



## Existence and uniqueness of solutions of uncertain linear systems

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**Abstract** This paper presents some new definitions and theorems about a system of linear uncertain differential equations. An existence and uniqueness of solutions of the system with initial condition will be proven. Also, it will be shown that the collection of solutions of the homogeneous uncertain system is a linear space.

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**Keywords.** Existence and uniqueness; Uncertain differential equations; Uncertain systems.

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### 1. INTRODUCTION

Differential equations is a branch of mathematics which is closely related to mathematical modeling that arises in real-world problems such as physics, engineering, biology, economics and other fields. Liu process and uncertain calculus were initialized by Liu (2009) to deal with differentiation and integration of functions of uncertain processes. Furthermore, uncertain differential equations, a type of differential equations driven by the Liu process, was defined by Liu (2008). Chen and Liu in [1] considered uncertain differential equation

$$dX_t = f(X, t)dt + g(X, t)dC_t, \quad (1.1)$$

and presented the following theorem about the existence and uniqueness of solutions of (1.1).

**Theorem 1.1.** ([1]) *The uncertain differential equation (1.1) has a unique solution if the coefficients  $f(x, t)$  and  $g(x, t)$  satisfy the Lipschitz condition*

$$|f(x, t) - f(y, t)| + |g(x, t) - g(y, t)| \leq L|x - y|, \quad \forall x, y \in \mathbb{R}, \quad t \geq 0,$$

*and linear growth condition*

$$|f(x, t)| + |g(x, t)| \leq L(1 + |x|), \quad \forall x \in \mathbb{R}, \quad t \geq 0,$$

*for some constants  $L$ . Moreover, the solution is sample-continuous.*

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In this paper we will consider the following system of linear uncertain differential equations

$$dX_t = [A(t)X_t + U(t)]dt + [B(t)X_t + V(t)]dC_t, \quad (1.2)$$

where  $X_t$  is an  $n$ -dimensional uncertain process,  $C_t$  is an  $n$ -dimensional Liu process,  $A(t)$  and  $B(t)$  are  $n \times n$  matrices of integrable uncertain real functions. We will prove the existence and uniqueness of solutions of system (1.2) with weaker conditions. After that we will show that the collection of solutions of the uncertain system with  $U(t) \equiv V(t) \equiv 0$  is a linear space. First, let us have some definitions and preliminaries.

**Definition 1.2.** ([5]) Let  $\Gamma$  be a nonempty set, and  $L$  be a  $\sigma$ -algebra over  $\Gamma$ . Each element  $A \in L$  is called an event. To measure uncertain event, uncertain measure was introduced as a set function satisfying the following axioms:

**Axiom 1.** (Normality)  $M\{\Gamma\} = 1$ .

**Axiom 2.** (Duality Axiom)  $M\{A\} + M\{A^c\} = 1$  for any event  $A$ .

**Axiom 3.** (Countable Subadditivity) For every countable sequence of events  $A_1, A_2, \dots$ , we have

$$M\left\{\bigcup_{i=1}^{\infty} A_i\right\} \leq \sum_{i=1}^{\infty} M\{A_i\}.$$

**Axiom 4.** (Product Axiom) Let  $(\Gamma_k, L_k, M_k)$  be uncertainty spaces for  $k = 1, 2, \dots$ . The product uncertain measure  $M$  is an uncertain measure satisfying

$$M\left\{\prod_{k=1}^{\infty} A_k\right\} = \bigwedge_{k=1}^{\infty} M_k\{A_k\},$$

where  $A_k$  are arbitrarily chosen events from  $L_k$  for  $k = 1, 2, \dots$ , respectively.

Let  $\Gamma$  be a nonempty set,  $L$  be a  $\sigma$ -algebra over  $\Gamma$ , and  $M$  be an uncertain measure. Then the triplet  $(\Gamma, L, M)$  is called an uncertainty space ([5]). Suppose  $T$  is a totally ordered set (e.g time). An uncertain process is a function  $X_t$  from  $T \times \Gamma$  to the set of real numbers such that  $\{\gamma \in \Gamma \mid X_t(\gamma) \in B\}$  is an event for any Borel set  $B$  of real numbers at each time  $t$  ([3]). Let  $X_t$  be an uncertain process, then for each  $\gamma \in \Gamma$ , the function  $X_t(\gamma)$  is called a sample path of  $X_t$  ([3]).

**Definition 1.3.** ([4]) An uncertain process  $C_t$  is said to be a Liu process if

- (i)  $C_0 = 0$  and almost all sample paths are Lipschitz continuous,
- (ii)  $C_t$  has stationary and independent increments,
- (iii) every increment  $C_{s+t} - C_s$  is a normal uncertain variable with expected value 0 and variance  $t^2$ .

Let  $X_t$  be an uncertain process and  $C_t$  be a Liu process. For any partition of closed interval  $[a, b]$  with  $a = t_1 < t_2 < \dots < t_{n+1} = b$ , the mesh is written as

$$\Delta = \max_{1 \leq i \leq n} |t_{i+1} - t_i|.$$

Then, Liu integral of  $X_t$  with respect to  $C_t$  is defined as

$$\int_b^a X_t dC_t = \lim_{\Delta \rightarrow 0} \sum_{i=1}^n X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i}),$$



provided that the limit exists almost surely and is finite. In this case, the uncertain process  $X_t$  is said to be Liu integrable ([4]). Suppose  $C_t$  is a Liu process, and  $f$  and  $g$  are two functions. Then

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t \tag{1.3}$$

is called an uncertain differential equation. A solution is an uncertain process  $C_t$  that satisfies (1.3) identically in  $t$  ([3]).

**Remark 1.4.** The uncertain differential equation (1.3) is equivalent to the uncertain integral equation

$$X_t = X_0 + \int_0^t f(s, X_s)ds + \int_0^t g(s, X_s)dC_s.$$

Our aim in the next section is to prove the existence and uniqueness of solutions of uncertain linear system with an initial condition under weaker conditions than Theorem 1.1.

## 2. UNCERTAIN LINEAR SYSTEMS

The system of linear uncertain differential equations are important both from theoretical point of view and from the practical dimension. Their application significance can be found in solving the problem of linear mechanical systems and linear electric circuits and other uncertain linear systems. Their other important application is in the initial approximation of nonlinear problems. In this section, we will present the initial concepts and then the existence and uniqueness theorem for uncertain linear systems.

**Definition 2.1.** ([8]) Let  $C_{it}, i = 1, 2, \dots, n$  be independent Liu processes. Then,  $C_t = (C_{1t}, C_{2t}, \dots, C_{nt})^T$  is called an  $n$ -dimensional Liu process.

**Definition 2.2.** Let  $t$  be a positive real variable and  $X_t = (X_{1t}, X_{2t}, \dots, X_{nt})^T$  is an  $n$ -dimensional uncertain process whose elements  $X_{jt}$  are integrable uncertain processes. Also,  $A(t) = [a_{ij}(t)]$ ,  $B(t) = [b_{ij}(t)]$  are  $n \times n$  matrices of integrable uncertain real functions and  $U_t = (u_1(t), u_2(t), \dots, u_n(t))$ ,  $V_t = (v_1(t), v_2(t), \dots, v_n(t))$  are  $n$ -component vectors of uncertain integrable real functions. Then

$$dX_t = [A(t)X_t + U(t)]dt + [B(t)X_t + V(t)]dC_t \tag{2.1}$$

is called a system of uncertain linear differential equations.

**Definition 2.3.** Let  $C_t = (C_{1t}, C_{2t}, \dots, C_{nt})^T$  be an  $n$ -dimensional Liu process and  $A(t) = [a_{ij}(t)]$  be an  $n \times n$  matrix of uncertain integrable real functions and  $X_t = (X_{1t}, X_{2t}, \dots, X_{nt})^T$  be an  $n$ -dimensional uncertain process whose elements  $X_{jt}$  and all  $a_{ij}(t)X_{jt}$  are integrable uncertain processes. Then, the Liu integral of  $A(t)X_t$  with respect to  $C_t$  on  $[a, b]$  is defined by



$$\int_a^b A(t)X_t dC_t = \begin{pmatrix} \sum_{j=1}^n \int_b^a a_{1j}(t)X_{jt} dC_{jt} \\ \sum_{j=1}^n \int_b^a a_{2j}(t)X_{jt} dC_{jt} \\ \vdots \\ \sum_{j=1}^n \int_b^a a_{mj}(t)X_{jt} dC_{jt} \end{pmatrix}.$$

In this case,  $A(t)X_t$  is said to be Liu integrable with respect to  $C_t$ .

**Remark 2.4.** The system (2.1) is equivalent to the uncertain integral equation

$$X_t = X_0 + \int_0^t [A(s)X_s + U(s)]ds + \int_0^t [B(s)X_s + V(s)]dC_s.$$

**Definition 2.5.** ([7]) Let  $A = [a_{ij}]$  be an  $m \times n$  matrix and  $X = (X_1, X_2, \dots, X_n)^T$  is a  $n$ -dimensional vector, then the infinite norm of A and X are defined as follows.

$$|A| = \bigvee_{i=1}^m \sum_{j=1}^n |a_{ij}|, \quad |X| = \bigvee_{i=1}^n |x_i|.$$

In [6], Yao Kai proved the following theorem about the linearity of multi-dimensional Liu integral.

**Theorem 2.6.** ([6]) Let  $C_t$  be an  $n$ -dimensional Liu process, and  $X_t$  and  $Y_t$  be two  $n$ -dimensional Liu integrable uncertain processes on  $[a, b]$ . Then,

$$\int_a^b (\alpha X_t + \beta Y_t) dC_t = \alpha \int_a^b X_t dC_t + \beta \int_a^b Y_t dC_t,$$

for any real numbers  $\alpha$  and  $\beta$ .

In [2] the authors, have proved the following theorem in the case of Liu integrable uncertain matrix process.

**Theorem 2.7.** ([2]) Let  $C_t$  be an  $n$ -dimensional Liu process, and  $Y_t$  be an  $m \times n$  Liu integrable uncertain matrix process. Then

$$\left| \int_a^b Y_t(\gamma) dC_t(\gamma) \right| \leq K(\gamma) \int_a^b |Y_t(\gamma)| dt, \quad \forall \gamma \in \Gamma$$

where  $K(\gamma)$  is the Lipschitz constant of the sample path  $C_t(\gamma)$ .

In the following, we will prove a theorem similar to the theorem above for uncertain linear systems. First, let us state some preliminaries that we will use to prove the theorem.

**Theorem 2.8.** Let  $X_t = (X_{1t}, X_{2t}, \dots, X_{nt})^T$  be a Liu integrable uncertain process,  $C_t$  be an  $n$ -dimensional Liu process and  $A(t)$  is an  $n \times n$  Liu integrable uncertain



matrix process on  $[a, b]$ . Then

(a)  $A(t)X_t$  is time integrable on  $[a, b]$ , and

$$\left| \int_a^b A(t)X_t(\gamma)dt \right| \leq \int_a^b |A(t)||X_t(\gamma)|dt,$$

for each sample path  $X_t(\gamma)$  and  $A(t)$ .

(b)  $A(t)X_t$  is Liu integrable on  $[a, b]$ , and

$$\left| \int_a^b A(t)X_t(\gamma)dC_t \right| \leq \int_a^b |A(t)||X_t(\gamma)|dC_t,$$

for each sample path  $X_t(\gamma)$  and  $A(t)$ .

*Proof.* (a) Since the uncertain process  $X_t$  is sample-continuous and  $A(t)$  is continuous, its almost all sample paths and  $A(t)X_t$  are continuous functions of  $t$ . Therefore, the limit

$$\lim_{\Delta \rightarrow 0} \sum_{i=1}^n A(t_i)X_{t_i}(t_{i+1} - t_i),$$

exists almost surely and is finite for any partition of the closed interval  $[a, b]$ , and  $A(t)X_t$  is integrable on  $[a, b]$ . Thus,

$$\begin{aligned} \left| \int_a^b A(t)X_t(\gamma)dt \right| &= \left| \sum_{i=1}^n \int_a^b a_{ij}(t)X_{jt}(\gamma)dt \right| \\ &\leq \sum_{i=1}^n \left| \int_a^b a_{ij}(t)X_{jt}(\gamma)dt \right| \\ &\leq \int_a^b \sum_{i=1}^n |a_{ij}(t)X_{jt}(\gamma)|dt \\ &\leq \int_a^b |A(t)||X_t(\gamma)|dt. \end{aligned}$$

(b) Since the uncertain Liu integrable process  $X_t$  is sample-continuous and  $A(t)$  is continuous, its almost all sample paths and  $A(t)X_t$  are continuous functions of  $t$ . Therefore, the limit

$$\lim_{\Delta \rightarrow 0} \sum_{i=1}^n A(t_i)X_{t_i}(C_{t_{i+1}} - C_{t_i}),$$



exists almost surely and is finite for any partition of the closed interval  $[a, b]$ , and  $A(t)X_t$  is Liu integrable with respect to  $C_t$  on  $[a, b]$ . Then,

$$\begin{aligned} \left| \int_a^b A(t)X_t(\gamma)dC_t \right| &= \bigvee_{i=1}^n \left| \sum_{j=1}^n \int_a^b a_{ij}(t)X_{jt}(\gamma)dC_{jt} \right| \\ &\leq \bigvee_{i=1}^n \sum_{j=1}^n \left| \int_a^b a_{ij}(t)X_{jt}(\gamma)dC_{jt} \right| \\ &\leq \int_a^b \bigvee_{i=1}^n \sum_{j=1}^n |a_{ij}(t)X_{jt}(\gamma)|dC_{jt} \\ &\leq \int_a^b |A(t)X_t(\gamma)|dC_t. \end{aligned}$$

and the proof is completed.  $\square$

**Theorem 2.9.** Let  $C_t$  be an  $n$ -dimensional Liu process,  $X_t$  be an  $n$ -component uncertain process which is Liu integrable on every subinterval of  $[a, b]$ , and  $A(t)$  is an  $n \times n$  matrix. If  $c \in [a, b]$ , then

$$\int_a^b A(t)X_t dC_t = \int_a^c A(t)X_t dC_t + \int_c^b A(t)X_t dC_t.$$

*Proof.* Since  $X_t$  is Liu integrable with respect to  $C_t$  on  $[a, b]$ , the uncertain process  $X_{it}$  is Liu integrable with respect to  $C_t$  on  $[a, b]$ . Thus,  $X_{it}$  is Liu integrable with respect to  $C_t$  on each subinterval of  $[a, b]$ . Therefore,

$$\int_a^b A(t)X_t dC_t = \begin{pmatrix} \sum_{j=1}^n \int_a^b a_{1j}X_{jt}dC_{jt} \\ \sum_{j=1}^n \int_a^b a_{2j}X_{jt}dC_{jt} \\ \vdots \\ \sum_{j=1}^n \int_a^b a_{nj}X_{jt}dC_{jt} \end{pmatrix}$$



$$\begin{aligned}
 &= \begin{pmatrix} \sum_{j=1}^n \int_a^c a_{1j} X_{jt} dC_{jt} + \sum_{j=1}^n \int_c^b a_{1j} X_{jt} dC_{jt} \\ \sum_{j=1}^n \int_a^c a_{2j} X_{jt} dC_{jt} + \sum_{j=1}^n \int_c^b a_{2j} X_{jt} dC_{jt} \\ \vdots \\ \sum_{j=1}^n \int_a^c a_{nj} X_{jt} dC_{jt} + \sum_{j=1}^n \int_c^b a_{nj} X_{jt} dC_{jt} \end{pmatrix} \\
 &= \begin{pmatrix} \sum_{j=1}^n \int_a^c a_{1j} X_{jt} dC_{jt} \\ \sum_{j=1}^n \int_a^c a_{2j} X_{jt} dC_{jt} \\ \vdots \\ \sum_{j=1}^n \int_a^c a_{nj} X_{jt} dC_{jt} \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^n \int_c^b a_{1j} X_{jt} dC_{jt} \\ \sum_{j=1}^n \int_c^b a_{2j} X_{jt} dC_{jt} \\ \vdots \\ \sum_{j=1}^n \int_c^b a_{nj} X_{jt} dC_{jt} \end{pmatrix} \\
 &= \int_a^c A(t) X_t dC_t + \int_c^b A(t) X_t dC_t.
 \end{aligned}$$

and the proof is completed. □

Now the following theorem, which is similar to Theorem 2.7, will be proven for uncertain linear systems.

**Theorem 2.10.** *Let  $C = (C_{1t}, C_{2t}, \dots, C_{nt})^T$  be an  $n$ -dimensional Liu process, and  $X_t$  be a Liu integrable  $n$ -component uncertain process and  $A(t) = [a_{ij}(t)]$  is an  $n \times n$  matrix whose elements are uncertain integrable functions. Then,*

$$\left| \int_a^b A(t) X_t(\gamma) dC_t(\gamma) \right| \leq K(\gamma) \int_a^b |A(t)| |X_t(\gamma)| dt, \quad \forall \gamma \in \Gamma,$$

where  $K(\gamma)$  is the Lipschitz constant of the sample path  $C_t(\gamma)$ .

*Proof.* For each  $\gamma \in \Gamma$ , we have

$$\begin{aligned}
 \left| \int_a^b A(t) X_t(\gamma) dC_t(\gamma) \right| &= \bigvee_{i=1}^n \left| \sum_{j=1}^n \int_a^b a_{ij}(t) X_{jt}(\gamma) dC_{jt}(\gamma) \right| \\
 &\leq \bigvee_{i=1}^n \sum_{j=1}^n \left| \int_a^b a_{ij}(t) X_{jt}(\gamma) dC_{jt}(\gamma) \right|.
 \end{aligned}$$

According to Theorem 2.7, for a sample path  $C_{jt}(\gamma)$  with a Lipschitz constant  $K(\gamma)$ , we have

$$\left| \int_a^b a_{ij}(t) X_{jt}(\gamma) dC_{jt}(\gamma) \right| \leq K(\gamma) \int_a^b |a_{ij}(t)| |X_{jt}(\gamma)| dt.$$

Hence,

$$\bigvee_{i=1}^n \sum_{j=1}^n \left| \int_a^b a_{ij}(t) X_{jt}(\gamma) dC_t \right| \leq K(\gamma) \int_a^b \bigvee_{i=1}^n \sum_{j=1}^n |a_{ij}(t)| |X_{jt}(\gamma)| dt.$$



Therefore, by the definition of  $|A|$ ,

$$\left| \int_a^b A(t)X_t(\gamma)dC_t(\gamma) \right| \leq K(\gamma) \int_a^b |A(t)||X_t(\gamma)|dt.$$

The proof is completed.  $\square$

Now, the main result about the existence and uniqueness of the solutions of system

$$dX_t = [A(t)X_t + U(t)]dt + [B(t)X_t + V(t)]dC_t \quad (2.2)$$

with initial value  $X_{t_0} = X_0$  will be proven.

**Theorem 2.11.** *Suppose that there exists a continuous function  $k(t)$  on  $[a, b]$  such that  $|A(t)| \leq k(t)$ ,  $|B(t)| \leq k(t)$ ,  $|U(t)| \leq k(t)$ , and  $|V(t)| \leq k(t)$  on  $[a, b]$ . Then, the initial value problem (2.2) has a unique solution  $X(t)$  on  $[a, b]$  in the following sense.*

$$X_t = X_{t_0} + \int_{t_0}^t [A(s)X_s + U(s)]ds + \int_{t_0}^t [B(s)X_s + V(s)]dC_s, \quad t \in [a, b].$$

*Proof.* We first prove the existence of a solution. Consider the following recursive sequence

$$X_t^{n+1} = X_{t_0}^0 + \int_{t_0}^t [A(s)X_s^n + U(s)]ds + \int_{t_0}^t [B(s)X_s^n + V(s)]dC_s.$$

Using Theorems 2.6, 2.8 and 2.10, we show that  $(X_t^n)$  converges uniformly.

$$\begin{aligned} |X_t^1 - X_t^0| &= \left| \int_{t_0}^t [A(s)X_s^0 + U(s)]ds + \int_{t_0}^t [B(s)X_s^0 + V(s)]dC_s \right| \\ &\leq \left| \int_{t_0}^t [A(s)X_s^0 + U(s)]ds \right| + \left| \int_{t_0}^t [B(s)X_s^0 + V(s)]dC_s \right| \\ &\leq \left| \int_{t_0}^t [A(s)X_s^0 + U(s)]ds \right| + K(\gamma) \int_{t_0}^t |B(s)X_s^0 + V(s)|ds \\ &\leq \int_{t_0}^t |A(s)||X_s^0| + |U(s)|ds + K(\gamma) \int_{t_0}^t |B(s)||X_s^0| + |V(s)|ds \\ &\leq \int_{t_0}^t k(s)|X_s^0| + k(s)ds + K(\gamma) \int_{t_0}^t k(s)|X_s^0| + k(s)ds \\ &= \int_{t_0}^t k(s)(|X_s^0| + 1)ds + K(\gamma) \int_{t_0}^t k(s)(|X_s^0| + 1)ds \\ &= (1 + K(\gamma))(1 + |X_0|) \int_{t_0}^t k(s)ds. \end{aligned}$$





Let  $p(t) = \left| \int_{t_0}^t k(s)ds \right|$ . Then,

$$|X_t^1 - X_t^0| \leq (1 + K(\gamma))(1 + |X_0|)p(t).$$

Suppose that

$$|X_t^n - X_t^{n-1}| \leq (1 + K(\gamma))^n(1 + |X_0|)\frac{p^n(t)}{n!}.$$

Then,

$$\begin{aligned} |X_t^{n+1} - X_t^n| &= \left| \int_{t_0}^t [A(s)X_s^n + U(s)]ds + \int_{t_0}^t [B(s)X_s^n + V(s)]dC_s \right. \\ &\quad \left. - \int_{t_0}^t [A(s)X_s^{n-1} + U(s)]ds - \int_{t_0}^t [B(s)X_s^{n-1} + V(s)]dC_s \right| \\ &= \left| \int_{t_0}^t A(s)[X_s^n - X_s^{n-1}]ds + \int_{t_0}^t B(s)[X_s^n - X_s^{n-1}]dC_s \right| \\ &\leq \left| \int_{t_0}^t A(s)[X_s^n - X_s^{n-1}]ds \right| + K(\gamma) \int_{t_0}^t |B(s)[X_s^n - X_s^{n-1}]|ds \\ &\leq \int_{t_0}^t |A(s)||X_s^n - X_s^{n-1}|ds + K(\gamma) \int_{t_0}^t |B(s)||X_s^n - X_s^{n-1}|ds \\ &\leq \int_{t_0}^t k(s)|X_s^n - X_s^{n-1}|ds + K(\gamma) \int_{t_0}^t k(s)|X_s^n - X_s^{n-1}|ds \\ &\leq \int_{t_0}^t k(s)(1 + K(\gamma))|X_s^n - X_s^{n-1}|ds \\ &\leq \int_{t_0}^t (1 + K(\gamma))\frac{(1 + K(\gamma))^n}{n!}(1 + |X_0|)p^n(s)k(s)ds \\ &\leq \frac{(1 + K(\gamma))^{n+1}}{n!}(1 + |X_0|) \int_{t_0}^t p^n(s)k(s)ds \\ &= \frac{(1 + K(\gamma))^{n+1}}{(n + 1)!}(1 + |X_0|)p^{n+1}(t). \end{aligned}$$

Therefore, the sequence  $(X_t^n)$  is uniformly converges.



Now assume that  $X_t$  and  $X_t^*$  are two solutions of the system with a common initial value  $X_{t_0} = X_0$ . Then  $\forall \gamma \in \Gamma$ , we have

$$\begin{aligned}
 & |X_t(\gamma) - X_t^*(\gamma)| \\
 &= \left| \int_{t_0}^t [A(s)X_s(\gamma) + U(s)]ds + \int_{t_0}^t [B(s)X_s(\gamma) + V(s)]dC_s(\gamma) \right. \\
 &\quad \left. - \int_{t_0}^t [A(s)X_s^*(\gamma) + U(s)]ds - \int_{t_0}^t [B(s)X_s^*(\gamma) + V(s)]dC_s(\gamma) \right| \\
 &= \left| \int_{t_0}^t [A(s)(X_s(\gamma) - X_s^*(\gamma))]ds + \int_{t_0}^t [B(s)(X_s(\gamma) - X_s^*(\gamma))]dC_s(\gamma) \right| \\
 &\leq \int_{t_0}^t |A(s)||X_s(\gamma) - X_s^*(\gamma)|ds + K(\gamma) \int_{t_0}^t |B(s)||X_s(\gamma) - X_s^*(\gamma)|ds \\
 &\leq \int_{t_0}^t k(s)|X_s(\gamma) - X_s^*(\gamma)|ds + K(\gamma) \int_{t_0}^t k(s)|X_s(\gamma) - X_s^*(\gamma)|ds \\
 &= (1 + K(\gamma)) \int_{t_0}^t k(s)|X_s(\gamma) - X_s^*(\gamma)|ds.
 \end{aligned}$$

Let  $|X_t(\gamma) - X_t^*(\gamma)| = W_t(\gamma)$  and  $1 + K(\gamma) = L(\gamma)$ . Then,

$$W_t(\gamma) < \varepsilon + L(\gamma) \int_{t_0}^t k(s)W_s(\gamma)ds, \quad \forall \varepsilon > 0.$$

By the Grönwall's inequality, we obtain  $0 \leq W_t(\gamma) < \varepsilon e^{\int_{t_0}^t k(s)ds}$ . Thus,

$$|X_t(\gamma) - X_t^*(\gamma)| = 0.$$

That means  $X_t = X_t^*$  almost surely. The uniqueness of the solution is verified.  $\square$

**Definition 2.12.** If  $U(t)$  and  $V(t)$  in (2.2) are identically 0, that is, if (2.2) has the form

$$dX_t = [A(t)X_t]dt + [B(t)X_t]dC_t, \quad (2.3)$$

then the equation is said to be homogeneous uncertain differential equations system.

**Theorem 2.13.** If  $X_t$  is a solution of (2.3) on  $[a, b]$  and if there exists  $t_0 \in [a, b]$  such that  $X_{t_0} = 0$ , then  $X_t = 0$  for all  $t \in [a, b]$ .

*Proof.* Let  $Y_t = 0$  for  $t \in [a, b]$ . Hence by the uniqueness of solutions of uncertain linear system,  $X_t = Y_t$  for all  $t \in [a, b]$ .  $\square$

**Theorem 2.14.** The collection of solutions of (2.3) is an  $n$ -dimensional linear space.

*Proof.* Let  $\alpha \in \mathbb{R}$ ,  $X_t$  and  $X_t^*$  be solutions of (2.3). Then

$$\begin{aligned}
 d(\alpha X_t + X_t^*) &= [A(t)(\alpha X_t + X_t^*)]dt + [B(t)(\alpha X_t + X_t^*)]dC_t \\
 &= [\alpha(A(t)X_t dt + B(t)X_t dC_t)] + [A(t)X_t^* dt + B(t)X_t^* dC_t] \\
 &= \alpha dX_t + dX_t^*.
 \end{aligned}$$



Thus,  $\alpha X_t + X_t^*$  is also a solution of (2.3) for every  $\alpha \in \mathbb{R}$ . Therefore, the collection of solutions is a linear space. Let us denote by  $I_i$  the  $n$ -dimensional vector where  $i$ -th component is equal to 1 and the remaining ones are equal to 0. By the existence theorem, for  $i = 1, \dots, n$  there are solutions  $X_t^{(i)}$  of (2.3) such that

$$X_t^{(i)} = I_i.$$

It will be shown that  $\{X_t^{(i)} : 1 \leq i \leq n\}$  is a basis for the collection of the solutions of (2.3). Suppose that  $\sum_{i=1}^n C_i X_t^{(i)} \equiv 0$ . Then  $\sum_{i=1}^n C_i X_{t_0}^{(i)} = \sum_{i=1}^n C_i I_i = 0$ . Hence,  $(C_1, C_2, \dots, C_n) = 0$  and therefore  $C_i = 0$  for  $1 \leq i \leq n$ ; that is, these solutions are independent. Thus, the collection of solutions is a linear space of dimension greater than or equal to  $n$ .

Now suppose  $Y_t^1, Y_t^2, \dots, Y_t^n, Y_t^{(n+1)}$  are linearly independent solutions of (2.3). Let  $t_0 \in [a, b]$ . Then by Theorem 2.13,  $Y_{t_0}^1, Y_{t_0}^{(2)}, \dots, Y_{t_0}^{(n)}, Y_{t_0}^{(n+1)}$  are linearly independent. That is, we have  $(n+1)$  linearly independent  $n$ -vectors. From linear algebra, this is impossible.  $\square$

### 3. CONCLUSION

In this paper, a system of linear uncertain differential equations was considered and some new definitions and theorems about the system were presented. Preparing some auxiliary theorems, the main contribution of this paper was to provide an existence and uniqueness of solutions of the system with initial condition. Also, it was shown that the collection of solutions of homogeneous uncertain linear system is a linear space.

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