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Regularization of a nonlinear inverse problem by discrete mollification method

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Abstract In this article, the application of discrete mollification as a regularization procedure for solving a nonlinear inverse problem in one dimensional space is considered. Ill-posedness is identified as one of the main characteristics of inverse problems. It is clear that if we have a noisy data, the inverse problem becomes unstable. As such, a numerical procedure based on discrete mollification and space marching method is applied to address the ill-posedness of the mentioned problem. The regularization parameter is selected by generalized cross validation (GCV) method. The numerical stability and convergence of the proposed method are investigated. Finally, some test problems, whose exact solutions are known, are solved using this method to show the efficiency.

Keywords. Nonlinear inverse problem, Discrete mollification, Space marching, Stability, Convergence.
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1. Introduction

The problem of identification of the unknown source term has been widely investigated from 1970s [2, 8, 16, 17]. For a source term of the form $F = F(u)$, Cannon and DuChateau considered a nonlinear diffusion equation and determined the unknown source term without assuming an a-priori functional form [2]. Fatullayev in [4] approximated the unknown source term by polygons linear pieces through a numerical

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procedure. In [15], the authors examined the existence and uniqueness of determining the unknown source term of the form $F = F(x,t)$. In [7], Isakov obtained some results for linear source term's inversion of parabolic equations. In [1], the authors used an approach based on the theory of inverse infinite-dimensional open dynamical systems (DS) or, in other words, input-state-output systems to solve a problem of identification of the heat transfer sources in nonlinear media. In all of these papers, the initial and boundary conditions are known functions and the determination of the unknown source term is investigated.

In the context of approximation method for inverse problem many approaches have been investigated. However, most of the essays in this issue have been limited to the linear problems [12, 13], even fewer articles concern the mollification method for nonlinear inverse problems. In [5], a nonlinear inverse problem is solved through mollification method, but the nonlinearity is limited to its boundary conditions.

In this work we consider the problem of identification of $q(t)$ and $u(x,t)$ satisfying the following inverse problem with a source term of the form $F(x,t,u)$:

$$u_t(x,t) - u_{xx}(x,t) = F(x,t,u(x,t)), \quad 0 < x < 1, \quad 0 < t < T, \tag{1.1}$$
$$u(x,0) = f(x), \quad 0 \le x \le 1, \tag{1.2}$$
$$u(0,t) = g(t), \quad 0 \le t \le T, \tag{1.3}$$
$$u(1,t) = q(t), \quad 0 \le t \le T, \tag{1.4}$$

with the following overspecified condition:

$$u_x(0,t) = p(t), \quad 0 \le t \le T, \tag{1.5}$$

where $F(x,t,u)$, $f(x)$, $g(t)$ and $p(t)$ are known functions. T is also given. When $q(t)$ is given, the problem (1.1)-(1.4) is called a direct problem. For this direct problem suppose that $f(x)$, $g(t)$ and $q(t)$ are continuously differentiable, $f(0) = g(0)$, $q(0) = f(1)$ and $F(x,t,u)$ satisfies the following conditions:
1) The function $F(x,t,u)$ is a contiuous function.
2) There exists a constant l such that

$$|F(x,t,u) - F(x,t,v)| \le l\,|u - v|\,.$$

3) F is a bounded and uniformly continuous function in u.
Then direct problem (1.1)-(1.4) has a unique solution [3]. It is shown in [14] that a linear version of inverse problem (1.1)-(1.5) is an ill-posed cauchy problem for a parabolic equation. As such our nonlinear cauchy problem (1.1)-(1.5) is ill-posed. To find the solution of the inverse problem (1.1)-(1.5), we will introduce a stable and convergent algorithm based on discrete mollification and the space marching method. Without loss of generality, we suppose that, instead of $f(x)$, $g(t)$ and $p(t)$ we have approximate amounts of these functions presented as $f^\varepsilon(x)$, $g^\varepsilon(t)$ and $p^\varepsilon(t)$ such that

$$\|f^\varepsilon(x) - f(x)\|_\infty \le \varepsilon,\ \|g^\varepsilon(t) - g(t)\|_\infty \le \varepsilon,\ \|p^\varepsilon(t) - p(t)\|_\infty \le \varepsilon.$$

Because of the presence of noise in the data, we first regularize problem by the discrete mollification method. Mollification method is recognized as a reliable regularization method based on convolution that has been widely applied to many ill-posed problems [9, 10, 11, 13]. The idea of this method is very simple [6]: if the data of the problem are not clear and only an approximate amount of data is accessible, it is recommended to find a sequence of mollification operators to map improper data into well-posed classes of the problem (mollify the improper data). Consequently, the intended problem will be a well-posed one. This paper is organized as follows: sSection 2 contains the preliminary concepts, theorems and notations of the discrete mollification method. In section 3, we regularize the intended inverse problem then we solve the regularized problem by the proposed method. The next section considers the stability and convergence proof of the space marching numerical algorithm. Finally, section 5 is devoted to the numerical solution of some examples, which are solved by the mentioned method.

2. Discrete mollification method

In this section, the basic idea of discrete mollification is introduced (see [13] for more details).
Let $G = \{g(x_j) = g_j\}_{j=1}^{M}$ be a discrete function defined on $K = \{x_j, j = 1, .., M\} \subset [0, 1]$ satisfying

$$0 \leq x_1 < x_2 < .. < x_{M-1} < x_M \leq 1.$$

Set

$$s_j = \begin{cases} 0, & j = 0 \\ \frac{1}{2}(x_j + x_{j+1}), & j = 1, .., M-1 \\ 1, & j = M. \end{cases}$$

Let $p > 0$ is given. Then for any $x \in I_\delta = [p\delta, 1 - p\delta]$, we define discrete mollification of G as follows

$$J_\delta G(x) = \sum_{j=1}^{M} \Big(\int_{s_{j-1}}^{s_j} \rho_{\delta,p}(x - s)ds\Big)g_j,$$

where

$$\rho_{\delta,p}(x) = \begin{cases} A_p\delta^{-1}\exp(-\frac{x^2}{\delta^2}), & |x| \leq p\delta \\ 0, & |x| > p\delta, \end{cases}$$

such that $A_p = (\int_{-p}^{p} \exp(-s^2)ds)^{-1}$. We usually take $p = 3$ and the radius of mollification, δ, is selected automatically by the GCV method (see [5] for more details). We note that

$$\sum_{j=1}^{M} \int_{s_{j-1}}^{s_j} \rho_{\delta,p}(x - s)ds = \int_{-p\delta}^{p\delta} \rho_{\delta,p}(s)ds = 1.$$

Set

$$\Delta x = \max_{1 \leq j \leq M-1} |x_{j+1} - x_j| .$$

The main properties of the discrete mollification method are as follows. (see [13] for more details).

Theorem 2.1. *([5])*
1. Let $g(x) \in C^{0,1}(R^1)$ and $G = \{g(x_j) = g_j\}_{j=1}^{M}$ be the discrete version of g and

let $G^\varepsilon = \{g_j^\varepsilon\}_{j=1}^M$ be the perturbed discrete version of g satisfying $\|G - G^\varepsilon\|_{\infty,K} \leq \varepsilon$. Then there exists a constant C, independent of δ, such that

$$\|J_\delta G^\varepsilon - J_\delta g\|_{\infty,I_\delta} \leq C(\varepsilon + \Delta x).$$

2. If $g'(x) \in C^{0,1}(R^1)$, let $G = \{g(x_j) = g_j\}_{j=1}^M$ and $G^\varepsilon = \{g_j^\varepsilon\}_{j=1}^M$ satisfying $\|G - G^\varepsilon\|_{\infty,K} \leq \varepsilon$, then

$$\|D(J_\delta G^\varepsilon) - (J_\delta g)'\|_{\infty,I_\delta} \leq \frac{C}{\delta}(\varepsilon + \Delta x) + C_\delta(\Delta x)^2.$$

3. Suppose that $G = \{g(x_j) = g_j\}_{j=1}^M$ be the discrete function defined on K, and D_0^δ be a differentiation operator defined by $D_0^\delta(G) = D(J_\delta G)(x)$ then

$$\left\|D_0^\delta(G)\right\|_{\infty,K} \leq \frac{C}{\delta}\,\|G\|_{\infty,K}.$$

In order to compute $J_\delta G(x)$ throughout the domain $[0,1]$, we have to either extend the discrete data function g to a bigger interval $I'_\delta = [-p\delta, 1 + p\delta]$ or confine this function to the interval $I_\delta = [p\delta, 1-p\delta]$. In this paper, the former approach described in [5] is applied. We seek constant extension g^* of g to the intervals $[-p\delta, 0]$ and $[1, 1 + p\delta]$, satisfying the conditions: $\|J_\delta g^* - g\|_{L_2[0,p\delta]}$ and $\|J_\delta g^* - g\|_{L_2[1-p\delta,1]}$ are minimum. The unique solution to this optimization problem at the boundary $t = 1$ is given by [12]:

$$g^* = \frac{\int_{1-3\delta}^{1} [g(t) - \int_0^1 \rho_\delta(t-s)g(s)ds][\int_1^{1+3\delta} \rho_\delta(t-s)g(t)ds]dt}{\int_{1-3\delta}^{1} \left[\int_1^{1+3\delta} \rho_\delta(t-s)ds\right]^2 dt}.$$

A similar result holds at the end point t = 0. A proof of these statements can be found in [12].
For each $\delta > 0$, the extended function is defined on the interval I'_δ and the corresponding mollified function is computed on I = [0, 1]. All the conclusions and error estimates hold in the subinterval I_δ. Details on the computation of mollified operators and mollification parameters can be found in [10, 11, 12].

3. Numerical procedure

In this section we discuss the application of discrete mollification and space marching method to solve the problem (1.1)-(1.5).
To this end, we first regularize the proposed problem. The regularized problem is governed by:

$$v_t(x,t) - v_{xx}(x,t) = F(x,t,v(x,t)), \quad 0 < x < 1, \quad 0 < t < T, \tag{3.1}$$
$$v(x,0) = J_{\delta_1} f^\varepsilon(x), \quad 0 \leq x \leq 1, \tag{3.2}$$
$$v(0,t) = J_{\delta_3} g^\varepsilon(t), \quad 0 \leq t \leq T, \tag{3.3}$$
$$v_x(0,t) = J_{\delta_2^0} p^\varepsilon(t), \qquad 0 \leq t \leq T. \tag{3.4}$$

The δ_1-mollification is taken with respect to x and the δ_2^0 and δ_3-mollifications are taken with respact to t. To compute the solution of the problem (3.1)-(3.4) through the space, marching method, we define the time and spatial steps, respectively, as

$$\Delta t = k = \frac{1}{N}, \quad \Delta x = h = \frac{1}{M},$$

where N and M are positive integers. Let us denote the numerical approximations of functions $v(jh, nk)$, $v_x(jh, nk)$ and $v_t(jh, nk)$ with $j = 0, .., M$ and $n = 0, .., N$ by U_j^n, Q_j^n and R_j^n, respectively. The space marching algorithm for this problem is as follows:

1. Choose the radii of mollification, δ_1, δ_2^0 and δ_3 using the GCV method.
2. Set
   $U_j^0 = J_{\delta_1} f^\varepsilon(jh), j = 1, .., M,$
   $Q_0^n = J_{\delta_2^0} p^\varepsilon(nk), n = 0, .., N,$
   $U_0^n = J_{\delta_3} g^\varepsilon(nk), n = 0, .., N,$
3. Perform a linear extrapolation to compute R_0^0.
4. Set
   $R_0^n = (D_0)_t(J_{\delta_3} g^\varepsilon(nk)), n = 1, .., N.$
5. For $j = 0$ to $j = M - 1$,
   For $n = 0$ to $n = N$

$$U_{j+1}^n = U_j^n + hQ_j^n, \tag{3.5}$$

$$Q_{j+1}^n = Q_j^n + h(R_j^n - F(jh, nk, U_j^n)), \tag{3.6}$$

$$R_{j+1}^n = R_j^n + h(D_0)_t(J_{\delta_2^j} Q_j^n), \tag{3.7}$$

where D_0 is the centered difference operator denoted by

$$D_0 f(t) \approx \frac{f(t + \Delta t) - f(t - \Delta t)}{2\Delta t}.$$

4. Stability and convergence analysis of the space marching algorithm

In this section, we analyze stability and convergence properties of the space marching algorithm (3.5)-(3.7).

Theorem 4.1. *(Stability theorem) There exists some constant M_1, such that*

$$\max\{|U_M|, |Q_M|, |R_M|, B\} \leq \exp(M_1) \max\{|U_0|, |Q_0|, |R_0|, B\}.$$

Proof. Set $B = \max_{\{x,t,u\}} |F(x, t, u)|$ and $|\delta|_{-\infty} = \min_j(\delta_2^j)$. From (3.5) and (3.6), we obtain

$$\left|U_{j+1}^n\right| \leq (1 + h) \max\{\left|U_j^n\right|, \left|Q_j^n\right|\}, \tag{4.1}$$

$$\left|Q_{j+1}^n\right| \leq (1 + h) \max\{\left|Q_j^n\right|, \left|R_j^n\right|, B\}. \tag{4.2}$$

By Theorem 2.1 and (3.7), we derive

$$\left|R_{j+1}^n\right| \leq (1 + h\frac{C}{|\delta|_{-\infty}}) \max\{\left|Q_j^n\right|, \left|R_j^n\right|\}. \tag{4.3}$$

Then, from (4.1)-(4.3), it is obtained that

$$\max\{|U_{j+1}|, |Q_{j+1}|, |R_{j+1}|, B\} \leq (1 + hM_1) \max\{|U_j|, |Q_j|, |R_j|, B\},$$

where

$$M_1 = \max\{1, \frac{C}{|\delta|_{-\infty}}\}.$$

After M iteration of the last inequality, we have

$$\max\{|U_M|, |Q_M|, |R_M|, B\} \leq (1 + hM_1)^M \max\{|U_0|, |Q_0|, |R_0|, B\},$$

which implies

$$\max\{|U_M|, |Q_M|, |R_M|, B\} \leq \exp(M_1) \max\{|U_0|, |Q_0|, |R_0|, B\}.$$

The proof is complete. □

Theorem 4.2. *(Convergence theorem) For fixed δ, as h, k and ε tend to zero, the numerical scheme (3.5)-(3.7) converges to the mollified exact solution.*

Proof. We let

$$\begin{aligned} \Delta U_j^n &= U_j^n - v(jh, nk), \\ \Delta Q_j^n &= Q_j^n - v_x(jh, nk), \\ \Delta R_j^n &= R_j^n - v_t(jh, nk), \end{aligned}$$

therefore, we have

$$\begin{aligned} \Delta U_{j+1}^n &= U_{j+1}^n - v((j+1)h, nk) \\ &= \Delta U_j^n + (U_{j+1}^n - U_j^n) - (v((j+1)h, nk) - v(jh, nk)) \\ &= \Delta U_j^n + h(Q_j^n - v_x(jh, nk)) + O(h^2) \\ &= \Delta U_j^n + h\Delta Q_j^n + O(h^2), \end{aligned} \tag{4.4}$$

$$\begin{aligned} \Delta Q_{j+1}^n &= Q_{j+1}^n - v_x((j+1)h, nk) \\ &= \Delta Q_j^n + (Q_{j+1}^n - Q_j^n) - (v_x((j+1)h, nk) - v_x(jh, nk)) \\ &= \Delta Q_j^n + h(R_j^n - F(jh, nk, U_j^n)) \\ &\quad - h(v_t(jh, nk) - F(jh, nk, v(jh, nk))) + O(h^2), \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \Delta R_{j+1}^n &= R_{j+1}^n - v_t((j+1)h, nk) \\ &= \Delta R_j^n + (R_{j+1}^n - R_j^n) - (v_t((j+1)h, nk) - v_t(jh, nk)) \\ &= \Delta R_j^n + h(D_0(J_{\delta_2^j} Q_j^n) - v_{xt}(jh, nk)) + O(h^2). \end{aligned} \tag{4.6}$$

Following (4.4) and (4.5), it is obtained that

$$\left|\Delta U_{j+1}^n\right| \leq \left|\Delta U_j^n\right| + h\left|\Delta Q_j^n\right| + O(h^2), \tag{4.7}$$

$$\begin{aligned} \left|\Delta Q_{j+1}^n\right| \leq \left|\Delta Q_j^n\right| &+ h\{\left|\Delta R_j^n\right| + \left|F(jh, nk, U_j^n) - F(jh, nk, v(jh, nk))\right|\} \\ &+ O(h^2). \end{aligned} \tag{4.8}$$

Applying Theorem 2.1 and equation (4.6), we derive

$$\left|\Delta R_{j+1}^{n}\right| \leq \left|\Delta R_{j}^{n}\right| + h(C\frac{\left|\Delta Q_{j}^{n}\right| + k}{|\delta|_{-\infty}} + C_{\delta}k^{2}) + O(h^{2}). \tag{4.9}$$

Let

$$\Delta_j = \max\{\left|\Delta U_j^n\right|, \left|\Delta Q_j^n\right|, \left|\Delta R_j^n\right|\},\ C_0 = \max\{1, l, \tfrac{C}{|\delta|_{-\infty}}\},\ C_1 = \tfrac{Ck}{|\delta|_{-\infty}} + C_\delta k^2,$$

then it is concluded that

$$\Delta_{j+1} \leq (1 + hC_0)\Delta_j + hC_1 + O(h^2).$$

After M iteration, we obtain

$$\Delta_M \leq (1 + hC_0)^M \Delta_0 + h(1 + hC_0)^{M-1} C_1 + ... + h(1 + hC_0)C_1 + hC_1. \tag{4.10}$$

Theorem 2.1 yields the following inequalities

$$\begin{aligned}
|\Delta U_0^n| \leq& C(\varepsilon + k),\\
|\Delta Q_0^n| \leq& C(\varepsilon + k),\\
|\Delta R_0^n| \leq& \frac{C}{|\delta|_{-\infty}}(\varepsilon + k) + C_\delta k^2,
\end{aligned}$$

therefore, as ε, h and k tend to zero, Δ_0 and the right hand side of (4.10) tend to zero and so does Δ_M and the proof is complete. □

5. Numerical examples

In this section, we apply discrete mollification combined with the space marching method which was briefly described in sections 3 and 4 to some test problems, then we present some numerical examples to demonstrate the effectiveness and stability of our proposed method. The stability of the method with respect to noise in the data is investigated using noisy data. The noisy discrete data functions are generated by adding a random perturbation to the exact data functions. For example, for $f(x)$, its discrete noisy version is

$$f_j^\varepsilon = f(x_j) + \varepsilon_j, \qquad j = 0, 1, .., M,$$

where the ε_j's are Gaussian random variables with variance ε^2. The radii of mollification are chosen automatically by the GCV method. For checking the accuracy of our algorithm, we use the relative weighted l_2-norm. In these examples, we take $T = 1$. We use Mathematica 10.3.1 for our computations.

Example 5.1. We first consider the problem (1.1)-(1.5) with

$$F(x, t, u) = \frac{1}{10} x \exp(\pi t) + \pi u,$$

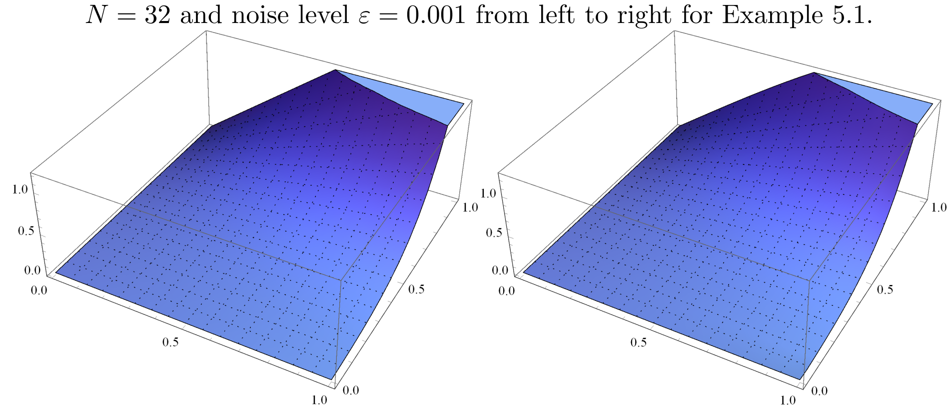
where the exact solution is:

$$u(x, t) = \frac{1}{10} xt \exp(\pi t).$$

TABLE 1. Relative l_2 error norms for Example 5.1.

ε	M	N	Relative l_2 error for u
0.001	40	40	0.0099563
0.001	60	60	0.0076954
0.001	80	80	0.0068603
0.01	40	40	0.0124064
0.01	60	60	0.0092491
0.01	80	80	0.0073056

FIGURE 1. Exact and approximate solutions for $u(x,t)$ with $M = N = 32$ and noise level $\varepsilon = 0.001$ from left to right for Example 5.1.



and the initial and boundary conditions can be obtained from the exact solution. We test the accuracy of the proposed method by solving this problem by the above-mentioned method with several values of M and N and different noise levels. In Table 1, relative l_2 error norms with two noise levels $\varepsilon = 0.001$ and 0.01 are listed. Figure 1 illustrates exact and approximate solutions for $u(x,t)$ with $M = N = 32$ and noise level $\varepsilon = 0.001$. Figures 2 and 3 demonstrate exact and regularized solutions for $q(t)$ with $\varepsilon = 0.001$, $M = N = 20$ and absolute errors for $q(t)$ with $\varepsilon = 0.01, 0.05, 0.1$, $M = N = 20$, respectively. From Table 1, it is observed that at fixed noise level ε, numerical results improved by increasing the number of nodes and for sufficiently large number of nodes, the agreement between numerical and exact solutions becomes uniformly good. Figures 1-3 reveal the efficiency of the discrete mollification method as a regularization procedure.

Example 5.2. Consider the problem (1.1)-(1.5) with

$$F(x,t,u) = \begin{cases} \pi\sqrt{|x^2 - u^2|}, & 0 \le t < \frac{1}{2} \\ -\pi\sqrt{|x^2 - u^2|}, & \frac{1}{2} \le t < 1 \end{cases}$$

with the boundary conditions

$$u(0,t) = 0,$$
$$u(1,t) = \sin(\pi t),$$

Figure 2. Exact and regularized solutions for $q(t)$ with noise level $\varepsilon = 0.001$ and $M = N = 20$ for Example 5.1.

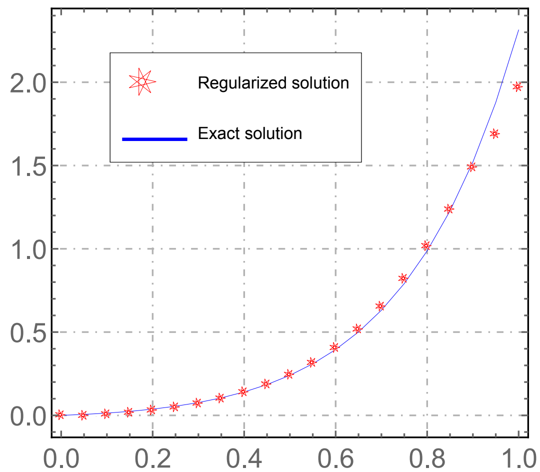


Figure 3. Absolute errors for $q(t)$ with $M = N = 20$ and noise levels $\varepsilon = 0.01, 0.05, 0.1$ for Example 5.1.

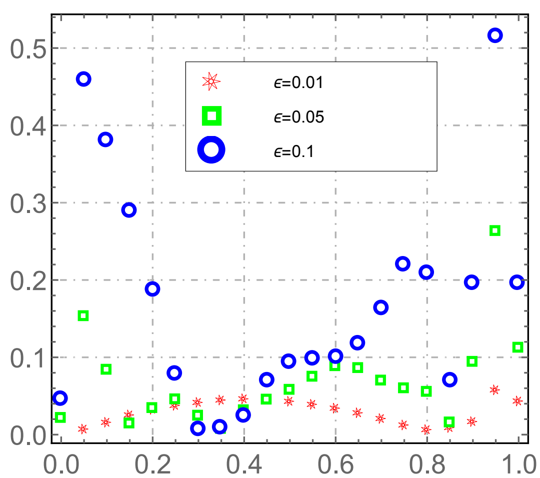
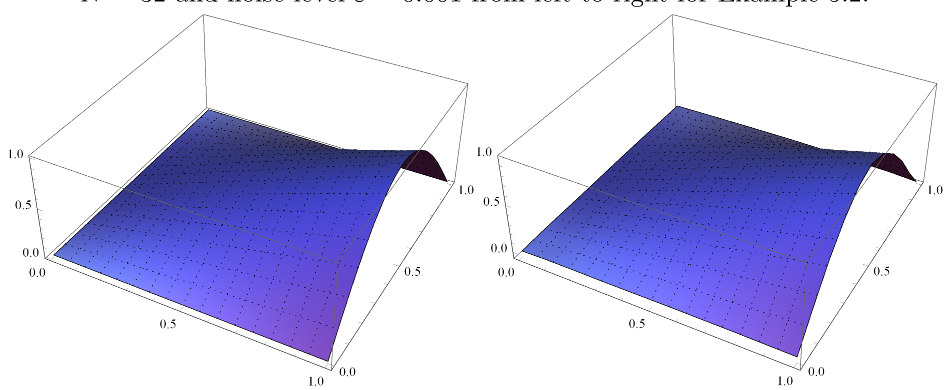


TABLE 2. Relative l_2 error norms for Example 5.2.

ε	M	N	Relative l_2 error for u
0.001	40	40	0.0126104
0.001	60	60	0.0087942
0.001	80	80	0.0063808
0.01	40	40	0.0256239
0.01	60	60	0.0109018
0.01	80	80	0.0093693

FIGURE 4. Exact and approximate solutions for $u(x,t)$ with $M = N = 32$ and noise level $\varepsilon = 0.001$ from left to right for Example 5.2.



and initial data

$$u(x,0) = 0,$$

with the overspecified condition

$$u_x(0,t) = \sin(\pi t),$$

where the exact solution is $u(x,t) = x\sin(\pi t)$.
The relative l_2 errors for this inverse problem are listed in Table 2 with two noise levels $\varepsilon = 0.001$ and 0.01. This table shows that at fixed noise level ε, as h and k decrease, the accuracy of the algorithm will improve. Figure 4 illustrates exact and approximate solutions for $u(x,t)$ with $M = N = 32$ and noise level $\varepsilon = 0.001$. Finally, the comparison between the exact solution and its regularized solution with the discrete mollification method for $q(t)$ are illustrated in Figures 5-6. These Figures demonstrate the effectiveness of the discrete mollification method.
Example 5.3. We consider the problem (1.1)-(1.5) with exact solution

$$u(x,t) = t^2\sin(x)$$

and the initial and boundary conditions can be obtained from the exact solution. With these assumptions, the source term $F(x,t,u)$ is $2t\sin(x) + u$. Table 3 illustrates relative l_2 errors with different noise levels $\varepsilon = 0.001$ and 0.01. From this table, we can

FIGURE 5. Exact and regularized solutions for $q(t)$ with noise level $\varepsilon = 0.001$ and $M = N = 20$ for Example 5.2.

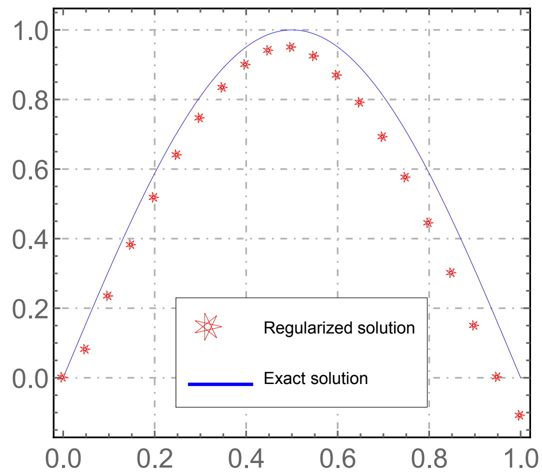


FIGURE 6. Absolute errors for $q(t)$ with $M = N = 20$ and noise levels $\varepsilon = 0.01, 0.05, 0.1$ for Example 5.2.

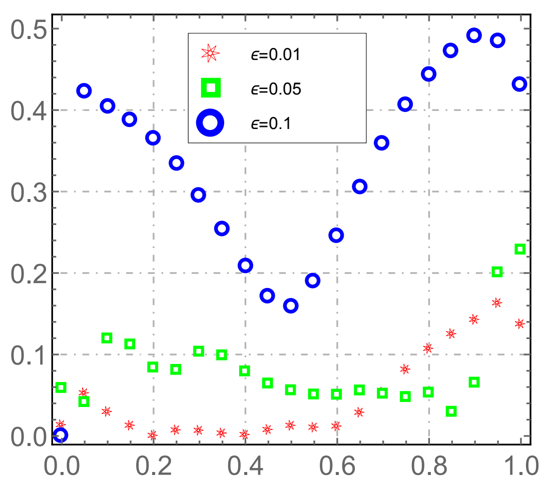
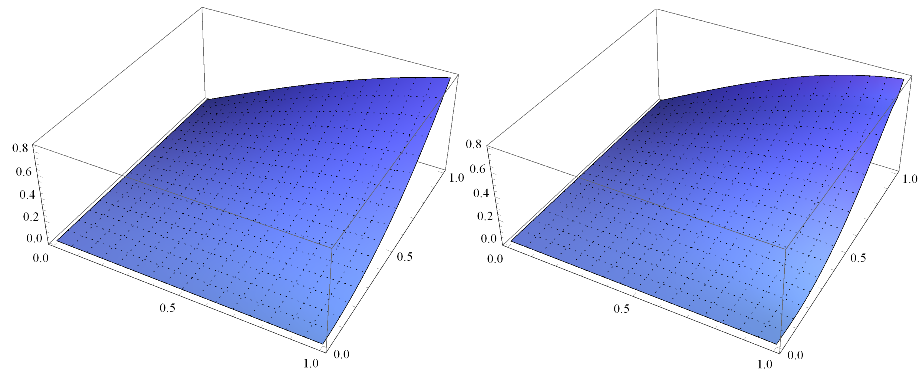


TABLE 3. Relative l_2 error norms for Example 5.3.

ε	M	N	Relative l_2 error for u
0.001	40	40	0.0105380
0.001	60	60	0.0080616
0.001	80	80	0.0079946
0.01	40	40	0.0154812
0.01	60	60	0.0093298
0.01	80	80	0.0081275

FIGURE 7. Exact and approximate solutions for $u(x,t)$ with $M = N = 32$ and noise level $\varepsilon = 0.001$ from left to right for Example 5.3.



observe that as h and k decrease, the accuracy of approximated solutions will enhance. Figure 7 shows exact and approximate solutions for $u(x,t)$ with $M = N = 32$ and noise level $\varepsilon = 0.001$. In order to investigate the influence of discrete mollification as a regularization procedure, we solve this problem by the space marching algorithm with discrete mollification. Then, we carry out the numerical results in Figures 8-9 for $q(t)$. These Figures reveal the efficiency of the discrete mollification method as a regularization procedure.

6. Conclusion

In this research, we have developed a regularization approach based on the discrete mollification and space marching method to numerically solve a nonlinear inverse problem in one dimensional space. The stability and convergence of the proposed algorithm have been proved. The theoritical analysis and numerical tests illustrate that the mollification is a suitable regularization method for determining the boundary condition in one-dimensional inverse problem.

Figure 8. Exact and regularized solutions for $q(t)$ with noise level $\varepsilon = 0.001$ and $M = N = 20$ for Example 5.3.

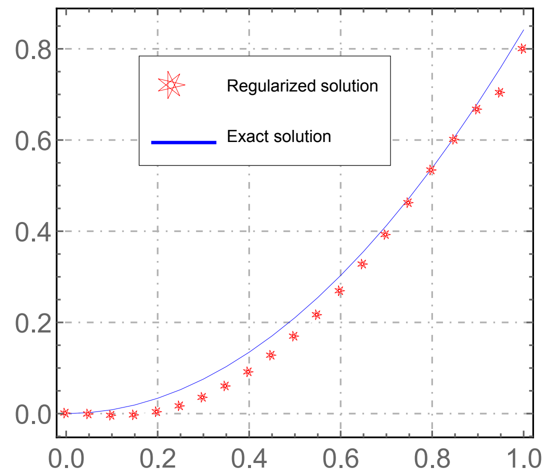
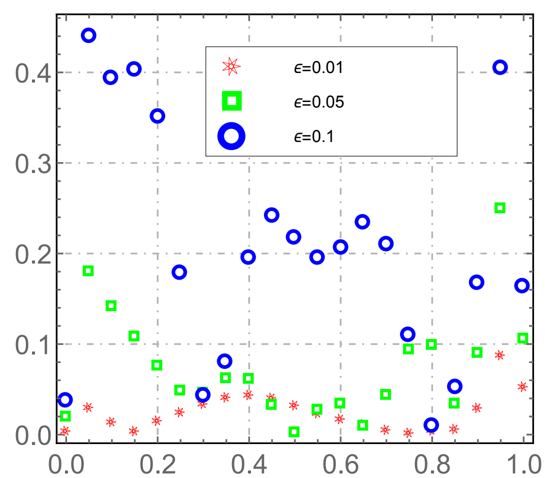


Figure 9. Absolute errors for $q(t)$ with $M = N = 20$ and noise levels $\varepsilon = 0.01, 0.05, 0.1$ for Example 5.3.



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