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Residual Method for Nonlinear System of Initial Value Problems

Meltem Evrenosoglu Adiyaman*
Department of Mathematics,
Faculty of Sciences, Dokuz Eylül University,
Tınaztepe, Buca, 35160 Izmir, Turkey.
E-mail: meltem.evrenosoglu@deu.edu.tr

Burcu Noyan
Department of Mathematics,
Faculty of Sciences, Dokuz Eylül University,
Tınaztepe, Buca, 35160 Izmir, Turkey.
E-mail: burcunoyan55@hotmail.com

Abstract In this paper, the nonlinear system of initial value problems are solved numerically by using Residual method which is based on the minimizing residual function by the Taylor's series expansion. The convergence analysis of the method is given. The significant feature of the method is reduction of nonlinear system of initial value problems to the system of linear equations. To emphasize the accuracy and potential of the method, we solve Lorenz system and primary HIV-1 infection problem numerically.

Keywords. Nonlinear initial value systems, Bernstein polynomials, Residual method, Lorenz system, primary HIV-1 infection problem.

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1. Introduction

In this research, we propose Residual method to approximate nonlinear system of initial value problems in the form

$$\mathbf{Y}'(x) = \mathbf{F}(x, \mathbf{Y}(x)), \quad \mathbf{Y}(a) = \mathbf{Y}_0, \tag{1.1}$$

where $\mathbf{Y} : [a, b] \to D \subset \mathbb{R}^m$, $\mathbf{F} \in C^n[a, b] \times C(D)$, $\mathbf{F}^{(k)}(x, \mathbf{Y})$ denotes the kth derivative of $\mathbf{F}$ with respect to x, $\mathbf{F}^{(k)}(x, \mathbf{Y})$ is Lipschitz with Lipschitz constant L_k with respect to $\mathbf{Y}$ on $[a, b] \times D$ for $k = 0, 1, \ldots, n-1$ and $\mathbf{Y}_0 \in D$.

Investigation of the exact and numerical solutions of nonlinear system of initial value problems have been focused by some researchers for many years. Most famous type of them are dynamical systems and chaotic problems.

For an example of dynamical systems, we investigate primary HIV-1 infection problem which is a basic mathematical model widely used to describe the virus dynamics

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∗ corresponding.

of primary HIV-1 infection and described by a system of differential equations given in [3]

$$\frac{dT}{dt} = s - d\,T - \beta\,V\,T, \qquad T(t_0) = T_0, \tag{1.2}$$

$$\frac{dI}{dt} = \beta\,V\,T - \delta I, \qquad I(t_0) = I_0, \tag{1.3}$$

$$\frac{dV}{dt} = p\,I - c\,V, \qquad V(t_0) = V_0, \tag{1.4}$$

where T is the concentration of target cells, s represents the constant influx rate of target cells, d is the target cell loss rate, β is the target cells infection rate constant, I is the concentration of infected cells, δ is their loss rate constant, V is the serum viral concentration, p is the viral production rate constant and c is the virus concentrate rate constant. Eq. (1.2) expresses the dynamics of the target cells, Eq. (1.3) describes the dynamics of the infected cells and Eq. (1.4) expresses the viral dynamics. For this model, it is concluded that the control of HIV-1 in the periphery is limited by the availability of susceptible target cells.

For an example of chaotic problems, we consider Lorenz system given in [7]

$$\begin{aligned}
\frac{dx_1}{dt} &= a(x_2 - x_1), && x_1(t_0) = x_{10}, \\
\frac{dx_2}{dt} &= -x_1x_3 + bx_1 - x_2, && x_2(t_0) = x_{20}, \\
\frac{dx_3}{dt} &= x_1x_2 - cx_3, && x_3(t_0) = x_{30},
\end{aligned} \tag{1.5}$$

where a, b, c are all greater than zero. These equations were derived by Lorenz in the modelling of two dimensional fluid cell between two paralel plates at different temperatures. Some examples of the numerical treatment of Lorenz system are given in [1, 4, 7, 8, 9].

In order to solve these nonlinear system of initial value problems numerically, we use Residual method given in [2] which is based on the minimization of a residual function using the Taylor's series expansion. In this method, interval $[a, b]$ is divided into N subintervals and approximate solution is constructed as a linear combination of Bernstein polynomials on each subinterval. Then unknown coefficients of Bernstein polynomials are obtained using Residual method. When the other methods are used to solve nonlinear system of initial value problems numerically, either the system of nonlinear equations or the system of linear equations containing too many unknowns is encountered. In the first case, a numerical method such as Newton's method is used to approximate the system of nonlinear equations. In order to solve system of linear equations, Gauss elimination, LU factorization or the methods given in [5, 6, 10] are used in the second case. Whereas the Residual method reduces the nonlinear system of initial value problems to lower triangular system with non-zero diagonals. Thus, there is no need to use another method to solve the system, the system can be solved directly. This is the most significant advantage of the proposed method.

In section 2, Residual method is described for nonlinear system of initial value problems. Convergence of the method is analysed in section 3. In section 4, numerical solutions of primary HIV-1 infection problem and Lorenz system are given. In the Conclusion section, summary of the study and our suggestions regarding future works are presented.

2. Residual method for nonlinear system of initial value problems

The interval $[a,b]$ is divided into N equally spaced subintervals $[a_{i-1},a_i]$, where $a_i = a+ih,\ i=0,...,N,\ h=(b-a)/N$ and N is a positive integer. The initial value problem (1.1) is written piecewisely as follows

$$\mathbf{Y}_i'(x) = \mathbf{F}(x,\mathbf{Y}_i(x)), \quad x \in S_i, \tag{2.1}$$

where $S_i = [a_{i-1},a_i]$ for $i=1,...,N$ and

$$\mathbf{Y}_1(a_0) = \mathbf{Y}_0, \quad \mathbf{Y}_i(a_{i-1}) = \mathbf{Y}_{i-1}(a_{i-1}) \text{ for } i=2,..,N. \tag{2.2}$$

In order to approximate the solution $\mathbf{Y}_i(x)$, we use nth degree Bézier curve on S_i

$$\mathbf{U}_i(x) = \sum_{j=0}^{n} \mathbf{C}_j^i B_j^n\left(\frac{x-a_{i-1}}{h}\right), \tag{2.3}$$

where

$$B_j^n\left(\frac{x-a_{i-1}}{h}\right) = \binom{n}{j}\frac{1}{h^n}(x-a_{i-1})^j(a_i-x)^{n-j}$$

are the Bernstein polynomials over the interval $[a_{i-1},a_i]$ and $\mathbf{C}_j^i$ are unknown control vectors. We must determine $(n+1)$ unknown control vectors to obtain the approximate solution $\mathbf{U}_i(x)$ over each subinterval S_i.

Applying the initial conditions (2.2) to the approximate solution, we have the following conditions

$$\mathbf{U}_1(a_0) = \mathbf{Y}_0, \quad \mathbf{U}_i(a_{i-1}) = \mathbf{U}_{i-1}(a_{i-1}), \quad i=2,\dots,N. \tag{2.4}$$

Thus, we guarantee the continuity of the approximate solution, i.e $\mathbf{U}(x) \in C[a,b]$. Using end point interpolation property of Bézier curves and (2.4), we get

$$\begin{aligned} &C_0^1 = Y_0, \\ &C_0^i = C_n^{i-1}. \end{aligned} \tag{2.5}$$

Thus, we reduce the number of the unknown control vectors from $n+1$ to n for each subinterval. Substitution of (2.3) into the differential equation (2.1) gives

$$\mathbf{R}(x) = R_i(x), \quad x \in S_i, \quad i=1,2,\dots,N,$$

where

$$\mathbf{R}_i(x) = \mathbf{U}_i'(x) - \mathbf{F}(x,\mathbf{U}_i(x)), \quad x \in S_i. \tag{2.6}$$

Our aim is to determine unknown control vectors by minimizing the residual function. For this minimization, we equate the first n terms in Taylor's expansion of $\mathbf{R}_i(x)$ at $x = a_{i-1}$ to zero, that is

$$\begin{aligned} \mathbf{R}_i(a_{i-1}) &= 0, \\ \mathbf{R}'_i(a_{i-1}) &= 0, \\ \mathbf{R}''_i(a_{i-1}) &= 0, \\ &\vdots \\ \mathbf{R}_i^{(n-1)}(a_{i-1}) &= 0, \end{aligned} \tag{2.7}$$

where $\mathbf{R}_i^{(k)}$ denotes the kth derivative of $\mathbf{R}$ with respect to x on $[a, b]$.

From (2.7), we get

$$\begin{aligned} \mathbf{0} &= \mathbf{R}_i^{(k)}(a_{i-1}) \\ &= \mathbf{U}_i^{(k+1)}(a_{i-1}) - \mathbf{F}^{(k)}(a_{i-1}, \mathbf{U}_i(a_{i-1})), \quad k = 0, \dots, n-1, \end{aligned}$$

which yields the following linear equations

$$\frac{n(n-1)\dots(n-k)}{h^{k+1}} \Delta^{k+1} \mathbf{C}_0^i - \mathbf{F}^{(k)}(a_{i-1}, \mathbf{C}_0^i) = 0, \quad i = 1, \dots, N. \tag{2.8}$$

When equations (2.8) are written as matrix system, the system encountered is a system consisting of lower triangular matrix. The solution of this system will yield unknown control vectors $\mathbf{C}_k^i$, $k = 1, \dots, n$. As a result, the approximate solutions $\mathbf{U}_i(x)$ in each subinterval S_i are obtained by minimizing $\mathbf{R}_i(x)$, for $i = 1, ..., N$.

3. Convergence of the method for nonlinear system of initial value problems

In the proofs of the following lemmas and theorem, the similar techniques which are used in the proofs of the lemmas and theorem given in [2] are used.

Lemma 3.1. *The residual functions $\boldsymbol{R}_i(x)$ are order of n for $i = 1, ..., N$.*

Proof. The Taylor's expansion of $\mathbf{R}_i(x)$ at $x = a_{i-1}$ is

$$\begin{aligned} \mathbf{R}_i(x) &= \mathbf{R}_i(a_{i-1}) + (x - a_{i-1})\mathbf{R}'_i(a_{i-1}) + \frac{(x - a_{i-1})^2}{2!}\mathbf{R}''_i(a_{i-1}) + \dots \\ &\quad + \frac{(x - a_{i-1})^{n-1}}{(n-1)!}\mathbf{R}_i^{(n-1)}(a_{i-1}) + \frac{(x - a_{i-1})^n}{n!}\mathbf{R}_i^{(n)}(\xi_i) \end{aligned}$$

where ξ_i is between (a_{i-1}, x) and $x \in [a_{i-1}, a_i]$. Using equations (2.7), we get

$$\begin{aligned} \left\|\mathbf{R}_i(x)\right\| &= \left|\frac{(x - a_{i-1})^n}{n!}\right| \left\|\mathbf{R}_i^{(n)}(\xi_i)\right\| \\ &\le \tilde{C}h^n, \end{aligned}$$

where $\tilde{C} = \dfrac{1}{n!} \max_{x \in [a_{i-1}, a_i]} \left\|\mathbf{R}_i^{(n)}(x)\right\|$. Therefore,

$$\left\|\mathbf{R}(x)\right\| = \mathcal{O}(h^n).$$

□

Lemma 3.2. *Let $\tilde{\boldsymbol{U}}_i(x)$ be the auxiliary approximate solution of piecewise initial value problem (2.1) with initial conditions (2.2), then*

$$\mathbf{Y}_i^{(k)}(a_{i-1}) = \tilde{\boldsymbol{U}}_i^{(k)}(a_{i-1}), \quad \text{for } k=0,\ldots,n \tag{3.1}$$

and

$$\left\| \boldsymbol{Y}_i(x) - \tilde{\boldsymbol{U}}_i(x) \right\| \leq K h^{n+1}, \ \forall x \in S_i, \tag{3.2}$$

where $\boldsymbol{Y}_i(x)$ is the corresponding exact solution and

$$K = \frac{1}{(n+1)!} \max_{x \in [a,b]} \left\| \boldsymbol{Y}_i^{(n+1)}(x) \right\|.$$

Proof. The auxilary approximate solution $\mathbf{U}(x)$ must satisfy the initial conditions (2.2) of (2.1), i.e.,

$$\tilde{\mathbf{U}}_1(a_0) = \mathbf{Y}_0 \text{ and } \tilde{\mathbf{U}}_i(a_{i-1}) = \mathbf{Y}_{i-1}(a_{i-1}). \tag{3.3}$$

Let the residual function for $\tilde{\mathbf{U}}_i(x)$

$$\tilde{\mathbf{R}}_i(x) = \tilde{\mathbf{U}}_i'(x) - \mathbf{F}(x, \tilde{\mathbf{U}}_i(x)), \ x \in S_i \tag{3.4}$$

be 0 at the point $x = a_{i-1}$ for the auxiliary approximate solution as in (2.7), that is

$$\tilde{\mathbf{R}}_i^{(k)}(a_{i-1}) = 0, \tag{3.5}$$

for $k = 0, \ldots, n-1$ and $i = 1, \ldots, N$. Then,

$$\begin{aligned} 0 &= \tilde{\mathbf{R}}_i^{(k)}(a_{i-1}) \\ &= \tilde{\mathbf{U}}_i^{(k+1)}(a_{i-1}) - \mathbf{F}^{(k)}(a_{i-1}, \tilde{\mathbf{U}}_i(a_{i-1})) \\ &= \tilde{\mathbf{U}}_i^{(k+1)}(a_{i-1}) - \mathbf{F}^{(k)}(a_{i-1}, \mathbf{Y}_i(a_{i-1})) \\ &= \tilde{\mathbf{U}}_i^{(k+1)}(a_{i-1}) - \mathbf{Y}_i^{(k+1)}(a_{i-1}). \end{aligned}$$

It gives (3.1).

From the Taylor's expansion of $\mathbf{Y}_i(x) - \tilde{\mathbf{U}}_i(x)$ at $x = a_{i-1}$, we get

$$\begin{aligned} \mathbf{Y}_i(x) - \tilde{\mathbf{U}}_i(x) &= \left(\mathbf{Y}_i(a_{i-1}) - \tilde{\mathbf{U}}_i(a_{i-1}) \right) + (x - a_{i-1}) \left(\mathbf{Y}_i'(a_{i-1}) - \tilde{\mathbf{U}}_i'(a_{i-1}) \right) \\ &\quad + \frac{(x - a_{i-1})^2}{2!} \left(\mathbf{Y}_i''(a_{i-1}) - \tilde{\mathbf{U}}_i''(a_{i-1}) \right) + \cdots + \frac{(x - a_{i-1})^n}{n!} \\ &\quad \times \left(\mathbf{Y}_i^{(n)}(a_{i-1}) - \tilde{\mathbf{U}}_i^{(n)}(a_{i-1}) \right) + \frac{(x - a_{i-1})^{n+1}}{(n+1)!} \mathbf{Y}_i^{(n+1)}(\xi_{i-1}) \end{aligned}$$

and using (3.1), we obtain

$$\begin{aligned} \left\| \mathbf{Y}_i(x) - \tilde{U}_i(x) \right\| &= \left| \frac{(x - a_{i-1})^{n+1}}{(n+1)!} \right| \left\| \mathbf{Y}_i^{(n+1)}(\xi_{i-1}) \right\| \\ &\leq K h^{n+1}, \end{aligned}$$

where $K = \dfrac{1}{(n+1)!} \max\limits_{x \in [a,b]} \left\| \mathbf{Y}_i^{(n+1)}(x) \right\|$ and $\xi_{i-1} \in (a_{i-1}, x)$. □

Lemma 3.3. *Let $\tilde{\boldsymbol{U}}_i(x)$ be the auxiliary approximate solution of piecewise initial value problem (2.1) with initial conditions (2.2) and $\boldsymbol{U}_i(x)$ be the approximate solution of (2.1) with initial conditions (2.4), then*

$$\left\| \boldsymbol{U}_i(x) - \tilde{\boldsymbol{U}}_i(x) \right\| \leq \left(1 + \tilde{K} \left(h + \frac{h^2}{2!} + \cdots + \frac{h^n}{n!} \right) \right) \left\| \boldsymbol{U}_i(a_{i-1}) - \tilde{\boldsymbol{U}}_i(a_{i-1}) \right\|, \tag{3.6}$$

where $\tilde{K} = \max\limits_{k=0\ldots,n-1} L_k$, L_k are Lipschitz constants of $\boldsymbol{F}^{(k)}(x, \boldsymbol{Y})$ with respect to $\boldsymbol{Y}$ on $[a,b] \times D$, for $k = 0, 1, \ldots, n-1$.

Proof. We write the Taylor's expansion of the nth degree polynomial $\tilde{\mathbf{U}}_i(x) - \mathbf{U}_i(x)$ about $x = a_{i-1}$,

$$\begin{aligned}
\tilde{\mathbf{U}}_i(x) - \mathbf{U}_i(x) &= \left(\tilde{\mathbf{U}}_i(a_{i-1}) - \mathbf{U}_i(a_{i-1}) \right) + (x - a_{i-1}) \left(\tilde{\mathbf{U}}_i'(a_{i-1}) - \mathbf{U}_i'(a_{i-1}) \right) \\
&+ \frac{(x - a_{i-1})^2}{2!} \left(\tilde{\mathbf{U}}_i''(x) - \mathbf{U}_i''(x) \right) + \ldots \\
&+ \frac{(x - a_{i-1})^n}{n!} \left(\tilde{\mathbf{U}}_i^{(n)}(x) - \mathbf{U}_i^{(n)}(x) \right).
\end{aligned}$$

Using (2.6), (2.7), (3.4) and (3.5), we obtain

$$\begin{aligned}
\tilde{\mathbf{U}}_i(x) - \mathbf{U}_i(x) &= \left(\tilde{\mathbf{U}}_i(a_{i-1}) - \mathbf{U}_i(a_{i-1}) \right) + (x - a_{i-1}) \\
&\times \left(\mathbf{F}(a_{i-1}, \tilde{\mathbf{U}}_i(a_{i-1})) - \mathbf{F}\left(a_{i-1}, \mathbf{U}_i(a_{i-1})\right) \right) \\
&+ \frac{(x - a_{i-1})^2}{2!} \left(\mathbf{F}^{(1)}(a_{i-1}, \tilde{\mathbf{U}}_i(a_{i-1})) - \mathbf{F}^{(1)}(a_{i-1}, \mathbf{U}_i(a_{i-1})) \right) + \ldots \\
&+ \frac{(x - a_{i-1})^n}{n!} \left(\mathbf{F}^{(n-1)}(a_{i-1}, \tilde{\mathbf{U}}_i(a_{i-1})) - \mathbf{F}^{(n-1)}(a_{i-1}, \mathbf{U}_i(a_{i-1})) \right)
\end{aligned}$$

Taking maximum norm of both sides and using Lipschitz property, we have

$$\begin{aligned}
\left\| \tilde{\mathbf{U}}_i(x) - \mathbf{U}_i(x) \right\| &\leq \left\| \tilde{\mathbf{U}}_i(a_{i-1}) - \mathbf{U}_i(a_{i-1}) \right\| + hL_0 \left\| \tilde{\mathbf{U}}_i(a_{i-1}) - \mathbf{U}_i(a_{i-1}) \right\| \\
&+ \frac{h}{2!} L_1 \left\| \tilde{\mathbf{U}}_i(a_{i-1}) - \mathbf{U}_i(a_{i-1}) \right\| + \ldots \\
&+ \frac{h^n}{n!} L_{n-1} \left\| \tilde{\mathbf{U}}_i(a_{i-1}) - \mathbf{U}_i(a_{i-1}) \right\| \\
&\leq \left(1 + \tilde{K} \left(h + \tfrac{h^2}{2} + \cdots + \tfrac{h^n}{n!} \right) \right) \left\| \tilde{\mathbf{U}}_i(a_{i-1}) - \mathbf{U}_i(a_{i-1}) \right\|
\end{aligned}$$

where $\tilde{K} = \max_{k=0\ldots,n-1} L_k$, L_k are Lipschitz constants of $\mathbf{F}^{(k)}(x, \mathbf{Y})$ with respect to $\mathbf{Y}$ on $[a, b] \times D$, for $k = 0, 1, \ldots, n-1$. $\square$

Theorem 3.4. *Let $\boldsymbol{Y}(x)$ be the exact solution of*

$$\boldsymbol{Y}' = \boldsymbol{F}(x, \boldsymbol{Y}(x)), \quad \boldsymbol{Y}(x_0) = \boldsymbol{Y}_0, \tag{3.7}$$

where $\boldsymbol{Y} : [a, b] \to D \subset \mathbb{R}^m$, $\boldsymbol{F} \in C^n[a, b] \times C(D)$, $\boldsymbol{F}^{(k)}(x, \boldsymbol{Y})$ denotes the kth derivative of $\boldsymbol{F}$ with respect to x, $\boldsymbol{F}^{(k)}(x, \boldsymbol{Y})$ is Lipschitz with Lipschitz constant L_k with respect to $\boldsymbol{Y}$ on $[a, b] \times D$, for $k = 0, 1, \ldots, n-1$, $\boldsymbol{Y}_0 \in D$ and $\boldsymbol{U}(x)$ be the nth degree approximate function of (3.7) as defined in (2.3), then

$$\|\boldsymbol{Y}(x) - \boldsymbol{U}(x)\| \leq M h^n, \quad x \in [a, b],$$

where M is a constant which is independent on h.

Proof. We will prove this theorem using mathematical induction. Let $\tilde{\mathbf{U}}_i(x)$ be the corresponding auxilary approximate solution for $x \in S_i$, then

$$\|\mathbf{Y}_i(x) - \mathbf{U}_i(x)\| \leq \|\mathbf{Y}_i(x) - \tilde{\mathbf{U}}_i(x)\| + \|\tilde{\mathbf{U}}_i(x) - \mathbf{U}_i(x)\|, \text{ for } i = 1, \ldots, N.$$

Using (3.2) and (3.6), we obtain

$$\begin{aligned} \|\mathbf{Y}_i(x) - \mathbf{U}_i(x)\| &\leq K h^{n+1} + \left(1 + C\left(h + \frac{h^2}{2} + \cdots + \frac{h^n}{n!}\right)\right) \\ &\times \|\tilde{\mathbf{U}}_i(a_{i-1}) - \mathbf{U}_i(a_{i-1})\|. \end{aligned} \tag{3.8}$$

Since for the first interval $\tilde{\mathbf{U}}_1(x) = \mathbf{U}_1(x)$, we get

$$\|\mathbf{Y}_1(x) - \mathbf{U}_1(x)\| \leq K h^{n+1}, \quad \forall x \in S_1,$$

where $K = \dfrac{1}{(n+1)!} \max_{x \in [a,b]} \|\mathbf{Y}_i^{(n+1)}(x)\|$.

Then using initial conditions (2.4) and (3.3), we have

$$\begin{aligned} \|\mathbf{Y}_2(x) - \mathbf{U}_2(x)\| &\leq K h^{n+1} + \left(1 + C\left(h + \frac{h^2}{2} + \cdots + \frac{h^n}{n!}\right)\right) \\ &\times \|\tilde{\mathbf{U}}_2(a_1) - \mathbf{U}_2(a_1)\| \\ &\leq K h^{n+1} + \left(1 + C\left(h + \frac{h^2}{2} + \cdots + \frac{h^n}{n!}\right)\right) \\ &\times \|\mathbf{Y}_1(a_1) - \mathbf{U}_1(a_1)\| \\ &\leq 2K h^{n+1} + \mathcal{O}(h^{n+2}). \end{aligned}$$

Suppose that inequality is true for $N-1$, that is,

$$\|\mathbf{Y}_{N-1}(x) - \mathbf{U}_{N-1}(x)\| \leq (N-1)\, K h^{n+1} + \mathcal{O}(h^{n+2}).$$

From (2.4), (3.3) and (3.8), we have

$$
\begin{aligned}
\left\|\mathbf{Y}_N(x)-\mathbf{U}_N(x)\right\| &\leq (N-1)\,Kh^{n+1}+\left(1+C\left(h+\tfrac{h^2}{2}+\cdots+\tfrac{h^n}{n!}\right)\right)\\
&\times\left\|\tilde{\mathbf{U}}_N(a_{N-1})-\mathbf{U}_N(a_{N-1})\right\|\\
&\leq Kh^{n+1}+\left(1+C\left(h+\tfrac{h^2}{2}+\cdots+\tfrac{h^n}{n!}\right)\right)\\
&\times\left\|\mathbf{Y}_{N-1}(a_{N-1})-\mathbf{U}_{N-1}(a_{N-1})\right\|\\
&\leq Kh^{n+1}+\left(1+C\left(h+\tfrac{h^2}{2}+\cdots+\tfrac{h^n}{n!}\right)\right)\\
&\times (N-1)\,Kh^{n+1}+\mathcal{O}(h^{n+2})\\
&\leq NKh^{n+1}+\mathcal{O}(h^{n+2}).
\end{aligned}
$$

Since $N=(b-a)/h$, we obtain

$$\left\|\mathbf{Y}(x)-\mathbf{U}(x)\right\|\leq Mh^n$$

, where M is a constant which is independent on h. □

4. Numerical results and discussion

We solve the following three examples numerically to illustrate the applicability of the proposed method.

Example 4.1. Consider the system

$$
\begin{aligned}
\frac{dx}{dt}&=x+2y-z,\\
\frac{dy}{dt}&=x+z,\\
\frac{dz}{dt}&=4x-4y+5z,
\end{aligned}
$$

with the initial conditions

$$x(0)=-1,\ y(0)=0,\ z(0)=0.$$

The exact solution of the given system is

$$
\begin{aligned}
x(t)&=-2e^{2t}+e^{3t},\\
y(t)&=e^{2t}-e^{3t},\\
z(t)&=4e^{2t}-4e^{3t}.
\end{aligned}
$$

We apply Residual method to above system of initial value problems to confirm the error analysis of the method. In Table 1, we give the obtained error values for $h = 0.1$ and $n = 4$. e_x, e_y and e_z denotes the errors of x, y and z, respectively.

TABLE 1. Error values of Example 4.1 for $h = 0.1$ and $n = 4$.

t	e_x	e_y	e_z
0.000	0	0	0
0.100	4.90964(-09)	-5.68949(-09)	-2.27579(-08)
0.200	1.33950(-08)	-1.54966(-08)	-6.19866(-08)
0.300	2.75375(-08)	-3.14144(-08)	-1.25658(-07)
0.400	4.73634(-08)	-6.15767(-08)	-2.46307(-07)
0.500	9.47288(-08)	-1.36299(-07)	-5.45196(-07)
0.600	1.47834(-07)	-4.33625(-08)	-1.7345(-07)
0.700	2.37894(-07)	-2.70111(-07)	-1.08044(-06)
0.800	4.0321(-07)	-8.92306(-08)	-3.56923(-07)
0.900	3.79802(-07)	-1.87333(-06)	-7.49334(-06)
1.000	1.25292(-06)	-1.50373(-06)	-6.01492(-06)

In Table 2, we give the observed orders, which are well confirmed with theoretical results, obtained using the following formula

$$ord(h) = \frac{\log(\dfrac{e_{x,h}}{e_{x,h/2}})}{\log 2},$$

where $e_{x,h}$ is the maximum error moduli of the function x obtained using step-size h.

TABLE 2. Observed orders of Example 4.1.

N	$n = 2$	$n = 3$	$n = 4$
Observed orders for x			
10/20	1.67186	2.621231	3.56238
20/40	1.84842	2.83064	3.80865
40/80	1.93098	2.9302	3.95203
Observed orders for y			
10/20	1.69408	2.63226	3.56882
20/40	1.8598	2.83634	3.81389
40/80	1.93674	2.93312	4.03471
Observed orders for z			
10/20	1.75459	2.72627	3.69628
20/40	1.88481	2.88012	3.87734
40/80	1.94136	2.93665	4.04074

Example 4.2. Consider the Lorenz system

$$\frac{dx}{dt} = 10(y - x),$$

$$\frac{dy}{dt} = -xz + 28x - y,$$

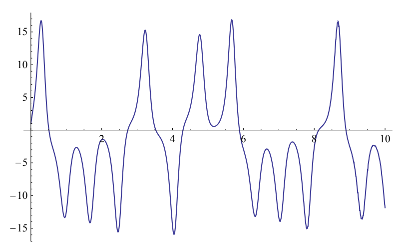
$$\frac{dz}{dt} = xy - 8/3z,$$

with the initial conditions

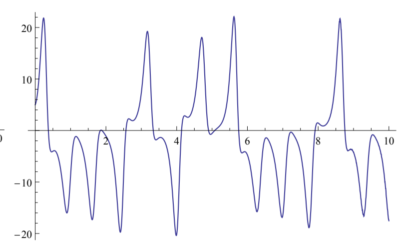
$$x(0) = 1, \ y(0) = 5, \ z(0) = 10.$$

We solve Lorenz system using Residual method and compare our results with the results obtained by piecewise successive linearization method (PSLM) given in [7]. Graphs of approximate solutions are given in Figure 4.2 and comparison of the results are given in Table 3.

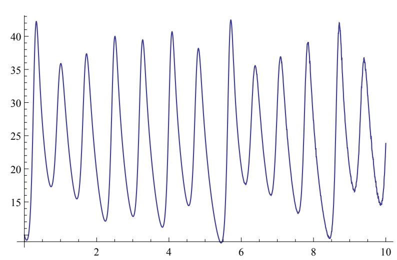
FIGURE 1. Graphs of the approximate solutions of Example 4.2 for $n = 4$



(a) Approximate solution of $x(t)$



(b) Approximate solution of $y(t)$



(c) Approximate solution of $z(t)$

TABLE 3. Numerical results of the Lorenz system obtained using PSLM [6] and RM.

t	$x(t)$		$y(t)$		$z(t)$	
	RM	REF.[6]	RM	REF.[6]	RM	REF.[6]
2	-1.44432	-1.444359	-1.07476	-1.074977	19.5172	19.517057
4	-14.6729	-14.675080	-20.1954	-20.189107	29.0463	29.063362
6	-2.88223	-2.883028	-4.76728	-4.763557	20.3578	20.355558
8	-2.69165	-2.679252	1.45007	1.429476	27.0352	27.105659
10	-11.847	-12.026645	-17.5237	-17.520281	23.8346	24.300154

Example 4.3. Consider the primary HIV-1 infection problem

$$\frac{dx}{dt} = 10^2 - 10^{-2}x - 1.3 \times 10^{-6} x\, z,$$
$$\frac{dy}{dt} = 1.3 \times 10^{-6} x\, z - \delta y,$$
$$\frac{dz}{dt} = 10^3 y - 3z,$$

with the initial conditions

$$x(0) = 10^4,\ y(0) = 0,\ z(0) = 10^{-6}.$$

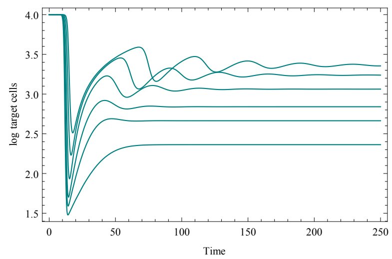
In the following graphs logarithm of approximate solution of x, y, z with base 10 are given in 250 days for different infected cell death rates $\delta = 0.1$, $\delta = 0.2$, $\delta = 0.3$, $\delta = 0.5$, $\delta = 0.75$ and $\delta = 1.0$.

It is seen from the graphs that the approximate solutions of Example 3 obtained using the Residual method coincide with the solutions given in [3].

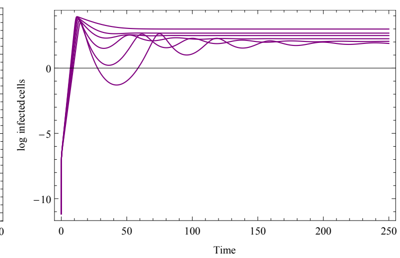
5. Conclusion

We use Residual method (RM) to approximate nonlinear system of initial value problems. Residual method reduces the nonlinear system of initial value problems to lower triangular system with non-zero diagonals. This is the significant advantage of the Residual method. As a result, we obtain high order accurate solutions. Observed orders obtained from numerical examples are in good agreement with the predicted ones in the theorem. Proposed method is illustrated with two examples, Lorenz system and primary HIV-1 infection problem. From numerical results, it might be seen that given method is efficient and has great potential. This method can be extended some special kind of nonlinear ordinary differential equations and partial differential equations.

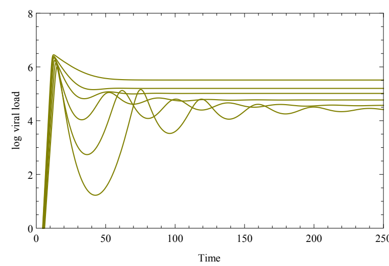
FIGURE 2. Graphs of the approximate solutions of Example 4.3



(a) Logarithm of approximate number of target cells x with base 10



(b) Logarithm of approximate number of infected cells y with base 10



(c) Logarithm of approximate number of viruses z with base 10

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