Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 9, No. 1, 2021, pp. 36-51
DOI:10.22034/cmde.2020.29444.1420

On the existence of positive solutions for a non-autonomous fractional differential equation with integral boundary conditions

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Abstract By using the Guo-Krasnoselskii's fixed point theorem, we investigate the existence of positive solutions for a non-autonomous fractional differential equations with integral boundary conditions of fractional order $\alpha \in (2, 3]$ in an ordered Banach space. The Fredholm integral equation has an important role in this article. Some examples are presented to illustrate the efficiency of the obtained results.

Keywords. Caputo fractional derivative, Integral boundary conditions, Fredholm integral equation.
2010 Mathematics Subject Classification. 26A33, 34B18, 34A08.

1. Introduction

Fractional differential equations play an important role in various fields of science and engineering. With the help of fractional calculus, we can describe several natural phenomena and mathematical models more accurately. Therefore, recently the factional differential equations have received much attention in theory and computations that is applications have been greatly developed [1, 7, 8, 9, 10, 11, 12, 21, 24, 25, 27, 30, 31, 32, 33, 38]. It is well-known that the initial and boundary value problems for nonlinear fractional differential equations arise in the study of models of control, porous media, electrochemistry, viscoelasticity, electromagnetic, etc. On the other hand, boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They have various applications in applied fields such as blood flow problems, chemical engineering, theorem-elasticity, underground water flow, population dynamics, and so forth. We refer the reader to the papers [2, 3, 4, 5, 6, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 26, 28, 34, 35, 36] and the references therein.

Received: 19 September 2018 ; Accepted: 14 December 2019.
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In this paper, we consider the following boundary value problem of nonlinear fractional differential equations (FDE) with integral boundary conditions

$$^{C}D^{\alpha}y(t) = f(t,\ y(t)), \quad t \in (0,\ 1), \quad \alpha \in (2,3], \tag{1.1}$$

subject to

$$y(0) - \beta_1 y'(0) = \int_0^1 g_0(s)y(s)ds, \tag{1.2}$$

$$y(1) - \beta_2 y'(1) = \int_0^1 g_1(s)y(s)ds, \tag{1.3}$$

and

$$y''(0) = 0, \tag{1.4}$$

where $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$, $g_0, g_1 \in C([0,1], \mathbb{R}^+)$ are positive and $\beta_1, \beta_2 > 1$ satisfy $1 + \beta_1 > \beta_2$. Here $^{C}D^{\alpha}$ represents the Caputo fractional operator of order α given by

$$^{C}D^{\alpha}y(t) = \frac{1}{\Gamma(k-\alpha)} \int_0^t (t-x)^{k-\alpha-1} y^{(k)}(x)dx, \tag{1.5}$$

where $k = \lceil \alpha \rceil$ is the smallest integer greater than or equal to k. Notice that the Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives, which have clear physical interpretations.
The main aim of this paper is to give existence result for positive solutions to (1.1)-(1.4). For this purpose, we are going to use of the Guo-Krasnoselskii's fixed point theorem presented in [22].
This paper is organized as follows: In Section 2, we introduce some necessary definitions and mathematical preliminaries of fractional calculus theories which are used further to achieve our target. The existence result for positive solutions to (1.1)-(1.4) is presented in Section 3. The section 4 consists numerical aspect of the manuscript.

2. Preliminaries

For sake of convenience to the readers, we firstly present the necessary definitions and some necessary facts in the fractional calculus theory and functional analysis. It should be noted that, within this study, $X = C([0,1])$ is the Banach space of all continuous real valued functions on the interval $[0,1]$ endowed with the norm: $\|y\| = \max\{|y(t)| : t \in [0,\ 1]\}$. Moreover, the nonempty convex closed subset P of X is called a cone in X if $ax \in P$ and $x + y \in P$ for all $x, y \in P$ and $a \geq 0; x \in X$ and $-x \in X$ imply $x = 0$. It is easy to show that the set $P = \{y \in C[0,1] : y(t) \geq 0\}$ is a positive cone in $C[0,1]$.

Definition 2.1. [30, 31] Let $y \in L_1[a,b]$. Then the Riemann–Liouville fractional integral $I^{\alpha}_{a^+}y$ of order $\alpha > 0$ is defined by

$$I^{\alpha}_{a^+}y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)ds,\ t > a.$$

The Caputo fractional derivative of order $\alpha > 0$ is defined as

$$^{C}D^{\alpha}_{a^+}y(t) = I^{n-\alpha}_{a^+}y^{(n)}(t),\ n = \lceil \alpha \rceil,$$

where $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to α.

For sake of simplicity, we will refer to $^{C}D^{\alpha}_{a^+}$ and $I^{\alpha}_{a^+}$ as D^{α} and I^{α}, respectively.

Lemma 2.2. [27] *If the fractional derivative $D^{\alpha}y(t)$ $(\alpha > 0)$ of a function y is integrable, then*

$$I^{\alpha}\left(D^{\alpha}y(t)\right) = y(t) - \sum_{j=0}^{[\alpha]} \frac{y^{(j)}(0)}{j!}t^{j},$$

where $[\alpha]$ denotes the integer part of α.

Definition 2.3. An equation of the form

$$y(t) = z(t) + \lambda \int_a^b \phi(t,\ s)y(s)ds, \tag{2.1}$$

is called a Fredholm integral equation of the second kind where z is a given function on $[a, b]$, y is an unknown function on $[a, b]$ and λ is a parameter. In addition, the kernel ϕ is a function on the square $G = [a, b] \times [a, b]$.

Theorem 2.4. [20] **(Successive Substitution)** *Let $z(t)$ be a continuous function defined on the interval $[a, b]$, and $\phi(t,\ s)$ be a continuous kernel defined on square $Q(a,\ b)$ and also bounded there by M. Let λ be a parameter. If $|\lambda|M(b-a) < 1$, then the unique solution to the Fredholm integral equation (2.1) is given by*

$$y(t) = z(t) + \lambda \int_a^b R(t, s; \lambda)z(s)ds,$$

where $R(t, s; \lambda)$ is the resolvent kernel given by

$$R(t, s; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1}\phi_m(x, t).$$

Lemma 2.5. [37] *Let X be a Banach space. If T be a bounded operator with $\|T\| < 1$ then $I - T$ is invertible.*

Theorem 2.6. [29] **(Arzella-Ascoli Theorem)** *A subset of $C[a, b]$ is compact if and only if it is closed, bounded and equicontinuous.*

Theorem 2.7. [22] **(Krasnoselskii's fixed point theorem).** *Let E be a Banach space, and let $C \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open sets in E with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let $S : C \bigcap(\overline{\Omega_2} \setminus \Omega_1) \longrightarrow C$ be a completely continuous operator such that either*

(1) $\|Su\| \geq \|u\|, u \in C \cap \partial\Omega_1$ *and* $\|Su\| \leq \|u\|, u \in C \cap \partial\Omega_2$, *or*
(2) $\|Su\| \leq \|u\|, u \in C \cap \partial\Omega_1$ *and* $\|Su\| \geq \|u\|, u \in C \cap \partial\Omega_2$.

Then S has a fixed point in $C \bigcap(\overline{\Omega_2} \setminus \Omega_1)$.

3. Main results

In this section, we discuss the existence of positive solutions to the nonlinear FDE (1.1) subject to the integral boundary conditions (1.2)-(1.4). Prior to presenting the main result, we discuss the following boundary value problem

$$^{C}D^{\alpha}y(t) = p(t), \quad t \in (0,\, 1), \tag{3.1}$$

satisfying the integral boundary conditions

$$y(0) - \beta_1 y'(0) = \int_0^1 g_0(s)y(s)(s)ds, \tag{3.2}$$

$$y(1) - \beta_2 y'(1) = \int_0^1 g_1(s)y(s)(s)ds, \tag{3.3}$$

$$y''(0) = 0, \tag{3.4}$$

where $p \in C\left[[0,\, 1], \mathbb{R}\right]$ and $\beta_1, \beta_2 \in \mathbb{R}$.

Lemma 3.1. *If $p \in C[[0,\, 1], \mathbb{R}]$ and $\beta_1, \beta_2 > 1$ such that $1 + \beta_1 > \beta_2$. Then the equation (3.1) with the integral boundary conditions (3.2)-(3.4) is equivalent to the following integral equation*

$$\begin{aligned} y(t) = &\int_0^1 G(t,s)p(s)ds - \frac{t+\beta_1-1}{1+\beta_1-\beta_2}\int_0^1 g_0(s)y(s)ds \\ &+\frac{t+\beta_1}{1+\beta_1-\beta_2}\int_0^1 g_1(s)y(s)ds, \end{aligned} \tag{3.5}$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \begin{cases} \frac{(\alpha-1)(\beta_2 t+\beta_1\beta_2)(1-s)^{\alpha-2}-(t+\beta_1)(1-s)^{\alpha-1}}{\Gamma(\alpha)(1+\beta_1-\beta_2)}, \\ 0 \le t \le s \le 1. \\ \\ \frac{(1+\beta_1-\beta_2)(t-s)^{\alpha-1}+(\alpha-1)(\beta_2 t+\beta_1\beta_2)(1-s)^{\alpha-2}-(t+\beta_1)(1-s)^{\alpha-1}}{(1+\beta_1-\beta_2)\Gamma(\alpha)}, \\ 0 \le s \le t \le 1. \end{cases} \tag{3.6}$$

Proof. Applying the operator I^{α} to both sides of (3.1) and using Lemma 2.2, we obtain

$$y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds + y(0) + y'(0)t + y''(0)\frac{t^2}{2}.$$

Notice that since $y''(0) = 0$ we deduce that

$$y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds + y(0) + y'(0)t, \tag{3.7}$$

and, therefore

$$y'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds + y'(0). \tag{3.8}$$

Satisfying the integral boundary conditions (3.2) and (3.3), we obtain the following linear system

$$y(0) - \beta_1 y'(0) = \int_0^1 g_0(s)y(s)ds, \tag{3.9}$$

$$y(0) + (1-\beta_2)y'(0) = \int_0^1 g_1(s)y(s)ds - C_1 + \beta_2 C_2, \tag{3.10}$$

where $C_1 = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds$ and $C_2 = \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds$ are constants. Therefore, as a result of solving the linear system (3.9)-(3.10), the values of $y(0)$ and $y'(0)$ are given by

$$y(0) = \frac{\beta_1\beta_2 C_2 - \beta_1 C_1}{1+\beta_1-\beta_2} + \frac{1-\beta_1}{1+\beta_1-\beta_2}\int_0^1 g_0(s)y(s)ds + \frac{\beta_1}{1+\beta_1-\beta_2}\int_0^1 g_1(s)y(s)ds,$$

and

$$y'(0) = \frac{\beta_2 C_2 - C_1}{1+\beta_1-\beta_2} - \frac{1}{1+\beta_1-\beta_2}\int_0^1 g_0(s)y(s)ds + \frac{1}{1+\beta_1-\beta_2}\int_0^1 g_1(s)y(s)ds.$$

Substitution of $y(0)$ and $y'(0)$ into Eq. (3.7) gives

$$\begin{aligned} y(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds - \frac{(\beta_1+t)}{\beta_1+1-\beta_2}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds \\ &+\frac{\beta_2(t+\beta_1)}{\beta_1+1-\beta_2}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds \\ &+\frac{\beta_1+t}{1+\beta_1-\beta_2}\int_0^1 g_1(s)y(s)ds - \frac{t+\beta_1-1}{1+\beta_1-\beta_2}\int_0^1 g_0(s)y(s)ds, \end{aligned}$$

which can be written as

$$\begin{aligned} &y(t) = \\ &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds - \int_0^t \frac{(\beta_1+t)}{\beta_1+1-\beta_2}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds \\ &-\int_t^1 \frac{(\beta_1+t)}{\beta_1+1-\beta_2}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)ds + \int_0^t \frac{\beta_2(\beta_1+t)}{\beta_1+1-\beta_2}\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds \\ &+\int_t^1 \frac{\beta_2(\beta_1+t)}{\beta_1+1-\beta_2}\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)ds + \frac{\beta_1+t}{\beta_1+1-\beta_2}\int_0^1 g_1(s)y(s)ds \\ &-\frac{t+\beta_1-1}{1+\beta_1-\beta_2}\int_0^1 g_0(s)y(s)ds. \end{aligned} \tag{3.11}$$

Finally, it can be easily verified that $y(t)$ in (3.11) is equivalent to the one in Eq. (3.5). This completes the proof. □

Remark 3.2. The Green's function $G(t,\ s)$ given in (3.6) is positive and continuous for all $(t,\ s) \in [0,\ 1] \times [0,\ 1]$.

Proof. The continuity is plain. It is enough to show that $G(t,\ s) > 0$. Since, $1+\beta_1 > \beta_2 > 1, \alpha \in [2,3)$ and $0 < s,t < 1$ we have $\beta_2(\alpha-1) > 1$. Hence $\beta_2(\alpha-1) > 1-s$ and $\beta_2(\alpha-1)(t+\beta_1) > (t+\beta_1)(1-s)$ or $(\alpha-1)(\beta_2 t+\beta_1\beta_2) > (t+\beta_1)(1-s)$. Then

$$(\alpha-1)(\beta_2 t+\beta_1\beta_2)(1-s)^{\alpha-2} - (t+\beta_1)(1-s)^{\alpha-1} > 0. \tag{3.12}$$

On other hand $(1+\beta_1-\beta_2)(t-s)^{\alpha-1} > 0$ when $0 \le s \le t \le 1$. Hence

$$\begin{aligned}(1+\beta_1-\beta_2)(t-s)^{\alpha-1} &+ (\alpha-1)(\beta_2 t+\beta_1\beta_2)(1-s)^{\alpha-2} \\ &- (t+\beta_1)(1-s)^{\alpha-1} > 0.\end{aligned} \tag{3.13}$$

Therefore, the result is concluded by (3.12) and (3.13). □

Lemma 3.3. *Let $\beta_1,\ \beta_2 > 1$ such that $1+\beta_1 > \beta_2$. Then there exists a positive continuous function $\gamma : [0,\ 1] \longrightarrow \mathbb{R}$ such that $G(t,s) \ge \gamma(t)G(s,\ s)$ for all $t,\ s \in [0,\ 1]$. Moreover, $\gamma_0 := \min\{\gamma(t)\ :\ t \in [0,1]\} > 0$.*

Proof. Define $k_1(t)$ and $k_2(t)$ as follows

$$k_1(t) = \frac{\beta_2(t+\beta_1)}{1+\beta_1-\beta_2} \ \text{ and } \ k_2(t) = \frac{t+\beta_1}{1+\beta_1-\beta_2}.$$

Let,

$$K := \max\{k_1(t), k_2(t) : t \in [0,1]\}$$

and

$$\gamma(t) = \frac{1}{K}\min\{k_1(t),\ k_2(t) : t \in [0,1]\}.$$

It is clear that

$$K = \frac{\beta_2(1+\beta_1)}{1+\beta_1-\beta_2},\ \ \gamma(t) = \frac{t+\beta_1}{\beta_2(1+\beta_1)} \ \text{ and } \ 0 < \gamma(t) < 1.$$

Moreover, for $t \le s$

$$\begin{aligned}&\gamma(t)G(s,s) = \\ &\frac{t+\beta_1}{\beta_2(1+\beta_1)}\frac{(\alpha-1)(\beta_2 s+\beta_1\beta_2)(1-s)^{\alpha-2} - (s+\beta_1)(1-s)^{\alpha-1}}{\Gamma(\alpha)(1+\beta_1-\beta_2)} \\ =&\frac{t+\beta_1}{\beta_2(1+\beta_1)}\frac{\beta_2(\alpha-1)(s+\beta_1)(1-s)^{\alpha-2} - (s+\beta_1)(1-s)^{\alpha-1}}{\Gamma(\alpha)(1+\beta_1-\beta_2)} \\ \le&\frac{s+\beta_1}{\beta_2(1+\beta_1)}\frac{(\alpha-1)(\beta_2 t+\beta_1\beta_2)(1-s)^{\alpha-2} - (t+\beta_1)(1-s)^{\alpha-1}}{\Gamma(\alpha)(1+\beta_1-\beta_2)} \\ \le&G(t,s)\end{aligned}$$

and similarly for $s \leq t$ we have

$$\begin{aligned} G(t,s) &\geq \gamma(t)\frac{-(s+\beta_1)(1-s)^{\alpha-1}+(\alpha-1)(\beta_2 s+\beta_1\beta_2)(1-s)^{\alpha-2}}{(1+\beta_1-\beta_2)\Gamma(\alpha)} \\ &= \gamma(t)G(s,\ s). \end{aligned}$$

Notice that, the continuous function $\gamma(t)$ on the compact interval $[0,1]$ has a minimum at some point t_0 in $[0,1]$. Hence $\gamma_0 = \gamma(t_0) > 0$. This completes the proof. □

Lemma 3.4. *Let $\beta_1,\ \beta_2 > 1$ such that $1+\beta_1 > \beta_2$. Suppose that $g_0,\ g_1$ are continuous and positive functions such that an auxiliary function $\phi : [0,1]\times[0,1] \to \mathbb{R}$ defined by*

$$\phi(t,s) = \frac{(t+\beta_1)}{1+\beta_1-\beta_2}g_1(s) + \frac{t+\beta_1-1}{1+\beta_1-\beta_2}g_0(s),$$

so that it satisfies

$$\begin{aligned} 0 \leq m &:= \min\{\phi(t,s) : t,\ s \in [0,1]\} \\ &\leq M := \max\{\phi(t,s) : t,\ s \in [0,1]\} < 1. \end{aligned}$$

If we define an operator $A : C[0,\ 1] \longrightarrow C[0,\ 1]$ by

$$(Ay)(t) = \int_0^1 \phi(t,s)y(s)ds. \tag{3.14}$$

Then A is a bounded linear operator and $A(P) \subset P$. Moreover, $(I-A)$ is invertible and $\|(I-A)^{-1}\| \leq \dfrac{1}{1-M}$.

Proof. It is clear that A is linear and $|(Ay)(t)| \leq M\|y\|$. This shows that A is a bounded linear operator. Let $y \in P$, then $y(s) \geq 0$ for every $s \in [0,\ 1]$. Since, $\phi(t,s) \geq 0$ it follows that $(Ay)(t) \geq 0$ for every $t \in [0,\ 1]$. Hence $A(P) \subset P$. Since $M < 1$ we have $\|Ay\| \leq M\|y\| < \|y\|$; i.e. $\|A\| < 1$. From Lemma 2.5, it may by concluded that $(I-A)$ is invertible.
Now we want to prove $\|(I-A)^{-1}\| \leq \frac{1}{1-M}$. It is necessary to find the expression for $(I-A)^{-1}$. For this purpose, we use the theory of Fredholm integral equations. We have $y(t) = (I-A)^{-1}z(t)$ if and only if $y(t) = z(t)+(Ay)(t)$ for each $t \in [0,\ 1]$. Using the operator (3.14) we get

$$y(t) = z(t) + \int_0^1 \phi(t,s)y(s)ds. \tag{3.15}$$

Since $M < 1$, then unique solution of Fredholm integral equation (3.15) is given by

$$y(t) = z(t) + \int_0^1 R(t,s)z(s)ds,$$

where the resolvent kernel $R(t,s)$ given by

$$R(t,\ s) = \sum_{m=1}^{\infty} \phi_m(t,s). \tag{3.16}$$

Here, $\phi_1(t,s)=\phi(t,s)$ and ϕ_m is given by

$$\phi_m(t,s)=\int_0^1 \phi(t,\tau)\phi_{m-1}(\tau,s)ds, \quad m=2,3,\cdots.$$

The series (3.16) is convergent because $|\phi(t,s)|\leq M<1$. Thus, we have

$$R(t,s)\leq M+M^2+\ldots=\frac{M}{1-M}. \tag{3.17}$$

Therefore, the expression of $(I-A)^{-1}$ is given by

$$(I-A)^{-1}z(t)=z(t)+\int_0^1 R(t,s)z(s)ds. \tag{3.18}$$

Equation (3.18) yields that

$$\begin{aligned}|(I-A)^{-1}z(t)|\leq&|z(t)|+\frac{M}{1-M}\int_0^1 z(s)ds\\ \leq&\|z\|\left\{1+\frac{M}{1-M}\right\}=\frac{1}{1-M}\|z\|.\end{aligned}$$

This completes the proof of Lemma 3.4. □

Remark 3.5. Since $\phi(t,s)\geq m$ for each $(t,s)\in[0,\ 1]\times[0,1]$, we can easily show that $R(t,s)\geq\frac{m}{1-m}$.

In the following, we study the existence of positive solution (1.1) with boundary conditions (1.2)-(1.4). For this purpose, we introduce some notations and basic assumptions.

Consider the boundary value problem (1.1)-(1.4) and integral equation

$$\begin{aligned}y(t)=&\int_0^1 G(t,s)f(s,y(s))ds-\frac{t+\beta_1-1}{1+\beta_1-\beta_2}\int_0^1 g_0(s)y(s)ds\\ &+\frac{t+\beta_1}{1+\beta_1-\beta_2}\int_0^1 g_1(s)y(s)ds\\ =&\int_0^1 G(t,s)y(s)ds+\int_0^1\phi(t,s)y(s)ds.\end{aligned} \tag{3.19}$$

Define the nonlinear operator $T:C[0,1]\longrightarrow C[0,1]$ by

$$Ty(t)=\int_0^1 G(t,s)f(s,y(s))ds. \tag{3.20}$$

In view of (3.14) and by applying (3.20), equation (3.19) can be expressed by

$$y(t)=Ty(t)+Ay(t), \quad t\in[0,\ 1]. \tag{3.21}$$

According to Lemma 3.1, the fixed point of $(T+A)y(t)$ coincide with the solutions of boundary value problem (1.1)-(1.4).

Consider the cone P_0 define by

$$P_0=\left\{y\in P\ :\ \min_{t\in[0,1]}y(t)\geq\frac{1-M}{1-m}\gamma_0\|u\|\right\},$$

where m and M are as in Lemma 3.4. From Lemma 3.1, we have y is a solution of (3.21) if and only if it is a solution of

$$y(t) = (I - A)^{-1}Ty(t), \tag{3.22}$$

i.e. y is a fixed point of the operator $S := (I - A)^{-1}T$.

We introduce following assumptions.
The nonlinear continuous function $f : [0, 1] \times [0, +\infty) \longrightarrow [0, +\infty)$ satisfies:
(A1) There exist $L_1 > 0$ and $u \in P$ with $\int_0^1 G(t, s)u(s)ds \leq 1$ such that

$$f(t, y) \leq yu(t)(1 - M), \quad \forall\, y \in (0, L_1],\ t \in [0,\ 1]. \tag{3.23}$$

(A2) There exist $L_2 > L_1$ and $v \in P$ with $\int_0^1 G(s, s)v(s)ds \geq \frac{1-m}{(1-M)\gamma_0}$ such that

$$f(t, y) \geq \frac{yv(t)(1 - m)}{\gamma_0}, \quad \forall\, y \geq L_2,\ t \in [0, 1]. \tag{3.24}$$

Now we show that the FDE (1.1) - (1.4) under the assumptions (A1) and (A2) has positive solution.

Theorem 3.6. *If $\beta_1,\ \beta_2 > 1$ such that $1+\beta_1 > \beta_2$ and Remark 3.4 is satisfied. Then FDE (1.1) - (1.4) under the assumptions (A1) and (A2) has positive solution.*

Proof. Note that, the solution of (3.19), (3.21) and (3.22) are equivalent. It follows from (3.18) that y is a solution of (3.22) if and only if

$$y(t) = (Ty)(t) + \int_0^1 R(t, s)(Ty)(s)ds,$$

or equivalently

$$y(t) = \int_0^1 G(t,\ s)f(s,\ y(s))ds + \int_0^1 R(t,\ s)\int_0^1 G(s,\ \tau)f(\tau,\ y(\tau))d\tau ds.$$

We want to show that the operator S, defined by

$$Sy(t) = \int_0^1 G(t, s)f(s, y(s))ds + \int_0^1 R(t, s)\int_0^1 G(s, \tau)f(\tau, y(\tau))d\tau ds \tag{3.25}$$

satisfies Theorem 2.7 with considering $E := C[0, 1]$ and $C := P_0$.
We continue the process in two stages as follows.
Step 1. We prove that, $S : C \longrightarrow C$ is a completely continuous operator. For this purpose we must prove that: (i) S is continuous and positive, (ii) S maps C into itself, (iii) S is bounded, and (iv) S is equicontinuous.
Since, $G(t, s)$ and f are continuous, hence S is continuous. If $y \in P_0$, then $Sy \geq 0$, because in view of Remark 3.2 and Lemma 3.4, $f, \phi(t, s), R(t, s)$ and $G(t, s)$ are

positive. From 3.3 and 3.5, we have

$$\begin{aligned}(Sy)(t) \geq & \int_0^1 \gamma_0 G(s,s) f(s,y(s)) ds \\ & + \int_0^1 R(t,s) \int_0^1 \gamma_0 G(s,\tau) f(\tau, y(\tau)) d\tau ds \\ \geq & \gamma_0 \left[1 + \frac{m}{1-m}\right] \int_0^1 G(s,s) f(s,y(s)) ds \\ = & \frac{\gamma_0}{1-m} \int_0^1 G(s,s) f(s,y(s)) ds. \end{aligned} \tag{3.26}$$

Moreover,

$$\begin{aligned}(Sy)(t) \leq & \int_0^1 G(t,s) f(s,y(s)) ds + \int_0^1 R(t,s) \int_0^1 G(s,\tau) f(\tau,y(\tau)) d\tau ds \\ \leq & \int_0^1 G(t,s) f(s,y(s)) ds + \frac{M}{1-M} \int_0^1 ds \int_0^1 G(s,\ \tau) f(\tau, y(\tau)) d\tau \\ \leq & \left(1 + \frac{M}{1-M}\right) \int_0^1 G(t,s) f(s,y(s)) ds \\ = & \frac{1}{1-M} \int_0^1 G(t,s) f(s,y(s)) ds. \end{aligned} \tag{3.27}$$

This implies that

$$\|Sy\| \leq \frac{1}{1-M} \int_0^1 G(t,s) f(s,y(s)) ds. \tag{3.28}$$

By combining (3.26) and (3.28) we obtain

$$(Sy)(t) \geq \frac{\gamma_0}{1-m} \geq \frac{1-M}{1-m} \gamma_0 \|Sy\|.$$

This shows that $Sy \in P_0$.
Let $\Lambda \subset P_0$ be bounded, which is to say there exists a positive constant $K > 0$ such that $\|y\| \leq K$ for all $y \in \Lambda$. Set

$$L = \max \{|f(t,\ y)| + 1 \ :\ 0 \leq t \leq 1,\ 0 \leq y \leq K\}.$$

Then for all $y \in \Lambda$, we have

$$\begin{aligned}|(Sy)(t)| = & |(I-A)^{-1} T y(t)| \leq \frac{1}{1-M} \int_0^1 G(t,\ s) f(s,\ y(s)) ds \\ \leq & \frac{L}{1-M} \int_0^1 G(t,\ s) ds. \end{aligned}$$

That is, the set $S(\Lambda)$ is bounded in P_0. On the Other hand, we have

$$\begin{aligned}
|(Ty)'(t)| =& \Big| \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,y(s))ds \\
&- \frac{1}{\beta_1+1-\beta_2} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))ds \\
&+ \frac{\beta_2}{\beta_1+1-\beta_1} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,y(s))ds \Big| \\
&\leq \frac{L}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} ds \\
&+ \frac{L}{(\beta_1+1-\beta_2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds \\
&+ \frac{Lb}{(\beta_1+1-\beta_2)\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \\
&\leq \frac{L}{\Gamma(\alpha)} + \frac{L}{(\beta_1+1-\beta_2)\Gamma(\alpha+1)} + \frac{L\beta_2}{(\beta_1+1-\beta_2)\Gamma(\alpha)} := N.
\end{aligned}$$

Then, for each $y \in \Lambda$ we have

$$\begin{aligned}
|Sy(t_1) - Sy(t_2)| &\leq \frac{1}{1-M} |Ty(t_1) - Ty(t_2)| \\
&\leq \frac{1}{1-M} \int_{t_1}^{t_2} |(Ty)'(s)| ds \leq \frac{N(t_2-t_1)}{1-M}.
\end{aligned}$$

Hence, $S(\Lambda)$ is equicontinuous. Therefore, by Arzela-Ascoli theorem, it is concluded that $\overline{S(\Lambda)}$ is compact. Therefore, $S : P_0 \longrightarrow P_0$ is a completely continuous operator.

Step 2. We have to prove that there are Ω_1 and Ω_2 in E such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and $S : C \bigcap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow C$ satisfying Theorem 2.7.
Let $y \in P_0$ with $\|y\| = L_1$. From (3.27) and (A1) we obtain

$$\begin{aligned}
(Sy)(t) \leq& \frac{1}{1-M} \int_0^1 G(s,s) f(s,y(s)) ds \\
\leq& \frac{1}{1-M} \int_0^1 G(t,s) u(s) y(s) (1-M) ds \\
\leq& \|y\| \int_0^1 G(t,s) u(s) ds \leq \|y\|.
\end{aligned}$$

Thus, $\|Sy\| \leq \|y\|$. Let $\Omega_1 := \{y \in C[0,1] \ : \ \|y\| < L_1\}$. Then, we have $\|Sy\| \leq \|y\|$ for $y \in P_0 \bigcap \partial\Omega_1$. Set

$$\widehat{L_2} = \max\left\{2L_1, \ \frac{1-m}{1-M}\frac{L_2}{\gamma_0}\right\} \ \text{ and } \ \Omega_2 := \left\{y \in C[0,1] \ : \ \|y\| < \widehat{L_2}\right\}.$$

For a $y \in P_0$ with $\|y\| = \widehat{L_2}$ we have

$$\begin{aligned}\min_{t\in[0,1]} y(t) &\geq \frac{(1-M)\gamma_0}{1-m}\|y\| \geq \frac{(1-M)\gamma_0}{1-m}\widehat{L_2} \\ &\geq \frac{(1-M)\gamma_0}{1-m}\frac{1-m}{(1-M)\gamma_0}L_2 \geq L_2.\end{aligned}$$

From (3.26) and **(A2)** we get

$$\begin{aligned}(Sy)(t) &\geq \frac{\gamma_0}{1-m}\int_0^1 G(s,s)f(s,y(s))ds \\ &\geq \frac{\gamma_0}{1-m}\int_0^1 G(s,s)\frac{v(s)y(s)(1-m)}{\gamma_0}ds \\ &\geq \int_0^1 G(s,s)v(s)y(s)ds.\end{aligned}$$

Since $y \in P_0$, we have $y(s) \geq \frac{(1-M)\gamma_0}{1-m}\|y\|$ for every $s \in [0,1]$. Then, the above inequality yields

$$(Sy)(t) \geq \frac{(1-M)\gamma_0}{1-m}\int_0^1 G(s,s)v(s)ds\|y\|.$$

Hence, $\|Sy\| \geq \|y\| \quad for\ y \in P_0 \bigcap \partial\Omega_2$. In view of Theorem 2.7 (2), the operator S has a fixed point in $P_0 \bigcap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$ and hence this fixed point is a solution of (1.1) with boundary conditions (1.2)-(1.4) which is positive. This completes the proof.
With L_1 and L_2 as in **(A1)** and **(A2)**, we assume that f satisfies:
(A3) There exists $\delta \in P$ with $\int_0^1 G(s,s)\delta(s)ds \geq \frac{1-m}{(1-M)\gamma_0}$ such that

$$f(t,y) \geq \frac{y(1-m)\delta(t)}{\gamma_0} \quad \text{for all } y \in (0, L_1] \text{ and } t \in [0,1].$$

(A4) There exists $\sigma \in P$ with $\|\sigma\| \leq \left(\int_0^1 G(t,s)ds\right)^{-1}$ such that

$$f(t,y) \leq \sigma(t)y(1-M) \quad \text{for all } y \geq L_2 \text{ and } t \in [0,1].$$

Now, under the assumptions **(A3)** and **(A4)**, we show that the FDE (1.1)-(1.4) has a positive solution. □

Theorem 3.7. *If $\beta_1, \beta_2 > 1$ such that $1 + \beta_1 > \beta_2$ and assume that Remark 3.4 is satisfied. Then (1.1) with the boundary conditions (1.2)-(1.4)under assumptions* **(A3)** *and* **(A4)** *has at least one positive solution.*

Proof. Since the existence of a solution for the FDE (1.1)-(1.4) is equivalent to the existence of a fixed point for the operator $S : P_0 \longrightarrow P_0$, defined in (3.25), which is a completely continuous operator. If $y \in P_0$ with $\|y\| = L_1$. Then, after making use of

(A3), (3.26) gives

$$\begin{aligned} Sy(t) \geq & \frac{\gamma_0}{1-m}\int_0^1 G(s,s)\frac{\delta(s)y(s)(1-m)}{\gamma_0}ds \geq \int_0^1 G(s,s)\delta(s)y(s)ds \\ \geq & \frac{(1-M)\gamma_0}{1-m}\int_0^1 G(s,s)\delta(s)ds\|y\| \geq \|y\|. \end{aligned}$$

Hence, $\|Sy\| \geq \|y\|$. Let $\Omega_1 = \{y \in C[0,\ 1] : \|y\| < L_1\}$. Therefore, $\|Sy\| \geq \|y\|$ for $y \in P_0 \bigcap \partial\Omega_1$. Since f is continuous and $\mu_f := \max\{f(t,\ y); t \in [0,1], y \in [0,\ L_2]\}$ is well defined. For $\overline{L}_2$ defined by

$$\overline{L}_2 = \max\left(2L_1,\ \frac{\mu_f}{(1-M)\|\sigma\|}\right),$$

then for $y \in P_0$ with $\|y\| = \overline{L}_2$, we have $y(t) \leq \overline{L}_2$ for all $t \in [0,\ 1]$ and $f(t,\ y) \leq \max(\mu_f, y(1-M)\|\sigma\|)$. Since $\mu_f \leq (1-M)\|\sigma\|\overline{L}_2$ it follows that

$$f(t,y) \leq (1-M)\|\sigma\|\overline{L}_2 = (1-M)\|\sigma\|\|y\|.$$

The last inequality, together with (3.27) and assumption (A4) yields

$$\begin{aligned} (Sy)(t) \leq & \frac{1}{1-M}\int_0^1 G(t,s)f(s,y(s))ds \\ \leq & \frac{1}{1-M}\int_0^1 G(t,s)(1-M)\|\sigma\|\|y\|ds \\ \leq & \left(\|\sigma\|\int_0^1 G(t,s)ds\right)\|y\| \leq \|y\|. \end{aligned}$$

Therefore $\|Sy\| \leq \|y\|$ for $y \in P_0$ with $\|y\| = \overline{L}_2$. Let $\Omega_2 = \{y \in C[0,1] : \|y\| < \overline{L}_2\}$. Then we have $\|Sy\| \leq \|y\|$ for $y \in P_0 \bigcap \partial\Omega_2$. Therefore, case (1) of Theorem 2.7 is satisfied. It follows that S has a fixed point in $P_0 \bigcap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$ and this fixed point is a solution of (1.1) which is positive. This completes the proof. □

4. Example

In this section, we provide two examples to illustrate the application of our results.

Example 4.1. Consider the fractional differential equation

$$D^{\frac{5}{2}}y(t) = \frac{t}{1+t}y^2(t), \quad 0 < t < 1, \tag{4.1}$$

satisfying the boundary conditions

$$y(0) - \beta_1 y'(0) = \int_0^1 \frac{1}{2}y(s)ds, \tag{4.2}$$

$$y(1) - \beta_2 y'(1) = \int_0^1 \frac{1}{3}y(s)ds, \tag{4.3}$$

and

$$y''(0) = 0. \tag{4.4}$$

In this example, we have $\beta_1 = 3, \beta_2 = 2$ and $g_0(t) = \frac{1}{2}, g_1(t) = \frac{1}{3}$. Then $\beta_1+1-\beta_2 = 2$ and $\phi(t,s) = \frac{t+7}{12}, m = \frac{7}{12}, M = \frac{8}{12}$. Also, $\gamma_0 = \frac{3}{8}$. The cone is given by

$$P_0 := \left\{ y \in P : \min_{t\in[0,\ 1]} y(t) \geq \frac{3}{8}\|y\| \right\}.$$

Now , let $f_0 = \lim_{y\longrightarrow 0^+} \frac{f(t,y)}{y}$ and $f_\infty = \lim_{y\longrightarrow \infty} \frac{f(t,y)}{y}$ uniformly in $t \in [0,1]$. Obviously, $f_0 = 0$. Then, there exists a constant L_1 such that $f(t,\ y) \leq \epsilon_1 y$ for all $0 \leq y \leq L_1$, where $\epsilon_1 \leq \frac{1}{12}$. Then, (A1) is satisfied with $u(t) = \frac{t+1}{4}$. On the other hand, since $f_\infty = \infty$, there exists a constant $L_2 > L_1 > 0$ such that $f(t,y) \geq \epsilon_2 y$ for all $y \geq L_2$, where $\epsilon_2 \geq \frac{5}{9}$. Then, (A2) is satisfied with $v(t) = \frac{t}{1+t}$.

Example 4.2. Consider the fractional differential equation

$$D^{\frac{5}{2}} y(t) = f(t, y(t)), \quad 0 < t < 1, \tag{4.5}$$

$$y(0) - \beta_1 y'(0) = \int_0^1 \frac{1}{3} y(s) ds, \tag{4.6}$$

$$y(1) - \beta_2 y'(1) = \int_0^1 \frac{1}{3} y(s) ds, \tag{4.7}$$

and

$$y''(0) = 0. \tag{4.8}$$

In this example, we have $\beta_1 = \beta_2 = 2$ and $g_0(t) = g_1(t) = \frac{1}{3}$. Then $\beta_1 + 1 - \beta_2 = 1$ and $\phi(t,s) = m = M = \frac{1}{3}$. Also, $\gamma_0 = \frac{1}{3}$ and $\int_0^1 G(s,s)ds = 0.6287$. The cone is defined by

$$P_0 := \left\{ y \in P : \min_{t\in[0,1]} y(t) \geq \frac{1}{3}\|y\| \right\}.$$

Now , let $f_0 = \lim_{y\longrightarrow 0^+} \frac{f(t,y)}{y}$ and $f_\infty = \lim_{y\longrightarrow \infty} \frac{f(t,y)}{y}$ uniformly in $t \in [0,1]$. Then, (**A3**) is satisfied with $\delta(t) = 5,\ f_0 \in [10,\ +\infty)$ and (A4) is satisfied with $\sigma(t) = \frac{1}{4}$ and $f_\infty \in [0, \frac{1}{6}]$.

5. Conclusions

In this work, we consider a class of nonlinear fractional differential equation with integral boundary conditions involving Caputo fractional derivative possessing a lower terminal at 0 in order to study the existence of positive solutions. To obtain the results of this article, the Guo-Krasnoselskii's fixed point theorem have been implemented. Although the present study provides some insights in the equations encountered in the existence of solutions, this existence theorem may be explored for other classes of fractional differential equations, like recent contributions in [16], which is a subject for future study.

Acknowledgment

This research was supported by Babol Noshirvani University of Technology with Grant program No. BNUT/389089/98. The authors are very grateful to the anonymous reviewers for their comments.

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