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An efficient numerical solution for time switching optimal control problems

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Abstract In this paper, an efficient computational algorithm for the solution of Hamiltonian boundary value problems arising from bang-bang optimal control problems is presented. For this purpose, at first, based on the Pontryagin's minimum principle, the first order necessary conditions of optimality are derived. Then, an indirect shooting method with control parameterization, in which the control function is replaced with a piecewise constant function with values and switching points taken as unknown parameters, is presented. Thereby, the problem is converted to the solution of the shooting equation, in which the values of the control function and the switching points as well the initial values of the costate variables are unknown parameters. The important advantages of this method are that, the obtained solution satisfies the first order optimality conditions, further the switching points can be captured accurately which is led to an accurate solution of the bang-bang problem. However, solving the shooting equation is nearly impossible without a very good initial guess. So, in order to cope with the difficulty of the initial guess, a homotopic approach is combined with the presented method. Consequently, no priori assumptions are made on the optimal control structure and number of the switching points, and sensitivity to the initial guess for the unknown parameters is resolved too. Illustrative examples are included at the end and efficiency of the method is reported.

Keywords. Switching controls, Hamiltonian boundary value problem, Control parameterization, Homotopic approach.

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1. Introduction

One of the most challenging subjects in the control theory is optimal control of bang-bang type problems, where the input control jumps from one boundary to another. This type of control problems frequently appears in many applications, such as biological sciences [16], biochemical reaction systems [24], atomic physics [27] and aerospace engineering [30]. When the control oscillating between it's lower and upper bounds and appears linearly in the objective function and the dynamical equations, the optimal control is often characterized by bang-bang type control [6]. The computation of optimal bang-bang controls is of particular interest because of difficulty in obtaining switching points and optimal solution.

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So, over the past several decades, significant progress has been reached in the development of both direct and indirect numerical methods for finding optimal bang-bang controls [4, 11, 12, 17, 29]. Direct methods, which are based on discretization and nonlinear programming, have been used extensively in a variety of optimal control problems, such as bang-bang type controls. Consequently, various direct algorithms for optimal bang-bang control have been reported. For instance, we can refer to the modified Legendre pseudospectral method [25], the time-scaling transformation technique [31], the modified control parametrization approach [19] and smooth regularization method [26]. It is noted that, some of these algorithms require the number of switching points and structure of the optimal control and some are sensitive to the initial guess, in which no specific method to initialize most of them is discussed.

On the other hand, the indirect methods corresponds to the use of the Pontryagin's minimum principle [15] and lead to the solution of a Hamiltonian boundary value problem (HBVP). The primary advantages of them, are their high accuracy in the solution and the assurance that the solution satisfies the first order optimality conditions [6]. The indirect shooting and multiple shooting methods [22, 23] are the most popular methods for numerical solution of HBVPs, specially, those arise from bang-bang optimal control problems. However, these methods tend to suffer the difficulties in dealing with bang-bang problems in two aspects. The first is the need for a priori knowledge about the structure of the optimal control and number of the switching points. The second is the extreme sensitivity to the initial guess in solving the associated HBVP, mainly because of the lack of an accurate initial guess for the switching points and the initial values for the costate variables which have no physical interpretation [4].

In this paper, an indirect approach for solving a HBVP arising from bang-bang optimal control problems is utilized such that the sensitivity to the initial guess is resolved and does not require a priori knowledge about the structure of the optimal control and the number of switching points. For this purpose, at first, based on the Pontryagin's minimum principle, the first order necessary conditions of optimality are derived. Then, without any knowledge about the structure of the optimal control, the control function is replaced with a piecewise constant function with values and the switching points taken as unknown parameters. Consequently, the problem is converted to the solution of the shooting equation [3], in which the values of the control function and the switching points as well the initial values of the costate variables are unknown parameters. It is noted that, solving the shooting equation, is nearly impossible without a very close initial guess for the unknown parameters. So, in order to resolve the need for a priori knowledge about the structure of the optimal control and for reducing the sensitivity to the initial guess, a homotopic approach is applied too.

Homotopic approaches are applied as useful tools for solving a wide variety of problems [1]. The principal idea of the method is to define a one-parameter family of problems such that, the first problem is an easy problem to solve and the last one, is the original difficult problem. Each intermediate problem then uses the solution of the previous problem as an initial guess in the family and this process continues until the solution of the original problem is achieved. It is noted that, many researchers have used homotopic approaches to initialize the shooting method for bang-bang problems. Some of which are indicated to below.

In [2], a perturbation approach called continuation smoothing technique for solving bang-bang optimal control problems, by introducing a perturbed energy term in the objective function to yield a continuous optimal control, is developed. The perturbed bang-bang problem is then solved by the shooting method with updating the perturbation parameter continuatively.

In [13, 14, 18], homotopic approaches are used to solve the low-thrust orbit transfer problems, which leads to a discontinuous control with a huge number of thrust arcs. The resolution method is based on the single shooting combined to a homotopic approach by defining a homotopy connecting between two problems, discontinuous control mass maximization and continuous control energy minimization. The point is that the energy problem is indeed much easier to solve than the mass problem, with a much better convergence radius for the shooting method. Then, providing a solution of the energy problem, the continuation procedure is done to reach the solution of the mass problem.

In [8], minimization of the fuel consumption of the spacecraft during the transfer, from the Earth to the Moon, is considered. Because of the resulting non-smoothness of the Hamiltonian two-point boundary value problem, it is difficult to use shooting methods to compute numerical solutions. To overcome these difficulties, two homotopies are introduced: One connects the investigated problem to the minimization of the L^2-norm of the control, while the other introduces an interior penalization in the form of a logarithmic barrier. The combination of shooting with these continuation procedures allows to compute fuel optimal transfers for medium or low thrusts in the Earth-Moon system.

In [9], an inexact parametric smoothing Newton algorithm for the numerical solution of constrained optimal control problems subject to mixed control-state constraints is developed. In particular, the parametric smoothing Newton method is able to find the switching structure of constrained optimal control problems without any priori assumptions on active or inactive parts.

In [5], a new double-homotopy method is developed to address the common failures of the traditional homotopy method. In this approach, the traditional homotopy is employed until it encounters a difficulty and stops moving forward. Another homotopy originally designed for finding multiple roots of nonlinear equations takes over at this point, and it finds a different solution to allow the traditional homotopy to continue on and this process is repeated whenever necessary.

In [20], an adaptive control parametrization method is combined with a homotopy continuation technique to derive a unified method for direct solution of bang-bang optimal control problems. Consequently, the optimal control problem is transformed into a nonlinear programming problem and thereby, by solving the nonlinear programming problem, the optimal solution of the problem is obtained.

It is noted that, in most of the existing works, a new class of almost everywhere regular controls by using different perturbed terms is built. In other words, smoothing techniques are used to solve bang-bang optimal control problems. In this paper, a different homotopic approach is used together with an indirect shooting method for solving bang-bang optimal control problems. For this purpose, the HBVP corresponding to the optimality conditions related to a bang-bang problem, is transferred to a family of HBVPs with a homotopy parameter μ, such that in the case $\mu = 0$, we find the simplified BVP related to a simplified bang-bang problem which we can easily obtain the corresponding solution and in the case $\mu = 1$, we meet the desired HBVP related to the desired bang-bang problem. Each HBVP in this family is solved by the presented shooting method, in such a way that the solution of the first is used to construct an initial guess for the next HBVP, and this process continues until the desired HBVP is solved. Furthermore, the present technique can change the number of the switching points during the homotopy procedure. Consequently, a priori knowledge about the switching structure of the optimal control is not required. It is worthwhile to note that, the presented method in this paper is similar to that of [20], except the approach in [20] employs the homotopy method to obtain a direct solution of optimal control problem, however, the proposed method in this paper is an indirect one which satisfies the first order optimality conditions of the problem.

This paper is organized as follows: In Section 2, a formulation of the bang-bang optimal control problems with control appearing linearly is introduced. Section 3 is devoted to the Pontryagin's minimum principle and control parameterization. In Section 4, the shooting method is applied to solve a HBVP arising from bang-bang optimal control problems. Section 5 is devoted to the formulation of the homotopic approach. In Section 6, the proposed combined method is applied to two examples arising in the optimal control theory and the advantages of our method are shown.

2. Bang-Bang optimal control problems

Without loss of generality, in this paper, the problem is to find a scalar control $u(t)$, the corresponding state vector $\mathbf{x}(t) = [x_1(t), \dots, x_p(t)]^T$, and possibly free terminal time t_f which minimize the cost functional

$$J = \Phi(\mathbf{x}(t_f), t_f),$$

subject to a system of p nonlinear differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{h}(\mathbf{x}(t), u(t), t), \quad 0 \le t \le t_f, \tag{2.1}$$

with the initial and terminal conditions

$$\mathbf{x}(0) = \mathbf{x}_0, \tag{2.2}$$

$$\mathbf{x}(t_f) = \mathbf{x}_f, \tag{2.3}$$

together with the box constraints

$$u(t) \in \{-1, +1\}. \tag{2.4}$$

Here, $\Phi : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}$ is the terminal cost and also assumed to be smooth, the state $\mathbf{x}$ is continuous, and the control is always bang-bang, with finite number of switching points. The vector function $\mathbf{h} : \mathbb{R}^{p+2} \to \mathbb{R}^p$ is assumed to be a smooth function of the variables $(\mathbf{x}, u, t)$.

It is worthwhile to note that, according to the Pontryagin's minimum principle, when the controls are bounded and appear linearly both in the dynamic equations and cost functional, then the non-singular optimal control solution is bang-bang. In summary, if Eq. (2.4) is replaced by

$$-1 \le u(t) \le 1,$$

and the dynamic equations in Eq. (2.1) are considered as

$$\dot{\mathbf{x}}(t) = f_1(\mathbf{x}(t), t) + F_1(\mathbf{x}(t), t)u(t),$$

then, the solution is bang-bang, if singularity doesn't occur.

In addition, a Mayer-type cost functional is considered. If the cost to be minimized is in integral form (Lagrange-type), then the problem can be converted to a Mayer-type [6]. For this purpose, suppose that a cost functional has the following form

$$\int_0^{t_f} g(\mathbf{x}(t),u(t),t)dt.$$

So, the problem can easily be converted to a Mayer-type by defining a new state variable $x_{p+1}(t)=\int_0^{t_f} g(\mathbf{x}(t),u(t),t)dt$, and the following new differential equation

$$\dot{x}_{p+1}(t)=g(\mathbf{x}(t),u(t),t),$$

with the initial condition $x_{p+1}(0)=0$, is appended to the system dynamics. Also, $x_{p+1}(t_f)$ is considered as free boundary condition alongside the boundary conditions (2.3).

3. Minimum principle and control parameterization

To formulate the first order necessary conditions of optimality, the Hamiltonian function is introduced as follows

$$\begin{aligned}\mathscr{H}(\mathbf{x}(t),\lambda(t),u(t)) &= \lambda^T(t)\mathbf{h}(\mathbf{x}(t),u(t),t)\\ &= \lambda^T(t)f_1(\mathbf{x}(t),t)+\lambda^T(t)F_1(\mathbf{x}(t),t)u(t),\end{aligned} \tag{3.1}$$

where $\lambda(t)=[\lambda_1(t),\dots,\lambda_p(t)]^T$ is called the costate variable. Also, the factor of the control $u(t)$ in the Hamiltonian function is called the switching function and denoted by

$$\sigma(t)=\sigma(\mathbf{x}(t),\lambda(t),t)=\lambda^T(t)F_1(\mathbf{x}(t),t).$$

It is noted that, the solution of the optimal control problem satisfies the following necessary conditions

$$\dot{\mathbf{x}}^* = [\partial\mathscr{H}/\partial\lambda]^T, \tag{3.2}$$

$$\dot{\lambda}^* = -[\partial\mathscr{H}/\partial\mathbf{x}]^T, \tag{3.3}$$

and based on the Pontryagin's minimum principle, an optimal control must minimize the Hamiltonian function with related to the control function [15]. Consequently, according to the bang-bang property, it readily implies the following control law

$$u^*(t)=\begin{cases} -1, & \text{if } \sigma(\mathbf{x},\lambda,t)>0,\\ +1, & \text{if } \sigma(\mathbf{x},\lambda,t)<0,\\ \text{undefined}, & \text{if } \sigma(\mathbf{x},\lambda,t)=0.\end{cases} \tag{3.4}$$

Note that, the bang-bang control is characterized by the fact that the switching function

$$\sigma(t)=\sigma(\mathbf{x}(t),\lambda(t),t),$$

has only isolated zeroes that are the candidates for the switching points of the control function [6]. So, the switching function $\sigma(t)$ determines the optimal control via

$$u^*(t)=-\text{sign}(\sigma(t)),$$

on $[0,t_f]$, except on the switching points. Now, the following approximation for control $u(t)$

$$u(t)=\begin{cases} b_0, & s_0\le t\le s_1,\\ b_1, & s_1<t\le s_2,\\ \vdots & \\ b_n, & s_n<t\le s_{n+1},\end{cases} \tag{3.5}$$

is intended, where $0=s_0\le s_1\le\cdots\le s_n\le s_{n+1}=t_f$, are considered as unknown switching and terminal points, in which on each subdomain, $u(t)$ takes unknown parameter b_k, for $k=0,\dots,n$. So, in view of (2.4), we have

$$b_k\in\{-1,+1\},$$

where $b_{k+1}=-b_k$. Consequently, we have

$$b_k=-\text{sign}(\sigma(t)),\;\; s_k\le t\le s_{k+1},$$

and

$$b_n=(-1)^n b_0. \tag{3.6}$$

This control structure yields the following switching conditions

$$\sigma(s_i)=0,\;\; i=1,\dots,n. \tag{3.7}$$

It is noted that, according to (3.5) and (3.6), we parameterized the control function $u(t)$ by a vector $\upsilon=(s_1,\dots,s_{n+1},b_0)$, in which

$$b_0=u(0)=-\text{sign}(\sigma(0)).$$

Consequently, we can denote the control function $u(t)$ in Eq. (3.5) by $u(t;\upsilon)$. Now, by replacing the control function $u(t;\upsilon)$ in Eqs. (3.2) and (3.3), a system of differential equations, with $u(t;\upsilon)$ in right-hand-side, that is based on $\mathbf{x}(t)$ and $\lambda(t)$ is found. We can express this system of differential equations as

$$\dot{\mathbf{y}}=\mathbf{f}(t,\mathbf{y},u(t;\upsilon)), \tag{3.8}$$

where $\mathbf{y}(t)=[\mathbf{x}(t),\lambda(t)]^T\in\mathbb{R}^{2p}$, and $\mathbf{f}:\mathbb{R}^{2p+2}\to\mathbb{R}^{2p}$.

It is noted that, when the final time t_f is free, the condition

$$\mathscr{H}(\mathbf{x}(t_f),\lambda(t_f),u(t_f))+\frac{\partial}{\partial t}\Phi(\mathbf{x}(t_f),t_f)=0, \tag{3.9}$$

as transversality condition, must be considered besides the conditions (2.2) and (2.3). Also, in the case, $\mathbf{x}(t_f)=$ free, and t_f is fixed, the conditions

$$\frac{\partial}{\partial\mathbf{x}}\Phi(\mathbf{x}(t_f),t_f)-\lambda(t_f)=\mathbf{0}, \tag{3.10}$$

must be added to the conditions (2.2) and (2.3). However, in the case that, both the final state $\mathbf{x}(t_f)$ and the final time t_f are free, both the Eqs. (3.9) and (3.10), are added beside the initial and boundary conditions of the problem [15].

Finally, the differential equations (3.8), beside the initial and boundary conditions (2.2) and (2.3), the switching conditions (3.7), and/or the transversality condition (3.9), and/or conditions (3.10), form a HBVP as the necessary optimality conditions. In summary, the resulted HBVP is represented by

$$\begin{cases} \dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}, u(t; \upsilon)), \quad 0 \le t \le t_f \\ \mathbf{y}(0) = \mathbf{y}_0, \\ \Psi(\mathbf{y}(t_f), s_1, \ldots, s_n, t_f) = \mathbf{0}, \end{cases} \tag{3.11}$$

where $\Psi : \mathbb{R}^{2p} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{l+n+1}$ is boundary function of the problem. It is worthwhile to note that, in the case t_f is free, this HBVP is named as boundary value problem with free end boundary [28], in which, the values of the control function and the switching and terminal points as well the initial values of the costate variables are unknown parameters. The next section, is devoted to the numerical solution of this HBVP.

4. The shooting method

In this section, we suppose that, HBVP (3.11) is a BVP with free final time t_f and $\mathbf{x}(t_f)$ is fixed. To apply the shooting method for solving this HBVP, considering again the terminal, switching, and transversality conditions in Eqs. (2.3), (3.7), and (3.9), the shooting function is constructed as following

$$\Psi(\cdot) = \begin{pmatrix} \mathbf{x}(t_f) - \mathbf{x}_f \\ \mathscr{H}(\mathbf{x}(t_f), \lambda(t_f), u(t_f)) + \frac{\partial}{\partial t}\Phi(\mathbf{x}(t_f), t_f) \\ \sigma(s_1) \\ \vdots \\ \sigma(s_n) \\ b_0^2 - 1 \end{pmatrix}, \tag{4.1}$$

where $\Psi(\cdot)$ is used instead of $\Psi(\mathbf{y}(t_f), s_1, \ldots, s_n, t_f)$ for simplicity in notation. The purpose is to solve the shooting equation

$$\mathbf{S}(\mathbf{z}) = \Psi(\cdot) = \mathbf{0}, \tag{4.2}$$

in which the column vector $\mathbf{z}$, is a vector of unknown parameters and considered as

$$\mathbf{z} = \Big(\upsilon, \lambda_1(0), \ldots, \lambda_p(0)\Big)^T \in \mathbb{R}^{n+p+2}.$$

By assigning the unknown vector $\mathbf{z}$, the mentioned HBVP is transformed to the following initial value problem (IVP)

$$\begin{cases} \dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}, u(t; \upsilon)), \quad 0 \le t \le t_f \\ \mathbf{y}(0) = \mathbf{y}_0. \end{cases} \tag{4.3}$$

We assume that, IVP in Eqs. (4.3) has a unique solution which is denoted by $\mathbf{y}(t; \mathbf{z})$. To obtain the solution of IVP in Eqs. (4.3) for a given $\mathbf{z}$, the interval $[0, t_f]$ is first divided into $n+1$ segments $[s_{i-1}, s_i]$, $i = 1, \ldots, n+1$. Then, we solve dynamic equations with the initial condition $\mathbf{y}_0$ in the first segment $[s_0, s_1]$. The obtained value at $t = s_1$ is then considered as an initial condition for the dynamic equations in the second segment $[s_1, s_2]$, and this process continues until the last segment $[s_n, s_{n+1}]$ is reached. It is worthwhile to note that, the shooting equation (4.2) is a system of Non-Linear Equations (NLE) for finding the desired $\mathbf{z}$, in which, $\mathbf{y}(s_{n+1}; \mathbf{z})$ is an approximation of $\mathbf{y}(t_f)$ and to evaluate it at a given $\mathbf{z}$, we need to solve IVP in Eqs. (4.3). Further, if any condition in NLE (4.1) is violated, then the new vector $\mathbf{z}$ should be evaluated and the above process will be resumed until the shooting equation is satisfied.

It is noted that, the IVP in Eqs. (4.3) can be solved by a high-order of accuracy ordinary differential equations solver which have the possibility of error control. Apart from various methods and softwares developed for solving IVP in Eqs. (4.3), in this paper, the MATLAB function `ode45` is applied. This function is based on an explicit Runge-Kutta (4, 5) formula with a variable time step for efficient numerical computation of differential equations. Furthermore, `ode45` controls the error by two parameters `RelTol` and `AbsTol`. By using these parameters, we can adjust the relative and absolute error tolerances. Also, the MATLAB function `fsolve` is used for solving the NLE in Eq. (4.2). It is noted that, in this solver, we can adjust the accuracy of the obtained solution by two parameters `TolFun` and `TolX`, in which, the former specifies the termination tolerance on function value and the latter specifies the termination tolerance on the parameter value.

However, solving the NLE in Eq. (4.2), is nearly impossible without a very close initial guess for the unknown parameters, mainly because of (i) the lack of the priori knowledge about the structure of the optimal control and number of the switching points and also (ii) the extreme sensitivity to the initial guess for the switching points and costate variables which have no physical interpretation. To overcome this drawback, a homotopic approach is combined with the shooting method in the next section, in such a way that the consideration of the optimal control structure as well as the sensitivity to the initial guess are resolved.

5. Applying the homotopic approach

In general, the obtained NLE in the shooting equation (4.2) must be solved numerically and for this purpose we need to provide an initial guess. So, in order to get the solution to converge properly, a good initial guess is required, otherwise, the NLE-solver fails to converge on the correct solution. On the other hand, finding a suitable initial guess can be extremely difficult in practice. To overcome this drawback, in this section, we use a numerical homotopic method in combining with the shooting method.

For this purpose, at first, we suppose that the desired HBVP has arisen from a bang-bang optimal control problem which has one switching point. More precisely, we suppose $n = 1$, in the desired HBVP, and denote it as (F) problem

$$(\mathrm{F}): \begin{cases} \dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}, u(t; \upsilon)), \quad 0 \le t \le t_f \\ \mathbf{y}(0) = \mathbf{y}_0, \\ \Psi(\mathbf{y}(t_f), s_1, t_f) = \mathbf{0}, \end{cases}$$

Then, we select two HBVPs that have arisen from simple bang-bang problems with both have one switching point, but have different structure in optimal control solution, in which we know the corresponding solutions or we can find their solutions easily, as starting problems. Suppose these starting problems, as

$$(\mathrm{G}): \begin{cases} \dot{\mathbf{y}} = \mathbf{g}(t, \mathbf{y}, u(t; \upsilon)), \quad 0 \le t \le t_f \\ \mathbf{y}(0) = \tilde{\mathbf{y}}_0, \\ \eta(\mathbf{y}(t_f), s_1, t_f) = \mathbf{0}, \end{cases}$$

It is noted that, starting problems must have the same dimension with the desired problem. Now, the homotopy problem is constructed to connect one of these starting problems with the desired problem as following

$$(H):\begin{cases}\dot{\mathbf{y}}=\mu\mathbf{f}(t,\mathbf{y},u(t;\upsilon))+(1-\mu)\mathbf{g}(t,\mathbf{y},u(t;\upsilon)),\\ \mu(\mathbf{y}(0)-\mathbf{y}_0)+(1-\mu)(\mathbf{y}(0)-\tilde{\mathbf{y}}_0)=\mathbf{0},\\ \mu\Psi(\mathbf{y}(t_f),s_1,t_f)+(1-\mu)\eta(\mathbf{y}(t_f),s_1,t_f)=\mathbf{0}.\end{cases}$$

It is seen that, for $\mu=0$, we have the starting problem and for $\mu=1$, we get the desired problem. It is supposed that, for every $\mu\in[0,1]$, the problem (H) has a solution. By applying the shooting method to the problem (H), the following NLE is obtained

$$\mu\mathbf{S}(\mathbf{z})+(1-\mu)\tilde{\mathbf{S}}(\mathbf{z})=\mathbf{0}, \tag{5.1}$$

where $\tilde{\mathbf{S}}(\mathbf{z})$ is the shooting function corresponding to the starting problem. In Eq. (5.1), the parameter μ is appeared as deformation parameter. For applying the homotopy procedure, at first, the points $\mu_0,\mu_1,\ldots,\mu_q$, are selected such that

$$0=\mu_0<\mu_1<\ldots<\mu_q=1.$$

Then, for $i=1,\ldots,q$, we attempt to solve the problem (H) with $\mu=\mu_i$, when the solution of (H) with $\mu=\mu_{i-1}$ are considered as an initial guess for the next problem. More precisely, the solution of the starting problem is used as an initial guess for the problem (H) with $\mu=\mu_1$ and the solution of the problem (H) with $\mu=\mu_1$ is used as an initial guess for the problem (H) with $\mu=\mu_2$. This process continues until the solution of the problem (H) with $\mu=\mu_q$, which is the solution of the desired HBVP, is obtained.

It is noted that, the switching structure and the number of switching points of the control function in the starting and desired problems may be different. On the other hand, we do not have a priori knowledge about the structure and number of the switching points of the control function in the desired problem. Consequently, the present method should have the ability to handle the various changes of the switching structure during the homotopy procedure. For this purpose, we consider the following strategy.

Assuming that $\mathbf{z}_k$ is the solution of the NLE in Eq. (5.1) for $\mu=\mu_k$. Then we try to solve NLE in Eq. (5.1) with $\mu=\mu_{k+1}$ by the NLE-solver, in which the solution of the k-th step is served as an initial guess. Now, the following two cases may occur.

- If the NLE-solver converges, then the structure of the control function as well as the number of switching points in the problem (H) is suitable and it is safe to continue with a new updated $\mathbf{z}$.
- If the NLE-solver doesn't converge, then at first, we must add a switching point to the switching point of the problem (H) and try to solve new obtained NLE from the problem (H) with $\mathbf{z}_k$ as an initial guess for the related unknown parameters and an arbitrary value for the new parameter. If the NLE-solver does not converge again, we must change the starting problem and retry the solution of the problem (H), with the new starting problem.

This process continues until the solution of the desired problem is reached. At the end, we summarize the presented method, in the Algorithm 1.

5.1. **Starting problems.** Starting problems are HBVPs that have arisen from simple bang-bang problems and we can find their solutions easily. It is obvious that, among all the bang-bang problems with a known solution, starting problems can be chosen. These starting problems can construct with every dimension corresponding to the desired problem. In this paper,

Algorithm 1 Homotopic approach for numerical solution of time switching optimal control problems

Input:
Steplength: $\Delta\mu$
Solution of NLE (5.1) with $\mu = 0$: $\mathbf{z}_0$
Output:
Optimal Solution: $\mathbf{z}^*$

```
Initialize:
Set n ← 1 (the number of switching points in the starting problem)
Set z ← z_0
repeat
    Set μ_new ← min{μ + Δμ, 1}
    Apply the NLE-solver on (5.1) with μ = μ_new
    if the NLE-solver converges then
        Set z ← the solution of NLE (5.1)
    else
        Set n ← n + 1
        resolve (5.1) with NLE-solver
        if the NLE-solver does not converge again then
            Change the starting problem with another one
        Set z ← the solution of NLE (5.1)
    Set μ ← μ_new
until μ < 1
```

the following simple time-optimal bang-bang problem is chosen for constructing two HBVPs as starting problems.

Minimize:

$$J = t_f,$$

such that:

$$\dot{x_1} = x_2,$$
$$\dot{x_2} = u.$$

Also, the constraints on the control are

$$-1 \leq u \leq 1.$$

We consider the following two cases for initial and terminal state constraints to construct two HBVPs with both have one switching point, but have different structure in optimal control solution.

$$\textbf{case I}: \mathbf{x}(0) = [\alpha, \beta]^T, \text{and } \mathbf{x}(t_f) = [0, 0]^T, \tag{5.2}$$

$$\textbf{case II}: \mathbf{x}(0) = [0, 0]^T, \text{and } \mathbf{x}(t_f) = [\gamma, 0]^T. \tag{5.3}$$

Obviously, from Eqs. (3.1)-(3.3), the dynamical equations corresponding to this simple bang-bang problem are described by

$$dx_1/dt = x_2, \tag{5.4}$$

$$dx_2/dt = u, \tag{5.5}$$

$$d\lambda_1/dt = 0, \tag{5.6}$$

$$d\lambda_2/dt = -\lambda_1, \tag{5.7}$$

in which, with the initial and terminal state constraints, consequently, lead to two analytical solutions given by [7]

case I:

$$x_1^*(t) = \begin{cases} -0.5t^2 + \beta t + \alpha, & t \le t_1, \\ 0.5t^2 - t_f t + 0.5t_f^2, & t > t_1, \end{cases}$$

$$x_2^*(t) = \begin{cases} \beta - t, & t \le t_1, \\ t - t_f, & t > t_1, \end{cases}$$

$$\lambda_1^*(t) = c_1,$$

$$\lambda_2^*(t) = -c_1 t + c_2,$$

$$u^*(t) = \begin{cases} -1, & t \le t_1, \\ +1, & t > t_1, \end{cases}$$

where, $t_1 = \beta + \sqrt{\alpha + 0.5\beta^2}$, $t_f = 2t_1 - \beta$, $c_1 = 1/(t_f - t_1)$, and $c_2 = c_1 t_1$.

case II:

$$x_1^*(t) = \begin{cases} 0.5t^2, & t \le t_1, \\ -0.5(t - t_f)^2 + \gamma, & t > t_1, \end{cases}$$

$$x_2^*(t) = \begin{cases} t, & t \le t_1, \\ t_f - t, & t > t_1, \end{cases}$$

$$\lambda_1^*(t) = d_1,$$

$$\lambda_2^*(t) = -d_1 t + d_2,$$

$$u^*(t) = \begin{cases} +1, & t \le t_1, \\ -1, & t > t_1, \end{cases}$$

where, $t_1 = \sqrt{\gamma}$, $t_f = 2\sqrt{\gamma}$, $d_1 = -1/(t_f - t_1)$, and $d_2 = d_1 t_1$. We use these HBVPs as starting problems in the next section.

6. Illustrative examples

In this section, two examples are given to demonstrate the applicability and accuracy of our method. For these examples, we solve the system of ODEs in Eqs. (4.3) by MATLAB function `ode45` and solve the NLE in Eq. (5.1) by MATLAB function `fsolve`. In addition, all computations are performed on a 2.53 GHz Core i5 PC Laptop with 4 GB of RAM running in MATLAB R2011a.

6.1. **Example 1: Optimal control of the Rayleigh problem.** The Rayleigh problem is concerned with electric circuits which given in [21]. This is a free terminal-time problem, where the system is described by

$$\dot{x}_1 = x_2, \tag{6.1}$$
$$\dot{x}_2 = -x_1 + (1.4 - 0.14x_2^2)x_2 + 4u, \tag{6.2}$$

with the boundary conditions

$$\mathbf{x}(0) = [-5, -5]^T, \tag{6.3}$$
$$\mathbf{x}(t_f) = [0, 0]^T. \tag{6.4}$$

The constraints on the control are

$$-1 \leq u \leq 1,$$

and, the objective is to determine a control u that minimizes

$$J = \int_0^{t_f} 1 = t_f.$$

Now, the Hamiltonian in (3.1) is given by

$$\mathscr{H}(\mathbf{x}, \lambda, u) = [1 + \lambda_1 x_2 + \lambda_2(-x_1 + (1.4 - 0.14x_2^2)x_2 + 4u)].$$

Thus, the costate equations (3.3) are given by

$$d\lambda_1/dt = \lambda_2, \tag{6.5}$$
$$d\lambda_2/dt = -(\lambda_1 + \lambda_2(1.4 - 0.42x_2^2)). \tag{6.6}$$

The switching function $\sigma(t) = 4\lambda_2(t)$ determines the optimal control, according to the control law (3.4), as follows

$$u(t) = \begin{cases} -1, & \text{if } \lambda_2(t) > 0, \\ +1, & \text{if } \lambda_2(t) < 0, \\ \text{undefined}, & \text{if } \lambda_2(t) = 0. \end{cases}$$

Now, suppose that the optimal control has one switching point, which is led to the following parameterization for optimal control

$$u(t) = \begin{cases} +b_0, & 0 \leq t \leq s_1, \\ -b_0, & s_1 < t \leq t_f, \end{cases} \tag{6.7}$$

where, $b_0 = -\text{sign}(\lambda_2(0))$. Hence, we have to impose the following switching condition

$$\sigma(s_1) = \lambda_2(s_1) = 0. \tag{6.8}$$

Also, the transversality condition (3.9) yields

$$0 = \mathscr{H}(\mathbf{x}(t_f), \lambda(t_f), u(t_f)) = 1 + 4\lambda_2(t_f)u(t_f). \tag{6.9}$$

After, replacing the control function (6.7), in Eq. (6.2), the Eqs. (6.1)-(6.6) form the desirable HBVP, with the switching and transversality conditions in Eqs. (6.8) and (6.9) respectively.

TABLE 1. (Rayleigh problem): The trace of Algorithm 1 with $\Delta\mu = 0.1$, with the starting HBVP in Eqs. (5.4)-(5.7) and the initial and terminal state constraints (5.2).

Step No.	μ	Convergence Status	Stepsize ($\Delta\mu$)
0	0	–	0.1
1	0.1	conv.	0.1
2	0.2	conv.	0.1
3	0.3	div.	–

TABLE 2. (Rayleigh problem): The trace of Algorithm 1 with $\Delta\mu = 0.25$, with the starting HBVP in Eqs. (5.4)-(5.7) and the initial and terminal state constraints (5.3).

Step No.	μ	Convergence Status	Stepsize ($\Delta\mu$)	CPU Times (s)
0	0	–	0.25	–
1	0.25	conv.	0.25	0.303599
2	0.5	conv.	0.25	0.235552
3	0.75	conv.	0.25	0.258028
4	1	div.	0.25	0.973023
5	1	conv.	0.25	1.552983

So, the shooting function is constructed as

$$\mathbf{S}(\mathbf{z}) = \begin{pmatrix} \mathbf{x}(t_f) \\ 1 + 4\lambda_2(t_f)u(t_f) \\ \lambda_2(s_1) \\ b_0^2 - 1 \end{pmatrix},$$

in which, the unknown vector $\mathbf{z}$ is considered as

$$\mathbf{z} = \Big(s_1, t_f, b_0, \lambda_1(0), \lambda_2(0)\Big)^T.$$

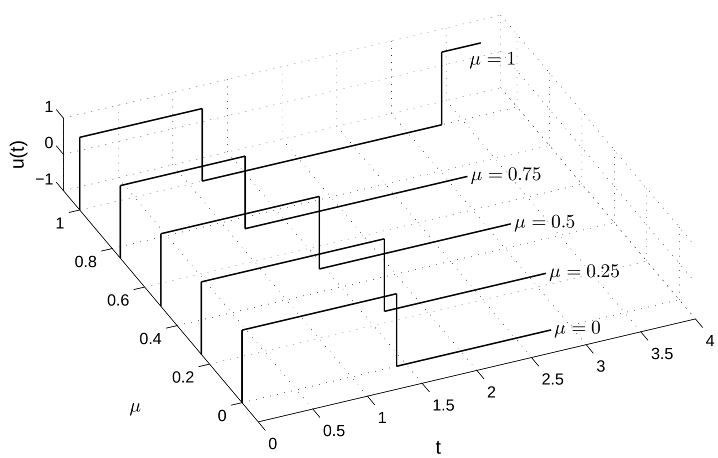
It is sufficient to consider an appropriate starting problem, which has the same dimension with the desired HBVP (6.1)-(6.6). Because, we do not have any knowledge about the structure of the optimal control and the number of switching points in this problem, we force to pick one of the starting HBVPs adventitiously. Here, at first, we solve this problem by using the proposed combined method with the starting HBVP in Eqs. (5.4)-(5.7) and the initial and terminal state constraints (5.2), where $\alpha = 1$ and $\beta = 3$. The results are summarized in Table 1. It is seen that, for $\mu = 0.3$, the Algorithm fails to converge to any solution and adding a switching point does not cause the convergence.

So, we change the starting problem. In particular, we choose the starting HBVP in Eqs. (5.4)-(5.7), but this time, with the initial and terminal state constraints (5.3), where $\gamma = 2$. The results are summarized in Table 2, along the computational time of homotopy steps. It is seen that, for $\mu = 1$ in step 4, the Algorithm fails to converge. However, adding a switching point in step 5, causes the convergence occurs. To show the accuracy and convergence of the method, the values of the switching and final times, and initial values of the costate variables are reported in Table 3, for some values of `fsolve` parameters `TolFun`, and `TolX`, together with two parameters of `ode45`, `RelTol` and `AbsTol`. Furthermore, control functions are

TABLE 3. (Rayleigh problem): The calculated values of switching and final times, and initial values of the costate variables for various values of fsolve and ode45 parameters.

TolFun	TolX	RelTol	AbsTol	s_1	s_2	t_f	$\lambda_1(0)$	$\lambda_2(0)$
1.0e-03	1.0e-03	1.0e-03	1.0e-03	1.12082177450	3.30998272240	3.66835975840	-0.12217138137	-0.08254042347
1.0e-06	1.0e-06	1.0e-06	1.0e-06	1.12050697317	3.31004714686	3.66817379568	-0.12234098070	-0.08265135354
1.0e-11	1.0e-11	1.0e-11	1.0e-11	1.12050659193	3.31004698385	3.66817338862	-0.12234128405	-0.08265161129
1.0e-13	1.0e-13	1.0e-13	1.0e-13	1.12050659193	3.31004698385	3.66817338862	-0.12234128406	-0.08265161129

FIGURE 1. (Rayleigh problem): Controls history obtained by using the proposed combined method.



shown in Figure 1. It is seen that, the number of switching points in the starting problem is one, whereas in the desired problem (in optimal control), we have two switching points.

6.2. **Example 2: Optimal control of a Van der Pol oscillator.** Consider the time-optimal control of a Van der Pol oscillator which has been treated by several papers [10]. The control problem is to minimize

$$J = \int_0^{t_f} 1 = t_f,$$

subject to

$$\dot{x}_1 = x_2, \tag{6.10}$$

$$\dot{x}_2 = -x_1 - (x_1^2 - 1)x_2 + u, \tag{6.11}$$

together with the following constraints on the control

$$-1 \le u \le 1,$$

and the boundary conditions

$$\mathbf{x}(0) = [-0.4, 0.6]^T, \tag{6.12}$$

$$\mathbf{x}(t_f) = [0.6, 0.4]^T. \tag{6.13}$$

Here, the Hamiltonian in (3.1) is given by

$$\mathscr{H}(\mathbf{x}, \lambda, u) = [1 + \lambda_1 x_2 + \lambda_2(-x_1 - (x_1^2 - 1)x_2 + u)].$$

Thus, the costate equations (3.3) are given by

$$d\lambda_1/dt = 2x_1 x_2 \lambda_2 + \lambda_2, \tag{6.14}$$

$$d\lambda_2/dt = -\lambda_1 - \lambda_2 + \lambda_2 x_1^2. \tag{6.15}$$

Now, suppose that the optimal control has one switching point, which is led to the following parameterization for optimal control

$$u(t) = \begin{cases} +b_0, & 0 \le t \le s_1, \\ -b_0, & s_1 < t \le t_f, \end{cases} \tag{6.16}$$

where, $b_0 = -\text{sign}(\lambda_2(0))$. After replacing the control function (6.16) in Eq. (6.11) and with the Eqs. (6.10), (6.14) and (6.15), we obtain the desired HBVP with the boundary conditions (6.12) and (6.13). Finally, with transversality and switching conditions, the shooting function is constructed as

$$\mathbf{S}(\mathbf{z}) = \begin{pmatrix} \mathbf{x}(t_f) - [0.6, 0.4]^T \\ 1 + 0.4\lambda_1(t_f) + \lambda_2(t_f)(-0.344 + u(t_f)) \\ \lambda_2(s_1) \\ b_0^2 - 1 \end{pmatrix},$$

in which, the unknown vector $\mathbf{z}$ is considered as

$$\mathbf{z} = \Big(s_1, t_f, b_0, \lambda_1(0), \lambda_2(0)\Big)^T.$$

Before applying the proposed combined method, and in order to illustrate the high sensitivity to initial guesses for unknown parameters, we apply the method without using the homotopic approach on the desired HBVP. The convergence and divergence regions of the method is shown in Figure 2. In this Figure, these regions are plotted for various values of initial guess $\mathbf{s} = [s_1, t_f]$, when, b_0, $\lambda_1(0)$ and $\lambda_2(0)$ are set to 1, -1.08 and -0.18, respectively. It is seen that, even the initial values for b_0, $\lambda_1(0)$ and $\lambda_2(0)$ are proximately considered, but if the initial guess for $\mathbf{s} = [s_1, t_f]$ is far from the optimal solution then the method, diverges quickly.

Now, this problem is solved by using the proposed combined method with the starting HBVP in Eqs. (5.4)-(5.7) and the initial and terminal state constraints (5.3), where $\gamma = 2$. In Figures 3 and 4, the obtained states and costates history are shown. Also, the corresponding control functions are plotted in Figure 5. In Table 4, the values of the switching and final times, and initial values of the costate variables are reported.

FIGURE 2. (Van der Pol oscillator): Convergence and divergence region of the method for various values of initial guess $\mathbf{s} = [s_1, t_f]$.

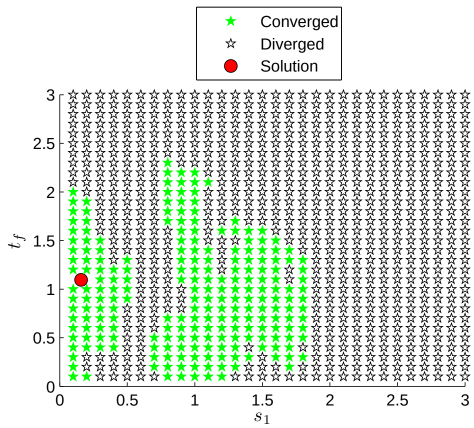
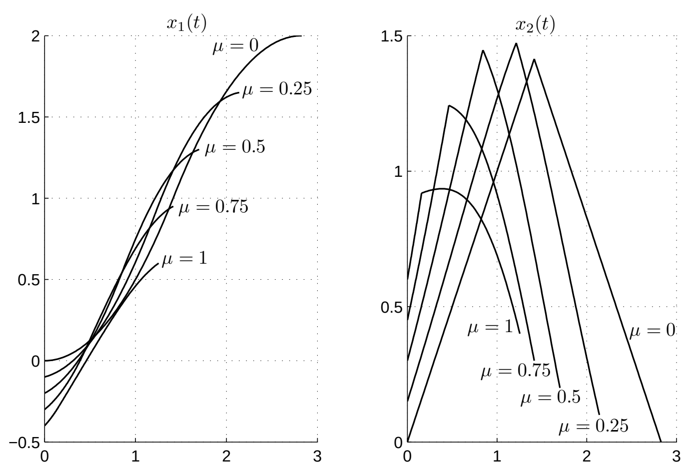


FIGURE 3. (Van der Pol oscillator): States history obtained by using the proposed combined method.



7. CONCLUSION

In the present work, the combined method was utilized in the indirect scheme for numerical solution of bang-bang optimal control problems. The present method satisfies the

FIGURE 4. (Van der Pol oscillator): Costates history obtained by using the proposed combined method.

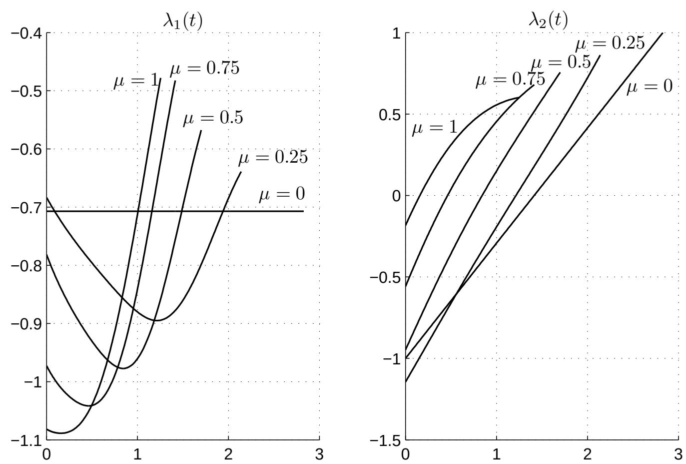
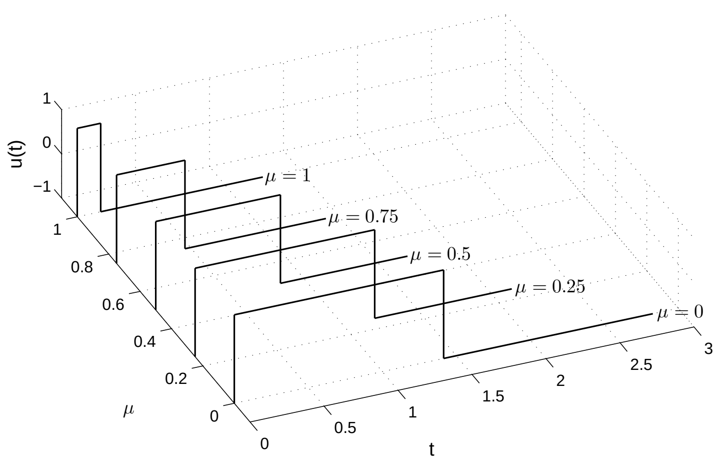


FIGURE 5. (Van der Pol oscillator): Control functions obtained by using the proposed combined method.



first order necessary conditions of optimality and provides an accurate solution of bang-bang problems. As well, it can capture the switching points accurately. Furthermore, the method does not require a priori knowledge about the optimal control solution. Illustrative examples were given to demonstrate the validity and applicability of the proposed method.

TABLE 4. (Van der Pol oscillator): The calculated values of switching and final times, and initial values of the costate variables for various values of fsolve and ode45 parameters.

TolFun	TolX	RelTol	AbsTol	s_1	t_f	$\lambda_1(0)$	$\lambda_2(0)$
1.0e-03	1.0e-03	1.0e-03	1.0e-03	0.15831409418	1.25407760021	-1.08160157382	-0.18434916499
1.0e-06	1.0e-06	1.0e-06	1.0e-06	0.15832013567	1.25407473151	-1.08160561560	-0.18436797416
1.0e-11	1.0e-11	1.0e-11	1.0e-11	0.15832014228	1.25407472933	-1.08160560669	-0.18436798108
1.0e-13	1.0e-13	1.0e-13	1.0e-13	0.15832014228	1.25407472933	-1.08160560669	-0.18436798108

REFERENCES

[1] E. L. Allgower and K. Georg, *Introduction to numerical continuation methods*, volume 45 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2003.

[2] R. Bertrand and R. Epenoy, *New smoothing techniques for solving bang-bang optimal control problems - Numerical results and statistical interpretation*, Optim. Control Appl. Methods, *23*(4) (2002), 171–197.

[3] J. T. Betts, *Survey of numerical methods for trajectory optimization*, J. Guid. Control. Dyn., *21*(2) (1998), 193–207.

[4] J. T. Betts, *Practical Methods for Optimal Control and Estimation Using Nonlinear Programming*, Cambridge University Press, New York, NY, USA, 2nd edition, 2009.

[5] P. Binfeng, L. Ping, P. Xun, and M. Yangyang, *Double-Homotopy Method for Solving Optimal Control Problems*, J. Guid. Control. Dyn., *39*(8) (2016), 1706–1720.

[6] A. E. Bryson and Y. -C Ho, *Applied optimal control: Optimization, estimation, and control*, Hemisphere Publishing Corp. Washington, D. C., 1975.

[7] D. N. Burghes and A. Graham, *Control and Optimal Control Theories With Applications*, Engineering science. Woodhead Publishing, 2004.

[8] J. -B. Caillau, B. Daoud, and J. Gergaud, *Minimum fuel control of the planar circular restricted three-body problem*, Celest. Mech. Dyn. Astron., *114*(1-2) (2012), 137–150.

[9] J. Chen and M. Gerdts, *Smoothig technique of nonsmooth newton methods for control-state constrained optimal control problems*, SIAM J. Numer. Anal., *50*(4) (2012), 1982–2011.

[10] E. Cristiani and P. Martinon, *Initialization of the Shooting Method via the Hamilton-Jacobi-Bellman Approach*, J. Optim. Theory Appl., *146*(2) (2010), 321–346.

[11] Z. Foroozandeh, M. Shamsi, and M. d. R. de Pinho, *A mixed-binary non-linear programming approach for the numerical solution of a family of singular optimal control problems*, Int. J. Control, *92*(7) (2019), 1551–1566.

[12] Z. Foroozandeh, M. Shamsi, and M. d. R. de Pinho, *A Hybrid Direct-Indirect Approach for Solving the Singular Optimal Control Problems of Finite and Infinite Order*, Iran. J. Sci. Technol. Trans. A Sci., *42*(3) (2018), 1545–1554.

[13] J. Gergaud and T. Haberkorn, *Homotopy method for minimum consumption orbit transfer problem*, ESAIM - Control. Optim. Calc. Var., *12*(2) (2006), 294–310.

[14] T. Haberkorn, P. Martinon, and J. Gergaud, *Low-thrust minimum-fuel orbital transfer: A homotopic approach*, J. Guid. Control. Dyn., *27*(6) (2004), 1046–1060.

[15] D. E. Kirk, *Optimal Control Theory*, Prentice-Hall, Englewood Cliffs, NJ, 1970.

[16] U. Ledzewicz and H. Schättler, *Analysis of a cell-cycle specific model for cancer chemotherapy*, J. Biol. Syst., *10*(3) (2002), 183–206.

[17] H. W. J. Lee, K. L. Teo, V. Rehbock, and L. S. Jennings, *Control parametrization enhancing technique for optimal discrete-valued control problems*, Automatica, *35*(8) (1999), 1401–1407.

[18] P. Martinon and J. Gergaud, *Using switching detection and variational equations for the shooting method*, Optim. Control Appl. Methods, *28*(2) (2007), 95–116.

[19] M. A. Mehrpouya and S. Fallahi, *A modified control parametrization method for the numerical solution of bang-bang optimal control problems*, JVC/Journal Vib. Control, *21*(12) (2015), 2407–2415.

[20] M. A. Mehrpouya, M. Shamsi, and M. Razzaghi, *A combined adaptive control parametrization and homotopy continuation technique for the numerical solution of bang-bang optimal control problems*, ANZIAM J., *56*(1) (2014), 48–65.

[21] M. R. Navabi, M. Shamsi, and M. Dehghan, *Numerical solution of the controlled Rayleigh nonlinear oscillator by the direct spectral method*, JVC/Journal Vib. Control, *14*(6) (2008), 795–806.

[22] H. J. Oberle and W. Grimm, *BNDSCO -A program for the numerical solution of optimal control problems*, Technical Report 515-89/22, Institute for Flight Systems Dynamics, DLR, Oberpfaffenhofen, Germany, 1989.

[23] M. R. Osborne, *On shooting methods for boundary value problems*, J. Math. Anal. Appl., *27*(2) (1969), 417–433.

[24] M. Sbarciog, M. Loccufier, and E. Noldus, *Optimality and stability in a class of bang-bang controlled biochemical reaction systems*, Int. J. Control, *81*(5) (2008), 836–850.

[25] M. Shamsi, *A modified pseudospectral scheme for accurate solution of Bang-Bang optimal control problems*, Optim. Control Appl. Methods, *32*(6) (2011), 668–680.

[26] C. Silva and E. Trélat, *Smooth regularization of bang-bang optimal control problems*, IEEE Trans. Automat. Contr., *55*(11) (2010), 2488–2499.

[27] D. Stefanatos, J. Ruths, and J.-S. Li, *Frictionless atom cooling in harmonic traps: A time-optimal approach*, Phys. Rev. A - At. Mol. Opt. Phys., *82*(6) (2010), 063422-1–063422-8.

[28] J. Stoer and R. Bulirsch, *Introduction to numerical analysis*, volume 12 of Texts in Applied Mathematics, Springer-Verlag, New York, third edition, 2002.

[29] K. L. Teo, H. W. J. Lee, and V. Rehbock, *Control parametrization enhancing technique for time optimal control and optimal three-valued control problems*, Dyn. Contin. Discret. Impuls. Syst. Ser. B Appl. Algorithm, *4*(4) (1998), 617–631.

[30] E. Trélat, *Optimal Control and Applications to Aerospace: Some Results and Challenges*, J. Optim. Theory Appl., *154*(3) (2012), 713–758.

[31] C. Yu, B. Li, R. Loxton, and K. L. Teo, *Optimal discrete-valued control computation*, J. Glob. Optim., *56*(2) (2013), 503–518.