 0.9$ and $h = 1/100$ for $t \in [0,1]$.



$$f(x,t) = x^2(1-x)^2 \left\{ \frac{6t^{3-2p}}{\Gamma(4-2p)} + \frac{12\alpha t^{3-p}}{\Gamma(4-p)} + \beta^2 t^3 \right\}$$

FIGURE 3. Comparison between the analytical and numerical solutions of example 1 at $p = 0.9$ and $h = 1/100$ for $t \in [0,1]$.

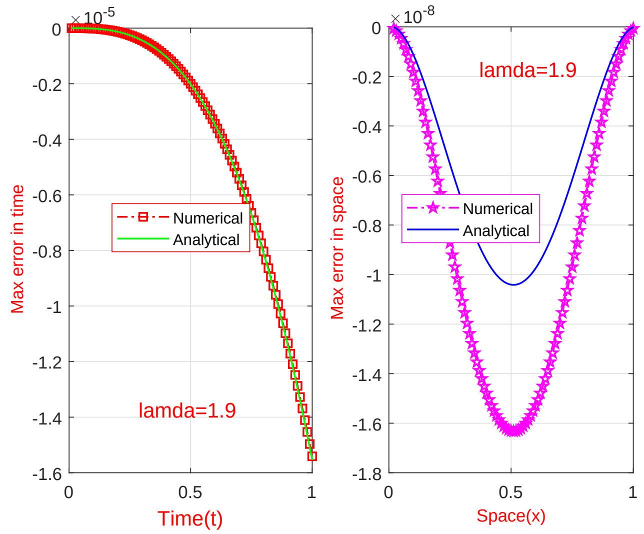


FIGURE 4. Numerical solution of example 1 for several λ at $h = 1/10$ and $t \in [0,1]$.

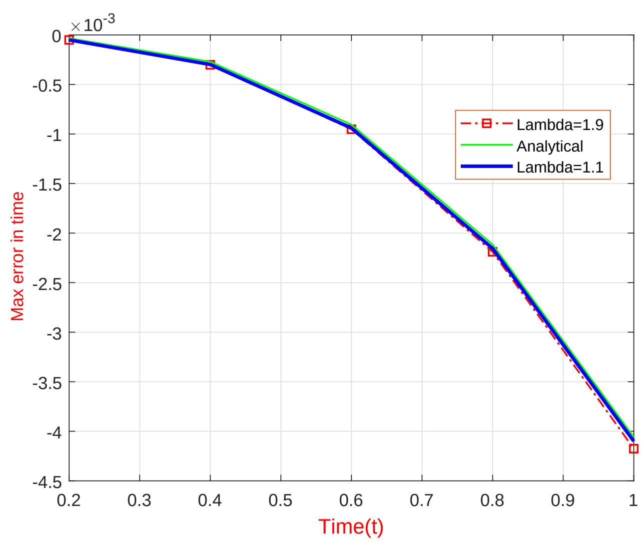
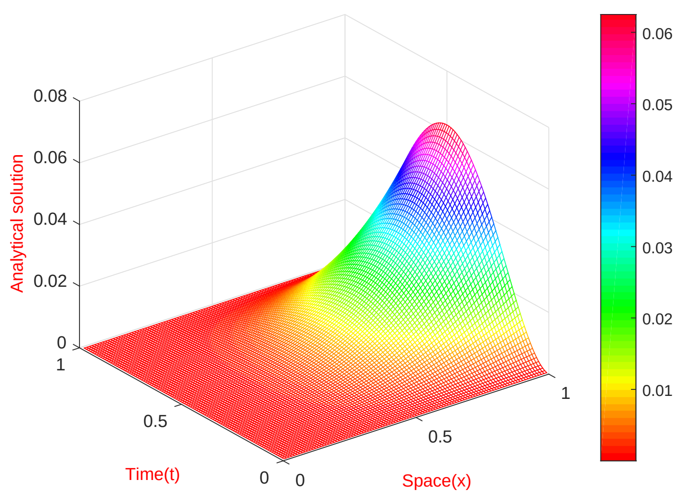


TABLE 2. Maximum errors and temporal convergence order of example 1 for $\lambda = 1.9$.

h=Δt	The maximum error	The temporal convergence order
.2	6.2521e-4	-
.1	1.6263e-4	1.94274
.05	4.0594e-5	2.00226
.025	1.0143e-5	2.00078
.0125	2.5350e-6	2.00042
.00625	6.3357e-7	2.00041

FIGURE 5. Analytical solution of example 2.



$$+\frac{\mu t^3}{2\cos\frac{\lambda\pi}{2}}\left\{\frac{\Gamma(5)}{\Gamma(5-\lambda)}(x^{4-\lambda}+(1-x)^{4-\lambda})-2\frac{\Gamma(4)}{\Gamma(4-\lambda)}(x^{3-\lambda}+(1-x)^{3-\lambda})\right.$$
$$\left.+\frac{\Gamma(3)}{\Gamma(3-\lambda)}(x^{2-\lambda}+(1-x)^{2-\lambda})\right\}.$$

The above equation has the exact solution $u(x,t) = x^2(1-x^2)t^3$.
We use the method of (5.6) to solve this problem for $\alpha = 25, \quad \beta = 10, \quad \mu = 1$.
In this test, corresponding with example 1 the computational results are tabulated in Table 2. We present the convergence behaviors of our method for various kinds of Riesz fractional order and plural of the word of h and t in Figs. 5,6,7 and 8. The error and convergence order in time are observed when the temporal step is chosen suitable.

FIGURE 6. Numerical solution of example 2 at $p = 0.9$ and $h = 1/100$ for $t \in [0,1]$.

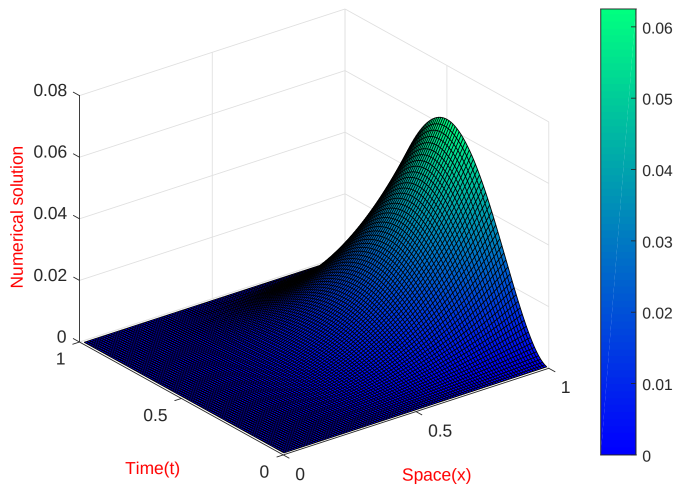


FIGURE 7. Comparison between the analytical and numerical solutions of example 2 at $p = 0.9$ and $h = 1/100$ for $t \in [0,1]$.

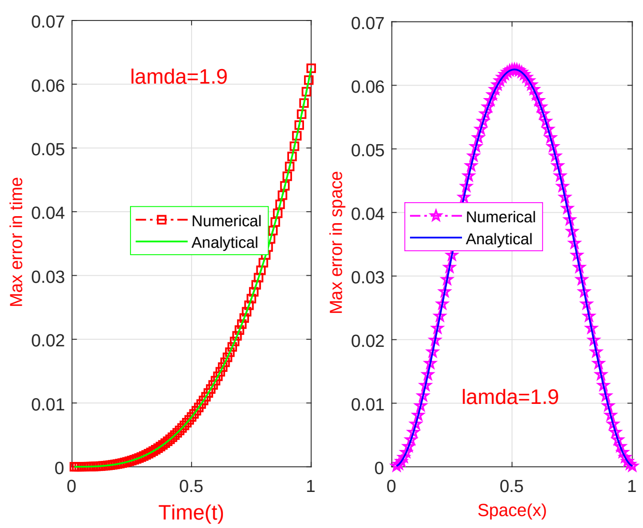
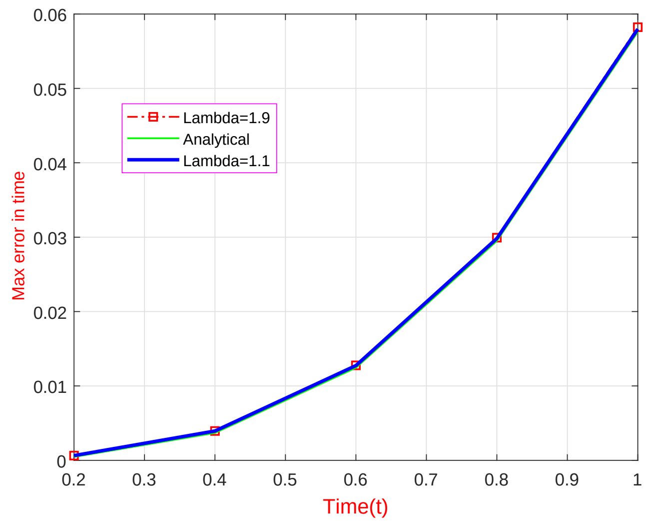


FIGURE 8. Numerical solution of example 2 for several λ at $h = 1/10$ and $t \in [0,1]$.



Example 7.3. Consider the following Riesz space fractional telegraph equation

$$\frac{\partial^p}{\partial t^p}\left(\frac{\partial^p}{\partial t^p}u(x,t)\right) + 2\alpha\frac{\partial^p}{\partial t^p}u(x,t) + \beta^2 u(x,t) = \mu\frac{\partial^\lambda}{\partial |x|^\lambda}u(x,t) + f(x,t),$$

subject to the initial and boundary value conditions

$$u(0,t) = u(1,t) = 0, \qquad 0 \le t \le T,$$

$$u(x,0) = 0, \quad \tfrac{\partial^p}{\partial t^p}u(x,0) = 0, \quad 0 \le x \le 1,$$

$$f(x,t) = x^6(1-x)^6\left\{\tfrac{\Gamma(1+2\sigma)}{\Gamma(1+2\sigma-2p)}t^{2\sigma-2p} + \tfrac{2\alpha\Gamma(1+2\sigma)}{\Gamma(1+2\sigma-p)}t^{2\sigma-p} + \beta^2 t^{2\sigma})\right\}$$

$$+\frac{\mu t^{2\sigma}}{2\cos(\frac{\lambda\pi}{2})}\left\{\tfrac{\Gamma(7)}{\Gamma(7-\lambda)}[(1-x)^{6-\lambda} + x^{6-\lambda}]\right.$$

$$-\tfrac{6\Gamma(8)}{\Gamma(8-\lambda)}(x^{7-\lambda} + (1-x)^{7-\lambda}) + \tfrac{15\Gamma(9)}{\Gamma(9-\lambda)}(x^{8-\lambda} + (1-x)^{8-\lambda})$$

$$-\tfrac{20\Gamma(10)}{\Gamma(10-\lambda)}(x^{9-\lambda} + (1-x)^{9-\lambda}) + \tfrac{15\Gamma(11)}{\Gamma(11-\lambda)}(x^{10-\lambda} + (1-x)^{10-\lambda})$$

$$\left.-\tfrac{6\Gamma(12)}{\Gamma(12-\lambda)}(x^{11-\lambda} + (1-x)^{11-\lambda}) + \tfrac{\Gamma(13)}{\Gamma(13-\lambda)}(x^{12-\lambda} + (1-x)^{12-\lambda})\right\}.$$

The above equation has the exact solution $u(x,t) = x^6(1-x)^6 t^{2\sigma}$.
We use the method of (5.6) to solve this problem For $\alpha = 15, \quad \beta = 14, \quad \mu = 1, \quad \sigma > p$.
The numerical solutions are shown in Table 3. Numerical results show that the convergence orders are almost second order in maximum error and temporal direction.

TABLE 3. Maximum errors and temporal convergence order of example 3.

p	σ	λ	h=Δt	The maximum error	The temporal convergence order
			.1	3.6142e-7	-
.8	1	1.2	.05	9.1493e-8	1.98192
			.025	2.2910e-8	1.99768
			.1	1.8324e-6	-
.8	1	1.8	.05	4.6258e-7	1.98594
			.025	1.1588e-7	1.99710
			.1	2.3114e-6	-
.8	1	1.9	.05	5.8136e-7	1.99127
			.025	1.4550e-7	1.99841
			.1	3.9783e-7	-
.8	1.2	1.2	.05	1.0063e-7	1.98311
			.025	2.5228e-8	1.99592
			.1	1.8270e-6	-
.8	1.2	1.8	.05	4.6134e-7	1.98554
			.025	1.1560e-7	1.99665
			.1	2.2957e-6	-
.8	1.2	1.9	.05	5.7767e-7	1.99058
			.025	1.4462e-7	1.99798
			.1	2.3850e-6	-
.9	1.8	1.9	.02	9.6215e-8	1.99472
			.004	3.8479e-9	2.00012

Figs. 9 and 10 shows the analytical and numerical solutions of $u(x,t) = x^6(1-x)^6 t^{2\sigma}$. Fig. 11 show the comparison between the analytical and numerical solutions of space and time for $h = 1/100, \quad \sigma = 1, \quad \lambda = 1.8$.

FIGURE 9. Analytical solution of example 3.

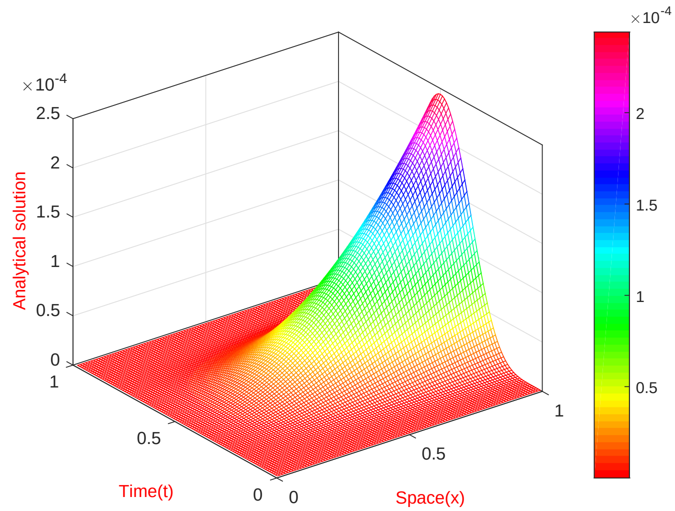


FIGURE 10. Numerical solution of example 3 at $p = 0.9$ and $h = 1/100$ for $t \in [0, 1]$.

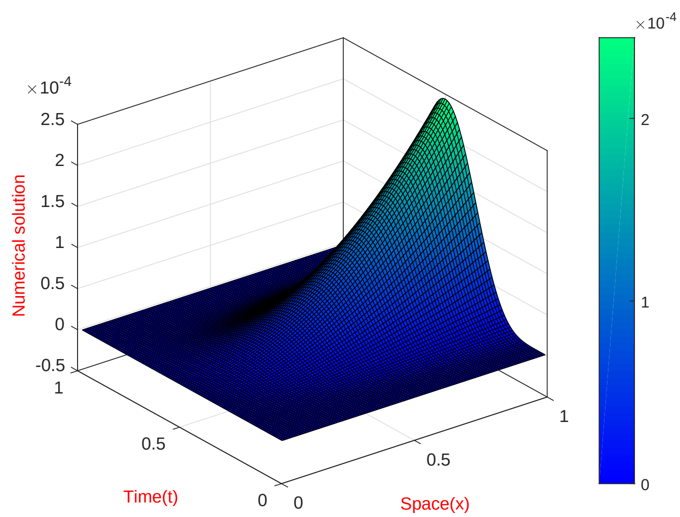


FIGURE 11. Comparison between the analytical and numerical solutions of example 3 at $p = .9$ and $h = 1/100$ for $t \in [0, 1]$.

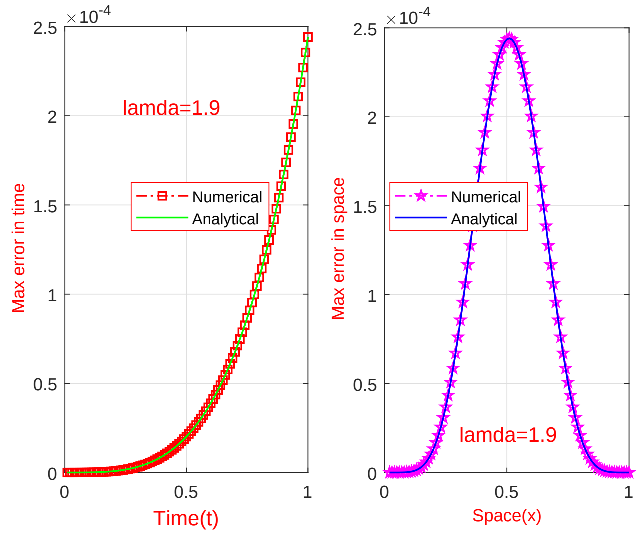
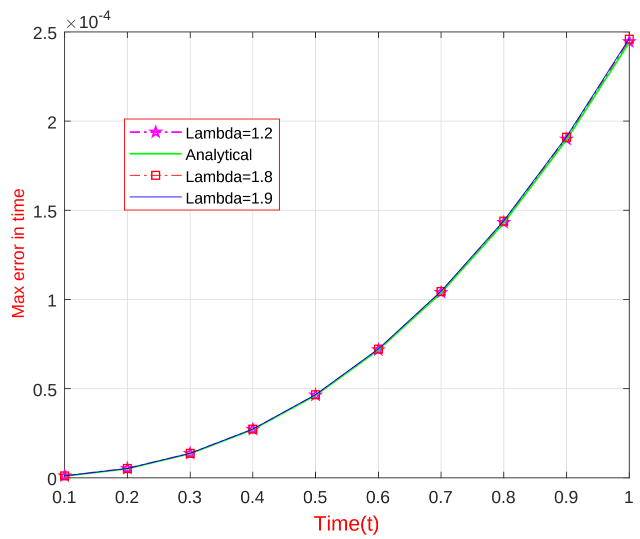


FIGURE 12. Numerical solution of example 3 for several λ at $h = 1/10$ and $t \in [0, 1]$.



8. Conclusion

This paper provides an iterative solution to the telegraph equation. We made a brief introduction to a approximation based on the piecewise linear interpolation that used for discretizing of Riesz space fractional telegraph equation. The approximate results approach in analytic form with order $O(\Delta t^2)$. The conclusions are verified and compared by three numerical examples. We believe that this approximation will be possible to have a better comprehension of the telegraph equation in electrical systems and transmission lines.

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