A New Error Evaluation for Singularly Perturbed Problem with Multi-Point Boundary Condition

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Abstract
We consider finite difference method to find best approximation of nonlinear singularly perturbed problem which contains multi-point boundary conditions. We surveyed the asymptotic estimates of the corresponding problem that needs to be solved with maximum principle. We constructed a finite difference scheme by using Bakhvalov mesh. Based on the error estimation, we proved that this method was first-order, uniformly convergent method with the discrete maximum norm. Finally, we conducted a numerical experiment in order to check the theoretical results.

Keywords. Singular perturbation, finite difference scheme, Bakhvalov mesh, uniformly convergence, multi-point boundary conditions.

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1. Introduction

The present study focuses the following nonlinear singularly perturbed problem that with multi-point boundary conditions:

\[-\varepsilon u''(x) + f(x, u) = 0, \quad 0 < x < 1,\]

\[u'(0) = \frac{A}{\varepsilon},\]  \hspace{1cm} (1.1)

\[u(1) - \varphi(u(l_1)) = 0, \quad 0 < l_1 < 1,\]  \hspace{1cm} (1.2)

where \(0 < \varepsilon << 1\) is a small perturbation parameter. \(A\) is given constant, and \(\varphi(u)\) is sufficiently smooth on \([0, 1]\). Furthermore

\[\left|\frac{\partial \varphi}{\partial u}\right| \leq m < 1,\]  \hspace{1cm} (1.3)

\[\frac{\partial f(x, u)}{\partial u} \geq \alpha > 0,\]  \hspace{1cm} (1.4)

and the solution, \(u(x)\), for the problem (1.1)-(1.3) has boundary layers near \(x = 0\) and \(x = 1\) points.
A singularly perturbed equation contains a very small parameter $\varepsilon$ which is multiplied by the highest derivative. Certain classical numerical methods lead to difficulties in solving such problems due to the small parameter $\varepsilon$. Therefore, it is considered appropriate to apply numerical methods such as the finite difference method, the finite element method, etc.

Singularly perturbed equations with a multi-point boundary are widely applied in fluid mechanics and other branches of applied mathematics [1, 10, 12, 13, 25, 29]. Examples include human pupil light reflex, problems in neurobiology, modeling physiological processes and diseases, optimal control theory, optically bistable devices, signal transmission, Reynold's number in fluid dynamics, heat transfer problem, hydrodynamics, transonic gas dynamics, chemical-reactor theory, control theory, oceanography, fluid mechanics, quantum mechanics, hydro mechanical problems, meteorology, electrical networks and other physical models. For further information, please refer to [9, 16, 21, 29, 31]. In recent years, various researchers developed existence-uniqueness solutions and the different numerical methods for the solution of these problems [4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 18, 19, 21, 22, 23, 25, 26, 27, 28, 29, 32, 33].

The aim of the present paper was to propose a new perspective using Bakhvalov mesh in the finite difference approach for numerical solution nonlinear singularly perturbed problem with multi-point boundary conditions. In particular, we know that it is hard to solve singularly perturbed problems exactly. Therefore, appropriate approximate and numerical methods should be used such as finite difference method, finite elements method. Also, many kinds of equations can be solved easily by finite difference method [2, 3, 20, 24, 30, 34]. The paper was constructed as follows: The Section 2 of the present study provides the asymptotic estimates of the solution with respect to the numerical solution analyses included in later sections. In Section 3, finite difference scheme was constructed with integral identities and quadrature rules. Furthermore, the remainder terms pertained the integral form. In section 4, the error analysis was provided to demonstrate the uniformly convergent first-order on Bakhvalov mesh. Finally, the numerical experiment is presented to support the theory.

In the following sections, $C$ and $C_0$ were used to refer to the positive constants independent of $\varepsilon$ and the mesh parameter, respectively.

2. Asymptotic behaviour estimates of the exact solution

In this section, a priori estimates for the exact solution of the problem (1.1)-(1.3) were provided for the analysis of the numerical solution in the next section.

**Lemma 2.1.** If $f(x, u) \in C^1[0,1]$ and $\frac{\partial f(x, u)}{\partial u} \geq \alpha > 0$, then, the solution of the problem (1.1)-(1.3) fulfills the following constraints:

$$\|u(x)\| \leq C_0, \quad (2.1)$$

where,

$$C_0 = \beta^{-1}(1+m(1-m)^{-1}|A|+(1-m)^{-1}|B|+\alpha^{-1}m(1-m)^{-1}\|f\|_\infty),$$

$$B = \varphi(0), \quad \beta > 0, \quad |\gamma| \leq m < 1,$$
and
\[ |u'(x)| \leq C \left\{ 1 + \frac{1}{\sqrt{\varepsilon}} \left[ \exp\left( -\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}} \right) + \exp\left( -\frac{\sqrt{\alpha}(1-x)}{\sqrt{\varepsilon}} \right) \right] \right\}. \tag{2.2} \]

**Proof.** Using the mean value theorem for nonlinear function \( f(x, u) \) in Eq. (1.1), we acquired,
\[ \frac{f(x, u) - f(x, 0)}{u - 0} = \partial f/\partial u(x, u_*) \quad u_* = \gamma u, \quad 0 < \gamma < 1, \]
and,
\[ F(x) = -f(x, 0), \quad a(x) = \partial f/\partial u(x, u_*). \]
Hence, the following linear equation was obtained,
\[ -\varepsilon u''(x) + a(x)u(x) = F(x), \tag{2.3} \]
where \( a(x) \geq \alpha > 0 \) and \( F(x) \) are sufficiently smooth functions.

Here in, it was also possible to use the mean value theorem for Equation (3.1) and the following form was obtained:
\[ \gamma = \frac{d\varphi}{du}(\eta u(l_1)), \quad 0 < \eta < 1, \quad u(1) - \gamma u(l_1) = B. \tag{2.4} \]

After using the maximum principle for the evaluation of the function \( u(x) \) in Eq. (2.3), we obtained,
\[ |u(x)| \leq \beta^{-1}|A| + |u(1)| + \alpha^{-1}\|F\|_\infty. \tag{2.5} \]
From boundary condition (2.4), we have
\[ |u(1)| \leq |B| + m|u(l_1)|. \tag{2.6} \]
By setting \( x = l_1 \) in (2.5), we obtain
\[ |u(l_1)| \leq \beta^{-1}|A| + |u(1)| + \alpha^{-1}\|F\|_\infty. \tag{2.7} \]
By substituting into (2.6), (2.7), we have
\[ |u(1)| \leq |B| + m(\beta^{-1}|A| + |u(1)| + \alpha^{-1}\|F\|_\infty), \tag{2.8} \]
the estimation (2.8) is written in Eq. (2.5) and so, we have
\[ |u(x)| \leq \beta^{-1}(1 + m(1 - m)^{-1})|A| + (1 - m)^{-1}|B| + \alpha^{-1}m(1 - m)^{-1}\|f\|_\infty, \]
where,
\[ C_0 = \beta^{-1}(1 + m(1 - m)^{-1})|A| + (1 - m)^{-1}|B| + \alpha^{-1}m(1 - m)^{-1}\|f\|_\infty, \]
which proves the inequality (2.1).
The proof of inequality (2.2) is almost identical to that of [13, 15] \( \Box \)
3. Difference approximation of the problem

Bakhvalov mesh was defined in the present section and the problem (1.1)-(1.3) was discretized with the finite difference method on Bakhvalov type mesh.

Notations: It is essential to remind that the following finite difference for any mesh function $g_i = g(x_i)$ was given on $\omega_N$ as,

\[
g_{x,i} = \frac{g_{i+1} - g_i}{h_i}, \
g_{x,i} = \frac{g_i - g_{i-1}}{h_i}, \
g_{x,i} = \frac{g_{x,i} + g_{x,i+1}}{2},
\]

\[
g_{x,i} = \frac{g_{x,i} - g_{x,i-1}}{2}, \quad h_i = \frac{h_i + h_{i+1}}{2},
\]

\[
\|g\|_\infty = \|g\|_{\infty, \omega_N} := \max_{0 \leq i \leq N} |g_i|.
\]

3.1. Bakhvalov mesh. The difference scheme should be $\varepsilon$- uniform; therefore, Bakhvalov mesh was used for the problem (1.1)-(1.3). For a positive integer $N$, the interval $[0, 1]$ was divided into three subintervals $[0, \sigma]$, $[\sigma, 1 - \sigma]$ and $[1 - \sigma, 1]$. In practice, $\sigma << 1$ was usually employed to make the mesh fine on the intervals $[0, \sigma]$ and $[1 - \sigma, 1]$ and coarse on the interval $[\sigma, 1 - \sigma]$. Here $\sigma$ depicts the transition point and could be written as follows:

\[
\sigma = \min \left\{ \frac{1}{4} - (\sqrt{\alpha})^{-1} \varepsilon \ln \varepsilon \right\}.
\]

A set of the mesh points $\bar{x}_N = \{x_i\}_{i=0}^{N}$ were introduced,

\[
x_i = \begin{cases} 
-\sqrt{\alpha}, & x_i \in [0, \sigma], \quad i = 0, \ldots, \frac{N}{4}, \\
-\sqrt{\alpha} \varepsilon \ln \left[1 - \left(1 - e^{-\frac{\alpha}{N}}\right)\frac{4}{N}\right], & x_i \in [0, \sigma], \quad i = 0, \ldots, \frac{N}{4}, \\
\sigma + (i - \frac{N}{4})h^{(1)}, & x_i \in [\sigma, 1 - \sigma], \quad i = \frac{N}{4} + 1, \ldots, \frac{3N}{4}, \\
1 - \sigma - \sqrt{\alpha} \varepsilon \ln \left[1 - \left(1 - e^{-\frac{\alpha}{N}}\right)\frac{4(i - \frac{N}{4})}{N}\right], & x_i \in [1 - \sigma, 1], \quad i = \frac{3N}{4} + 1, \ldots, \frac{5N}{4}, \\
1 - \sigma = \frac{4}{2}, & x_i \in [1 - \sigma, 1], \quad i = \frac{5N}{4} + 1, \ldots, N.
\end{cases}
\]

The step-size was set as $h_i = x_i - x_{i-1}$ for $i = 1, 2, \ldots, N$.

3.2. Construction of the difference scheme on Bakhvalov mesh. Non-uniform mesh on the interval $[0, 1]$ was denoted by $\omega_N$ in the following equations,

\[
\omega_N = \{0 < x_1 < x_2 < \ldots < x_{N-1} < 1\},
\]

and

\[
\bar{\omega}_N = \omega_N \cup \{x_0 = 0, x_N = 1\}.
\]
Eq. (1.1) was integrated over \((x_{i-1}, x_{i+1})\) to construct the difference scheme for this equation as follows,

\[
\int_{x_{i-1}}^{x_{i+1}} -\varepsilon u''(x)\varphi_i(x) \, dx + \int_{x_{i-1}}^{x_{i+1}} f(x, u)\varphi_i(x) \, dx = 0, \quad i = 1, ..., N - 1,
\]

(3.1)

here \(\{\varphi_i(x)\}_{i=1}^{N-1}\) represent the basis functions,

\[
\varphi_i(x) = \begin{cases} 
\varphi_i^{(1)}(x) = \frac{x - x_{i-1}}{h_i}, & x_{i-1} < x < x_i, \\
\varphi_i^{(2)}(x) = \frac{x_{i+1} - x}{h_{i+1}}, & x_i < x < x_{i+1}, \\
0, & x \notin (x_{i-1}, x_{i+1})
\end{cases}
\]

and the solution of the problem for \(\varphi_i^{(1)}(x)\) and \(\varphi_i^{(2)}(x)\) were determined respectively as,

\[
-\varepsilon \varphi_i^{(1)}'' = 0, \quad x_{i-1} < x < x_i,
\]

\[
\varphi_i(x_{i-1}) = 0, \quad \varphi_i(x_i) = 1,
\]

\[
-\varepsilon \varphi_i^{(2)}'' = 0, \quad x_i < x < x_{i+1},
\]

\[
\varphi_i(x_i) = 1, \quad \varphi_i(x_{i+1}) = 0.
\]

Due to several calculations in Eq. (3.1), using the interpolating quadrature rules from [4] with weight functions \(\varphi_i(x)\) on subintervals \((x_{i-1}, x_{i+1})\), it was possible to obtain,

\[
\int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi_i'(x) \, dx + \int_{x_{i-1}}^{x_{i+1}} f(x, u)\varphi_i(x) \, dx = 0,
\]

(3.2)

and

\[
\int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi_i'(x) \, dx + f(x_i, u_i) + R_i = 0,
\]

(3.3)

where \(R_i\) was the local truncation error and it was in the following form

\[
R_i = h_i^{-1} \int_{x_{i-1}}^{x_{i+1}} dx \varphi_i(x) \int_{x_{i-1}}^{x_{i+1}} \frac{d}{dx} f(x, u(\xi)) K^*_0(x, \xi) d\xi.
\]

(3.4)

In order to obtain the approximation for the boundary condition (1.2), we integrated (1.1) over \((x_0, x_1)\)

\[
\int_{x_0}^{x_1} -\varepsilon u''(x)\varphi_0(x) \, dx + \int_{x_0}^{x_1} f(x_0, u_0)\varphi_0(x) \, dx = 0, \quad i = 1, ..., N - 1,
\]

(3.5)
and
\[ \varepsilon u'(x_0) + \int_{x_0}^{x_1} [u'(x)\varphi'_0(x) + f(x_0, u_0)\varphi_0(x)] \, dx = 0. \] (3.6)

Due to several arrangements in Eq. (3.6), we obtained,
\[ \varepsilon u'(x_0) - \varepsilon u_{x,0} + \theta_0 f(x_0, u_0) + r_0 = 0, \] (3.7)
with the reminder term
\[ r_0 = \int_{x_0}^{x_1} dx \varphi_0(x) \int_{x_0}^{x_1} \frac{d}{dx} f(x, u(x)) K_0(x, \xi) d\xi. \] (3.8)

The approximation for the boundary condition (1.3) was obtained as
\[ u_N - \varphi(u_{N_0}) + r_1 = 0, \] (3.9)
with \( r_1 \) reminder term as follows
\[ r_1 = [u(l_1) - u(x_{N_0})] \varphi'(_1), \] (3.10)
where \( \xi \) was the intermediate point between \( u(x_{N_0}) \) and \( u(x_{l_1}) \).

The following difference scheme for the problem (1.1)-(1.3) was proposed via neglecting \( R_i, r_0 \) and \( r_1 \) in Eqs. (3.4), (3.8) and (3.10), respectively.
\[ -\varepsilon z_{x,i} + f(x_i y_i) = 0, \quad i = 1, \ldots, N - 1, \] (4.1)
\[ \varepsilon y_{x,0} - A - \theta_0 f(x_0, y_0) = 0, \] (4.2)
\[ y_N - \varphi(y_{N_0}) = 0. \] (4.3)

4. CONVERGENCE ANALYSIS

The present section provides the convergence analysis of the method for the problem (1.1)-(1.3). Given the error function \( z_i = y_i - u_i, \quad i = 0, 1, \ldots, N \) and \( z_i \) as the solution of the discrete problem, it was possible to write down the following equations:
\[ -\varepsilon z_{x,i} + [f(x_i, y_i) - f(x_i, u_i)] = R_i, \quad i = 1, \ldots, N - 1, \] (4.1)
\[ \varepsilon z_{x,0} - \theta_0 [f(x_0, y_0) - f(x_0, u_0)] = r_0, \] (4.2)
\[ z_N - [\varphi(y_{N_0}) - \varphi(u_{N_0})] = r_1. \] (4.3)

**Lemma 4.1.** The following estimate was valid for the solution of \( z_i \) in the discrete problem (4.1)-(4.3)
\[ \|z\|_{\infty, \omega_N} \leq C \{ \|R\|_{\infty, \omega_N} + |r_0| + |r_1| \}, \] (4.4)
holds.
Proof. The discrete problem (4.1)-(4.3) could be rewritten as

\[-\varepsilon z_{x,i} + \bar{a}_i z_i = R_i, \quad i = 1, \ldots, N - 1, \quad (4.5)\]

\[\varepsilon z_{x,0} - \theta_0 \tilde{b} z_0 = r_0, \quad (4.6)\]

\[z_N - \bar{\gamma} z_{N_0} = r_1, \quad (4.7)\]

where

\[\bar{a}_i = \frac{\partial f}{\partial u}(x_i, \bar{y}_i), \quad \bar{b} = \frac{\partial f}{\partial u}(x_0, \bar{y}_0), \quad \bar{\gamma} = \varphi'(\bar{y}_{N_0}), \]

\[\bar{y}_i, \quad \bar{y}_0, \quad \bar{y}_{N_0} - \text{intermediate values}.\]

Based on the maximum principle for (4.5)-(4.7), we obtained,

\[|z_i| \leq \beta^{-1}|r_0| + |z_N| + \alpha^{-1}\|R\|_{\infty, \omega_N}, \quad (4.8)\]

and

\[|z_N| \leq |r_1| + k|z_{N_0}|, \quad (4.9)\]

where

\[\bar{\gamma} \leq k < 1.\]

In Eq. (4.8), \(i = N_0\) yielded,

\[|z_{N_0}| \leq \beta^{-1}|r_0| + |z_N| + \alpha^{-1}\|R\|_{\infty, \omega_N}, \quad (4.10)\]

and applying (4.8)-(4.10), it was possible to obtain,

\[|z_i| \leq \beta^{-1}|r_0| + (1 - k)^{-1}\left\{k\beta^{-1}|r_0| + |r_1| + k\alpha^{-1}\|R\|_{\infty, \omega_N}\right\}, \quad (4.11)\]

Due to several calculations in Eq. (4.11), we obtained,

\[|z_i| \leq C\left\{|R|_{\infty, \omega_N} + |r_0| + |r_1|\right\}, \quad (4.12)\]

which, along with (4.12), proved Lemma 4.1. \qed

Lemma 4.2. Based on the assumptions for Lemma 2.1 in Section 2, for the error functions \(R_i, r_0\) and \(r_1\) we obtained,

\[\|R\|_{\infty, \omega_N} \leq CN^{-1}, \quad (4.13)\]

\[r_0 \leq CN^{-1}, \quad (4.14)\]

\[r_1 \leq CN^{-1}. \quad (4.15)\]
Proof. The error functions $R_i$, $r_0$ and $r_1$ were evaluated for the intervals $[0, \sigma]$, $[\sigma, 1 - \sigma]$ and $[1 - \sigma, 1]$ on Bakhvalov mesh, respectively.

From the previous expression (3.4) for $R_i$ and (2.2), we deduced that

$$|R_i| \leq \int_{x_{i-1}}^{x_{i+1}} \varphi(x) dx \left| \frac{\partial f(\xi, u(\xi))}{\partial \xi} + \frac{\partial f}{\partial u} \frac{du(\xi)}{d\xi} \right| d\xi$$

$$\leq C \int_{x_{i-1}}^{x_{i+1}} (1 + u'(\xi)) d\xi$$

$$\leq C \left\{ \int_{x_{i-1}}^{x_{i+1}} (1 + \frac{1}{\sqrt{\epsilon}}) \left[ \exp(-\frac{\sqrt{\alpha}x_{i-1}}{\sqrt{\epsilon}}) + \exp(-\frac{\sqrt{\alpha}(1-x)}{\sqrt{\epsilon}}) \right] d\xi \right\}. \tag{4.16}$$

1) The error function $R_i$ was evaluated for $\sigma < \frac{1}{4}$ and $x_i \in [0, \sigma]$:

$$x_{i-1} = -(\sqrt{\alpha})^{-1} \epsilon \ln \left[ 1 - (1 - \epsilon) \frac{4(i-1)}{N} \right], \tag{4.17}$$

$$h_i = -(\sqrt{\alpha})^{-1} \epsilon \ln \left[ 1 - (1 - \epsilon) \frac{4i}{N} \right] + (\sqrt{\alpha})^{-1} \epsilon \ln \left[ 1 - (1 - \epsilon) \frac{4(i-1)}{N} \right]. \tag{4.18}$$

It was possible to write the following equations,

$$h_i = \frac{4(1 - \epsilon)N^{-1}}{1 - 4i(1 - \epsilon)N^{-1}} \leq CN^{-1}, \tag{4.19}$$

and

$$h_i \leq CN^{-1}. \tag{4.20}$$

Based on the mean value theorem for (4.18).

Also, a simple calculation (4.16) yielded,

$$\exp(-\frac{\sqrt{\alpha}x_{i-1}}{\sqrt{\epsilon}}) + \exp(-\frac{\sqrt{\alpha}(1-x)}{\sqrt{\epsilon}}) \leq CN^{-1}, \tag{4.21}$$

and

$$\exp(-\frac{\sqrt{\alpha}(1-x_{i+1})}{\sqrt{\epsilon}}) + \exp(-\frac{\sqrt{\alpha}(1-x_{i-1})}{\sqrt{\epsilon}}) \leq CN^{-1}. \tag{4.22}$$

Hence, (4.16), (4.21) and (4.22) led to,

$$|R_i| \leq CN^{-1}, \ i = 0, ..., \frac{N}{4}.$$  

2) The error function $R_i$ was evaluated for $\sigma = \frac{1}{4}$ and $x_i \in [0, \sigma]$:

$$x_{i-1} = -(\sqrt{\alpha})^{-1} \epsilon \ln \left[ 1 - (1 - e^{-\frac{\sqrt{\alpha}}{4}}) \frac{4(i-1)}{N} \right], \tag{4.23}$$
\[ h_i = -(\sqrt{\alpha})^{-1} \varepsilon \ln \left[ 1 - (1 - e^{\frac{-\sqrt{\alpha}}{N}})^{4i/N} \right] \]
\[ + (\sqrt{\alpha})^{-1} \varepsilon \ln \left[ 1 - (1 - e^{\frac{-\sqrt{\alpha}}{N}})^{4(i-1)/N} \right]. \]  
(4.24)

Once the mean value theorem in (4.24) was used, it was possible to obtain,
\[ h_i = (\sqrt{\alpha})^{-1} \varepsilon \frac{\ln \left[ 1 - (1 - e^{-\sqrt{\alpha}/4 \varepsilon})^{4(i-1)/N} \right]}{\ln \left[ 1 - (1 - e^{-\sqrt{\alpha}/4 \varepsilon})^{4i/N} \right]} \leq CN^{-1}. \]  
(4.25)

Therefore, based on (4.16) and (4.25), it was possible to write,
\[ |R_i| \leq CN^{-1}, \quad i = 0, ..., \frac{N}{4}. \]

3) The error function \( R_i \) was evaluated for \( x_i \in [\sigma, 1 - \sigma] \):
\[ h^{(1)} = \frac{2(1 - 2\sigma)}{N} \leq CN^{-1}, \]  
(4.26)

and applying (4.16) and (4.26), it was possible to obtain,
\[ |R_i| \leq C h^{(1)} \leq CN^{-1}, \quad i = \frac{N}{4} + 1, ..., \frac{3N}{4}. \]

4) The error function \( R_i \) was evaluated for \( x_i \in [1 - \sigma, 1] \):
\[ x_{i-1} = 1 - \sigma - (\sqrt{\alpha})^{-1} \varepsilon \ln \left[ 1 - (1 - \varepsilon)^{\frac{4(i-1)-\frac{3N}{4}}{N}} \right], \]  
(4.27)
\[ h_i = -(\sqrt{\alpha})^{-1} \varepsilon \ln \left[ 1 - (1 - \varepsilon)^{\frac{4(i-1)-\frac{3N}{4}}{N}} \right] \]
\[ + (\sqrt{\alpha})^{-1} \varepsilon \ln \left[ 1 - (1 - \varepsilon)^{\frac{4(i-1)-\frac{3N}{4}}{N}} \right]. \]  
(4.28)

When the mean value theorem in (4.28) was applied, we obtained,
\[ h_i \leq CN^{-1}. \]  
(4.29)

Thus, the following inequality could be written based on (4.16) and (4.29):
\[ |R_i| \leq CN^{-1}, \quad i = \frac{3N}{4} + 1, ..., N. \]

5) The error function \( R_i \) was evaluated for \( 1 - \sigma = \frac{3}{4} \):
\[ x_{i-1} = 1\sigma - (\sqrt{\alpha})^{-1} \varepsilon \ln \left[ 1 - (1 - e^{\frac{-\sqrt{\alpha}}{N}})^{4(i-1)/N} \right], \]  
(4.30)
Once the mean value theorem in (4.31) was used, it was possible to deduce that

\[ h_i = -\left(\frac{\sqrt{\alpha}}{\sqrt{\epsilon}}\right) \frac{1}{\ln \left[ 1 - \left( 1 - e^{-\frac{\sqrt{\alpha}}{\sqrt{\epsilon}}} \right)^{4(i - \frac{3N}{4})} \right]} \leq C N^{-1}. \]  

(4.32)

And, together with (4.16), it yielded,

\[ |R_i| \leq C N^{-1}, \quad i = \frac{3N}{4} + 1, \ldots, N. \]

Herein, it was as well possible to estimate the error function \( r_0 \) based on the explicit expression in (3.8), we obtained,

\[ |r_0| \leq \left( \frac{\sqrt{\alpha}}{\sqrt{\epsilon}} \right) \int_{x_0}^{x_1} \phi_0(x) dx \int_{x_0}^{x_1} \frac{\partial f(x, u)}{\partial x} + \frac{\partial f}{\partial u} \frac{du}{dx} \, dx \]

\[ \leq C \int_{x_0}^{x_1} (1 + u'(x)) \, dx \]

\[ \leq C h_1 \int_{x_0}^{x_1} \left( 1 + \frac{1}{\sqrt{\epsilon}} \left[ \exp\left( -\frac{\sqrt{\alpha}}{\sqrt{\epsilon}} x_0 \right) + \exp\left( -\frac{\sqrt{\alpha}}{\sqrt{\epsilon}} (1 - x) \right) \right] \right) \, dx, \]  

(4.33)

in which

\[ \exp\left( -\frac{\sqrt{\alpha}}{\sqrt{\epsilon}} x_0 \right) + \exp\left( -\frac{\sqrt{\alpha}}{\sqrt{\epsilon}} (1 - x) \right) \leq C N^{-1}, \]  

(4.34)

and

\[ \exp\left( -\frac{\sqrt{\alpha}}{\sqrt{\epsilon}} (1 - x) \right) + \exp\left( -\frac{\sqrt{\alpha}}{\sqrt{\epsilon}} (1 - x_0) \right) \leq C N^{-1}. \]  

(4.35)

The inequalities (4.33), (4.34) and (4.35) enabled the following equation:

\[ |r_0| \leq C h^{(1)} \left\{ C h^{(1)} + 2 C N^{-1} \right\} \leq C N^{-1}. \]
Finally, it was possible to estimate the error function \( r_1 \). Based on (3.10), (4.34) and (4.35), we obtained

\[
|r_1| \leq |\varphi' (\xi)| \int_{x_{N_0}}^{l_1} u'(\eta) d\eta \\
\leq l_1 - x_{N_0} + \exp(-\sqrt{\alpha x_{N_0}}) + \exp(-\sqrt{\alpha l_1}) \\
+ \exp(-\sqrt{\alpha (1 - l_1)}) + \exp(-\sqrt{\alpha (1 - x_{N_0})}) \\
\leq h_i + C N^{-1} \\
\leq CN^{-1}.
\]

Thus, it would be possible to confidently yield the numerical solution of the problem (1.1)-(1.3), using the following theorem on \( \varepsilon \)-uniform convergence for the presented method. With these expressions the proof of Lemma 4.2 could be obtained.

The following theorem demonstrates the convergence result of the present study.

**Theorem 4.3.** Let \( u(x) \) be the exact solution of the problem (1.1)-(1.3) and \( y_i \) be the discrete solution of the difference scheme (3.11)-(3.13). Then the following estimates yields,

\[
\|y - u\|_{\infty, \omega_{N}} \leq CN^{-1}.
\]

**Proof.** The proof of this theorem could be obtained by using the Lemma 4.1 and Lemma 4.2. The proof is highly clear and easy. □

5. Algorithm and numerical results

This section gives on the demonstration of the following quasi-linearization procedure for the difference scheme (3.11)-(3.13). Moreover, the effectiveness of the presented method was confirmed by applying it to a nonlinear problem (1.1)-(1.3) for \( i = 1, \ldots, N - 1 \):

\[
\begin{align*}
\left( \frac{\varepsilon}{h_i h_i} \right) y_{i-1} - \left( \frac{2 \varepsilon}{h_i h_{i+1}} - \frac{\partial f}{\partial y}(x_i, y_i^{(n-1)}) \right) y_i + \left( \frac{\varepsilon}{h_i h_{i+1}} \right) y_{i+1} = -f_i, \\
\varepsilon y_{x,0}^{(n)} = A + f(x_0 - y_0^{(n-1)}) + \frac{\partial f}{\partial y}(x_0, y_0^{(n-1)})(y_0^{(n)} - y_0^{(n-1)})), \\
y_N^{(n)} = \varphi(y_{N_0}^{(n-1)}) + \frac{\partial \varphi}{\partial y}(y_{N_0}^{(n-1)})(y_0^{(n)} - y_{N_0}^{(n-1)})), \\
A_i = \frac{\varepsilon}{h_i h_i}, B_i = \frac{\varepsilon}{h_i h_{i+1}}, C_i = \frac{2 \varepsilon}{h_i h_{i+1}} - \frac{\partial f}{\partial y}(x_i, y_i^{(n-1)}), \\
F_i = f(x_i - y_i^{(n-1)}) - \frac{\partial f}{\partial y}(x_i, y_i^{(n-1)}),
\end{align*}
\]
\[ \alpha_1 = 1, \quad \beta_1 = \varepsilon^{-1} h_1(A + f(x_0, y_0^{(n-1)})) + \frac{\partial f}{\partial y}(x_0, y_0^{(n-1)})(y_0^{(n)} - y_0^{(n-1)}) , \]

\[ \alpha_{i+1} = \frac{B_i}{C_i - A_i \alpha_i}, \quad \beta_{i+1} = \frac{F_i + A_i \beta_i}{C_i - A_i \alpha_i}, \quad i = 1, \ldots, N - 1, \]

\[ y_1^{(n)} = \alpha_{i+1} y_1^{(n)} + \beta_{i+1}, \quad i = N - 1, \ldots, 2, 1. \]

This algorithm was stable due to \( A_i > 0, B_i > 0, C_i > A_i + B_i, i = 1, 2, \ldots, N \).

Subsequently, the following problem was taken into consideration in order to prove that the presented method worked:

\[ -\varepsilon u''(x) - \exp(-u(x)) = 0, \quad 0 < x < 1, \]

\[ u'(0) = \frac{1}{\varepsilon}, \quad u(1) - \varphi(u(\frac{1}{2})) = 0, \quad \varphi(u(x)) = \frac{1}{2}u(x) - 2. \]

The exact solution of this problem remained unavailable. Therefore, it was necessary to use the double mesh method [8] which was expected to provide the errors and rates of convergence for the present example.

The \( \varepsilon \)-uniform convergence rates were calculated using the following expression:

\[ P^N = \frac{\ln \left( e^N / e^{2N} \right)}{\ln 2}. \]

The error estimates were also denoted by

\[ e^N = \| y^{\varepsilon \cdot N} - y^{\varepsilon \cdot 2N} \|_{\infty, \omega_N}, \quad e^N = \max_{\varepsilon} e^N. \]

**Table 1. Errors \( e^N \) and rates of convergence \( p^N \) for test problem**

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( N = 32 )</th>
<th>( N = 64 )</th>
<th>( N = 128 )</th>
<th>( N = 256 )</th>
<th>( N = 512 )</th>
<th>( N = 1024 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2(^{-1} )</td>
<td>0.2137822</td>
<td>0.1078726</td>
<td>0.0542131</td>
<td>0.0271136</td>
<td>0.0135880</td>
<td>0.0067903</td>
</tr>
<tr>
<td>0.98</td>
<td>0.99</td>
<td>0.99</td>
<td>0.99</td>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3(^{-1} )</td>
<td>0.2514692</td>
<td>0.1305050</td>
<td>0.0665973</td>
<td>0.0336397</td>
<td>0.0168998</td>
<td>0.0084705</td>
</tr>
<tr>
<td>0.94</td>
<td>0.97</td>
<td>0.98</td>
<td>0.99</td>
<td>0.99</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4(^{-1} )</td>
<td>0.3316944</td>
<td>0.1764517</td>
<td>0.0912917</td>
<td>0.0464764</td>
<td>0.0234520</td>
<td>0.0117883</td>
</tr>
<tr>
<td>0.91</td>
<td>0.95</td>
<td>0.97</td>
<td>0.98</td>
<td>0.99</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5(^{-1} )</td>
<td>0.4114538</td>
<td>0.218725</td>
<td>0.1143295</td>
<td>0.0586237</td>
<td>0.0296941</td>
<td>0.0150992</td>
</tr>
<tr>
<td>0.91</td>
<td>0.93</td>
<td>0.96</td>
<td>0.98</td>
<td>0.97</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6(^{-1} )</td>
<td>0.4834303</td>
<td>0.2566565</td>
<td>0.1360977</td>
<td>0.0702530</td>
<td>0.0357047</td>
<td>0.0179924</td>
</tr>
<tr>
<td>0.91</td>
<td>0.91</td>
<td>0.95</td>
<td>0.97</td>
<td>0.98</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7(^{-1} )</td>
<td>0.5383700</td>
<td>0.2925638</td>
<td>0.1568564</td>
<td>0.0814925</td>
<td>0.0416001</td>
<td>0.0204012</td>
</tr>
<tr>
<td>0.88</td>
<td>0.89</td>
<td>0.94</td>
<td>0.97</td>
<td>1.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( e^N )</td>
<td>0.5383700</td>
<td>0.2925638</td>
<td>0.1568564</td>
<td>0.0814925</td>
<td>0.0416001</td>
<td>0.0204012</td>
</tr>
<tr>
<td>( p^N )</td>
<td>0.88</td>
<td>0.89</td>
<td>0.94</td>
<td>0.97</td>
<td>1.02</td>
<td></td>
</tr>
</tbody>
</table>
The method of this paper focused on was applied to the above example. When $N$ takes increasing values, it was observed in table and graphs that the convergence rate of the smooth convergence speed $p^N$ was first order. When the $\varepsilon$ is smaller, approximate solution curve is closer to the axes as presented in Figure 1. Hence, the numerical results indicated that the proposed scheme was working effectively.

6. Conclusion

The present study examined the nonlinear singularly perturbed problem that contains multi-point boundary conditions through the finite difference method. It was revealed that the method displayed uniform convergence with respect to the perturbation parameter $\varepsilon$ in the discrete maximum norm. Furthermore, the results of the theory and the numerical experiment provided results consistent with each other. Another main finding of the study was the potential of the proposed method to be used for more complicated nonlinear singularly perturbed problems. As a suggestion, with the help of this study, the convergence rate can be increased to higher degrees.

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REFERENCES


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