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Approximate distributed controllability of nonlocal Rayleigh beam

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Abstract This paper investigates the distributed controllability of nonlocal Rayleigh beam. The mathematical problem is formulated as an abstract differential equation. It is shown that a sequence of eigenfunction of nonlocal Rayleigh beam forms Riesz basis. Based on Riesz basis properties and theory of abstract differential equation, it is proved that a vibrating nonlocal Rayleigh beam is approximately controllable under suitable distributed control force while it is not exponentially stable.

Keywords. Controllability, Nanorod, Rayleigh.

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1. Introduction

The fourth industrial revolution is related to nanotechnology. Nanoelectromechanical systems (NEMSs) are widely used in industrial equipments. On the one hand, the vibration analysis of NEMSs is one of the key issues in design and manufacturing of them. On the other hand, the vibration analysis of a structure depends on its geometry and the type of vibration (transverse, axial or longitudinal, and torsional). In the manufacturing of NEMSs, the nanorods with axial oscillations play a significant role. Many researchers work on this subject. These researchers have studied the effects of various factors like the elastic medium [2, 11], the shape of nanorod cross section (uniform [16, 20], non-uniform [7, 19], tapered [6, 13], cone-like [8]), the magnetic field [18], the type of rod theory [22, 23], crack [3], the number of coupled nanorods [12, 21], discontinuity [15], controllability [9], the number of nanotube walls [26], and the type of nanorod material properties [1, 4, 14, 17, 24]. As known, the

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type of theory for the axial vibration analysis of nanorods depends on their thickness to length ratio. For the small thickness to length ratios (thin nanorods), the simple theory is proposed. In this theory, it is assumed that 1) the cross sections of the bar is originally plane and remains plane during deformation, and 2) the displacement components in the bar (except for the component parallel to the bar's longitudinal axis) are negligible. For the larger thickness to length ratios (thicker nanorods), these assumptions are not valid. Therefore, it should be used other modified theories like Rayleigh theory. In the Rayleigh theory, the inertia of the lateral motions by which the cross sections are extended or contracted in their own planes is considered [25]. It is expected that the effect of inertia changes the mechanical behaviors of nanorods. Therefore, it is necessary to study the various mechanical behaviors of nanorods for achieving a precise design and manufacturing of NEMSs. Since this issue does not covered in the previous studies, this study aims to consider the controllability problem of nonlocal Rayleigh beam. The rest of paper is organized as follows. In Section 2, the mathematical model of physical problem is introduced. The abstract formulation is given in Section 3. The approximate controllability is investigated in Section 4. A conclusion is given in Section 5.

2. Problem formulation

Consider the nanorod in the physical domain determined by a thin elastic rod of length l. Let $\Omega = (0, l)$ be a bounded open set in $\mathbb{R}$, $T > 0$ and $D = \Omega \times (0, T)$. The PDE describing the axial vibration of a nanorod over domain D can be expressed as follows

$$-EA\frac{\partial^2 u}{\partial x^2} + \rho A\frac{\partial^2 u}{\partial t^2} - \mu\rho A\frac{\partial^4 u}{\partial x^2 \partial t^2} - \rho\vartheta^2 I_p\frac{\partial u^4}{\partial x^2 \partial t^2} + \mu\rho\vartheta^2 I_p\frac{\partial^6 u}{\partial x^4 \partial t^2} = b(x)f(t), \tag{2.1}$$

where $u(x, t)$ is axial displacement, E is the conventional Young's modulus, A is the cross-sectional area of the nanorod, μ is nonlocal parameter, ϑ denoting poisson's ratio, ρ is density, I_p is the polar moment of the inertia of the cross section, $b(x)$ represents the shaping function around the control point x_0 and $f(t)$ is the axially distributed external control force. The equation (2.1) is obtained by using nonlocal continuum theory of Eringen [22]. The initial and boundary conditions are:

$$u(x, 0) = f_1(x), \tag{2.2}$$

$$\frac{d}{dt}u(x, 0) = f_2(x), \tag{2.3}$$

$$u(0, t) = u(l, t) = 0, \tag{2.4}$$

$$\frac{\partial^2 u}{\partial x^2}(0, t) = \frac{\partial^2 u}{\partial x^2}(0, t) = 0, \tag{2.5}$$

where $f_1(x)$ and $f_2(x)$ are given real valued functions.

3. Semigroup formulation

The system of equations (2.1)-(2.5) can be transformed to an abstract differential equation. As the state space we choose $\mathcal{H}$ in the following form

$$\mathcal{H} = H^4(\Omega) \cap \left\{ f | f(x) = \sum_{n=1}^{K} \sin\left(\frac{n\pi x}{l}\right) \right\}, \tag{3.1}$$

where K is a large enough constant number. Therefore, it is assumed that the given initial conditions are approximated by Fourier series. This condition is satisfied in most real physical problems and does not restrict the obtained results. For the sake of simplicity, we assume that $\mathbb{N} = \{1, \cdots, K\}$.

The operator J is defined as follows

$$J = (-\mu\rho A - \rho\vartheta^2 I_p)\frac{\partial^2}{\partial x^2} + \rho A + \mu\rho\vartheta^2 I_p \frac{\partial^4}{\partial x^4}. \tag{3.2}$$

On this state space, it is easy to show that J is an bounded and invertible operator. We write (2.1)-(2.5) in the following abstract form:

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} u \\ u_t \end{pmatrix} = \mathbb{A}\begin{pmatrix} u \\ u_t \end{pmatrix} + Bf, \\ \begin{pmatrix} u \\ u_t \end{pmatrix}|_{t=0} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \end{cases} \tag{3.3}$$

with $u_t = \frac{du}{dt}$, $B = \begin{pmatrix} 0 \\ J^{-1}(b(x)) \end{pmatrix}$ and $\mathbb{A}$ is given by

$$\mathbb{A}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ J^{-1}\left(EA\frac{\partial^2 u_1}{\partial x^2}\right) \end{pmatrix}, \tag{3.4}$$

where J^{-1} denotes the inverse of the operator J.

Definition 3.1. let $\mathcal{A}$ be a closed linear operator on the Hilbert space Z with simple eigenvalues $\{\gamma_n, n \geq 1\}$ and its corresponding eigenfunctions $\{\phi_n, n \geq 1\}$ form a Riesz-basis in Z. If the closure of $\{\gamma_n, n \geq 1\}$ is totally disconnected, then $\mathcal{A}$ is called Riesz-spectral operator.

The properties of Riesz-spectral operators have been widely studied on mathematical systems theory [5, 9, 10]. In the following, we will prove that $\mathbb{A}$ is Riesz-spectral operator.

Theorem 3.2. *The operator $\mathbb{A}$ as defined in (3.4) is the infinitesimal generator of a strongly continuous semigroup on $\mathcal{H}$.*

Proof. It is enough to show that the operator $\mathbb{A}$ is a Riesz-spectral operator. It can be shown that the operator $\mathbb{A}$ is closed and densely defined on $\mathcal{H}$. Let Λ denotes the set of eigenvalues $\mathbb{A}$. It suffices to show that the eigenvalues of A are simple, totally disconnected, satisfy

$$\sup Re(\lambda) < \infty, \tag{3.5}$$

and its eigenfunctions form Riesz basis. Thus, we begin by calculating the eigenvalues and eigenfunctions of $\mathbb{A}$. From (3.4) we have that

$$\mathbb{A}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\lambda\begin{pmatrix}u_1\\u_2\end{pmatrix}\Leftrightarrow\begin{cases}u_2=\lambda u_1,\\ J^{-1}\left(EA\frac{\partial^2 u_1}{\partial x^2}\right)=\lambda u_2.\end{cases} \tag{3.6}$$

Therefore, $u_2=\lambda u_1$ and

$$J^{-1}\left(EA\frac{\partial^2 u_1}{\partial x^2}\right)=\lambda^2 u_1\Leftrightarrow\left(EA\frac{\partial^2 u_1}{\partial x^2}\right)=J(\lambda^2 u_1), \tag{3.7}$$

which is equivalent to

$$\begin{cases}u_1\in\mathcal{H},\\ \left(EA\frac{\partial^2 u_1}{\partial x^2}\right)=\left((-\mu\rho A-\rho\vartheta^2 I_p)\frac{\partial^2}{\partial x^2}+\rho A+\mu\rho\vartheta^2 I_p\frac{\partial^4}{\partial x^4}\right)\lambda^2 u_1.\end{cases} \tag{3.8}$$

We want to find all solutions of (3.8). Therefore, we first obtain a set of solutions. It is easily seen that $\varphi_n=\sin(\frac{n\pi x}{l}),\quad 1\leq n\leq K$ lies in $\mathcal{H}$. Furthermore, it satisfies (3.8) if and only if λ_n satisfies

$$\lambda^2=\frac{-EA(\frac{n\pi}{l})^2}{(\frac{n\pi}{l})^2(\mu\rho A+\rho\vartheta^2 I_p)+\rho A+\mu\rho\vartheta^2 I_p(\frac{n\pi}{l})^4}. \tag{3.9}$$

The solution of above equation is denoted as follows

$$\lambda_{+n}=\sqrt{\frac{EA(\frac{n\pi}{l})^2}{(\frac{n\pi}{l})^2(\mu\rho A+\rho\vartheta^2 I_p)+\rho A+\mu\rho\vartheta^2 I_p(\frac{n\pi}{l})^4}}\,i, \tag{3.10}$$

$$\lambda_{-n}=-\sqrt{\frac{EA(\frac{n\pi}{l})^2}{(\frac{n\pi}{l})^2(\mu\rho A+\rho\vartheta^2 I_p)+\rho A+\mu\rho\vartheta^2 I_p(\frac{n\pi}{l})^4}}\,i. \tag{3.11}$$

For $\lambda_{\pm n}$ defined by (3.10) and (3.11), it is easy to see that

$$\varphi_{\pm n}(x)=\begin{pmatrix}\sin(\frac{n\pi x}{l})\\ \lambda_{\pm n}\sin(\frac{n\pi x}{l})\end{pmatrix}, \tag{3.12}$$

lies in the domain of $\mathbb{A}$, and satisfies $\mathbb{A}\varphi_{\pm n}=\lambda_{\pm n}\varphi_{\pm n}$. Hence, $\varphi_{\pm n}$ is an eigenfunction of $\mathbb{A}$.

Lemma 3.3. *The normalized set of eigenfunctions* $\{\frac{\varphi_{+n}}{\|\varphi_{+n}|},\frac{\varphi_{-n}}{\|\varphi_{-n}\|},n\in\mathbb{N}\}$ *forms a Riesz basis of* $\mathcal{H}$.

Proof. It is well-known that $\{\frac{l}{\sqrt{\mu_n}}\sin(\frac{n\pi x}{l}),n\in\mathbb{N}\}$, with

$$\mu_n=\frac{l}{2}(\frac{n\pi}{l})^8, \tag{3.13}$$

forms an orthonormal basis of $\mathcal{H}$. Let $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathcal{H}$. By the above, there exist $\{c_{1,n}\}_{n\in\mathbb{N}}$ and $\{c_{2,n}\}_{n\in\mathbb{N}}$ in l^2 such that

$$w_1(x) = \sum_{n=1}^{K} c_{1,n} \frac{l}{\sqrt{\mu_n}} \sin(\frac{n\pi x}{l}), \tag{3.14}$$

$$w_2(x) = \sum_{n=1}^{K} c_{2,n} \frac{l}{\sqrt{\mu_n}} \sin(\frac{n\pi x}{l}). \tag{3.15}$$

Using the normalized eigenfunctions, (3.14) and (3.15) can be written in the following form

$$w = \sum_{n=1}^{\infty} d_{+n} \frac{\varphi_{+n}}{\|\varphi_{+n}\|} + d_{-n} \frac{\varphi_{-n}}{\|\varphi_{-n}\|}, \tag{3.16}$$

with

$$\begin{cases} \frac{d_{+n}}{\|\varphi_{+n}\|} + \frac{d_{-n}}{\|\varphi_{-n}\|} = \frac{c_{1,n}}{\sqrt{\mu_n}}, \\ \lambda_{+n} \frac{d_{+n}}{\|\varphi_{+n}\|} + \lambda_{-n} \frac{d_{-n}}{\|\varphi_{-n}\|} = \frac{c_{2,n}}{\sqrt{\mu_n}}. \end{cases} \tag{3.17}$$

We write (3.17)in a matrix notation as follows

$$\begin{pmatrix} c_{1,n} \\ c_{2,n} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{\mu_n}}{\|\varphi_{+n}\|} & \frac{\sqrt{\mu_n}}{\|\varphi_{-n}\|} \\ \frac{\sqrt{\mu_n}\lambda_{+n}}{\|\varphi_{+n}\|} & \frac{\sqrt{\mu_n}\lambda_{-n}}{\|\varphi_{-n}\|} \end{pmatrix} \begin{pmatrix} d_{+n} \\ d_{-n} \end{pmatrix}. \tag{3.18}$$

The set $\{\frac{\varphi_{+n}}{\|\varphi_{+n}|}, \frac{\varphi_{-n}}{\|\varphi_{-n}\|}, n \in \mathbb{N}\}$ forms a Riesz basis of $\mathcal{H}$ if $\{d_{\pm n}\}_n \in l^2$ whenever $\{c_{\pm n}\}_n \in l^2$. This holds if and only if the matrix in (3.18) is (uniformly) bounded and (uniformly) boundedly invertible. Since $\lambda_{+n} \neq \lambda_{-n}$ we have that for all n the matrix is invertible. Using (3.1), (3.10), (3.11) and (3.13), it is easily seen that the coefficients and determinant of the matrix in (3.18) are (uniformly) bounded and away from zero, which implies that the matrix is uniformly bounded and boundedly invertible. □

Since the normalized eigenfunctions $\{\frac{\varphi_{+n}}{\|\varphi_{+n}|}, \frac{\varphi_{-n}}{\|\varphi_{-n}\|}, n \in \mathbb{N}\}$ form a Riesz basis of $\mathcal{H}$, we have that they are all the eigenfunctions. Using (3.10) and (3.11), it is clear that eigenvalues of $\mathbb{A}$ are simple, totally disconnected and satisfy (3.5). This finishes the proof. □

4. Approximate Controllability

Consider the PDE (2.1) with initial and homogeneous boundary conditions as defined in equations (2.2)-(2.5). We assume that the shaping function $b(x)$ around the control point x_0 is as follows

$$b(x) = \frac{1}{2\epsilon} \mathcal{X}_{(x_0-\epsilon, x_0+\epsilon)}, \tag{4.1}$$

for some $\epsilon > 0$, where $\mathcal{X}_{\mathbb{J}}$ denotes the characteristic function of a set $\mathbb{J}$. If ϵ be close to zero, which we may assume, then $b(x)$ is approximation of Dirac delta distribution around x_0. This form is used in practical engineering applications.

Theorem 4.1. *Consider the system (2.1)-(2.5) , where $b(x)$ is defined by (4.1). The system is approximately controllable if and only if the following condition is satisfied*

$$\sin(\frac{n\pi x_0}{l})\sin(\frac{n\pi\epsilon}{l}) \neq 0, \forall n = 1, 2, \cdots, K. \tag{4.2}$$

Proof. Consider the abstract formulation of system (2.1)-(2.5) as defined in (3.3). Using[5] Theorem 4.2.3, shows that (3.3) is approximately controllable if and only if for all $n \in \mathbb{N}$

$$\int_{x_0-\epsilon}^{x_0+\epsilon} \sin(\frac{n\pi x}{l})dx \neq 0, n = 1, 2, \cdots, K. \tag{4.3}$$

The proof is now completed by direct calculation of integral (4.3). □

5. Conclusion and future works

This paper considers the vibration of nonlocal Rayleigh beam where a distributed control force acts on a specified subdomain. The control problem is formulated in abstract differential equation form. It is proved that the operator $\mathbb{A}$ generates C_0-semigroup and it's eigenfunctions form a Riesz basis. It is shown that the vibration of nonlocal Rayleigh beam can be controlled approximately using distributed control force. Relations (3.10) and (3.11) show that eigenvalues of A have zero real part. Thus, axial vibration of a nanorod is stable phenomena but not exponentially stable. For the future work, one can investigates feedback stabilization problem. In addition, boundary control problem is an another interesting topic for the future work.

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