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Development of non polynomial spline and New B-spline with application to solution of Klein-Gordon equation

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Abstract In this paper we develop a non polynomial cubic spline function which we called "TS spline", based on trigonometric functions. The convergence analysis of this spline is investigated in details. The definition of B-spline basis function for TS spline is extended and "TS B-spline" is introduced. This paper attempts to develop collocation method based on this B-spline for the numerical solution of the nonlinear Klein-Gordon equation. The convergence analysis of this approach is discussed, the second order of convergence is proved consequently. The proposed method is applied on some test examples and the numerical results are compared with those already available in literature. Observed errors in the solutions show the efficiency and numerical applicability of the proposed method.

Keywords. Non-polynomial spline function,B-spline function, Nonlinear Klein-Gordon equation, Convergence Analysis.

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1. Introduction

In the theory of approximation, cubic spline function have received much attention in approximating, interpolating and curve fitting. The non-polynomial tension spline function which contains a non negative tension parameter were first studied by Schweikert, as an alternative of cubic spline [26]. Later Spath modified tension spline by choosing different values of tension parameter in various regions of domain and proposed the exponential spline [28]. The convergence properties of the exponential spline have derived by Pruess [18], then, he established that the exponential spline can produce co-convex and co-monotone interpolants properties [19]. Exponential spline from a theoretical point of view has discussed by Mccartin in details, he introduced the cardinal spline basis and the B-spline basis for the space of exponential splines [17].The construction of exponential B-spline has studied in [11]. There are

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several kinds of B-splines over non polynomial spaces in the literature, such as uniform algebraic trigonometric B-spline basis (UAT B-spline)[35], nonuniform algebraic trigonometric B-spline basis (NUAT B-spline)[36] and uniform hyperbolic polynomial B-spline [16]. The cubic trigonometric spline is applied for the solution of hyperbolic problems in [1].

In the first part of this paper, another kind of non-polynomial spline function is studied, which we called "TS spline function" and "TS B-spline" basis is derived.

In the second part of this paper, we have developed collocation method based on the proposed B-spline for approximating the nonlinear Klein-Gordon equation. The Klein-Gordon equation arises in many scientific and engineering fields such as relativistic quantum mechanics, nonlinear optics and solid state physics [33]. For the numerical solution of the Klein-Gordon equation various methods have been presented, such as second order finite difference methods [8, 13], variational iteration methods [27, 34] and Differential transform method [10]. Bratsos applied a predictor corrector scheme based on rational approximation to the matrix-exponential term in a three-time level recurrence relation [3]. Dehghan and Shokri used thin plate spline (TPS) radial basis function (RBF), for the solution of Klein-Gordon equation with quadratic and cubic nonlinearities [5] and Lakestani and Dehghan used cubic spline scaling functions [12]. Rashidinia et al. derived a three time level scheme, based on tension spline approximation [20], and also developed a method using wavelet polynomials[22]. Using cubic B-spline collocation method, Rashidinia et al. [21] and also Kuri et al. [9] approximated the solution of the nonlinear Klein-Gordon equation. For generalized nonlinear Klein-Gordon equation, a collocation method is developed which possesses spectral accuracy in both space and time [7], also a meshless method presented with a well-posed moving least squares approximation[14]. For a class of non linear Klein-Gorden equations, a family of small amplitude periodic solutions are obtained [15]. Recently a numerical method based on the Crank–Nicolson scheme and the Tau method in [25], and also a new approach for the Legendre wavelet-based approximation method in [31] are developed for the solution of Klein-Gordon equation.

In this paper we have developed collocation method using TS B-spline functions for the solution of nonlinear Klein-Gordon equation. The convergence analysis of this method is established. Finally numerical experiments are included and compared with those methods which are comparable, to demonstrate the viability and efficiency of the proposed method.

2. Trigonometric spline function

Consider the partition $\triangle := \{a = x_0 < x_1 < ... < x_{N-1} < x_N = b\}$ of the interval $I = [a, b]$, with mesh spacing $h_i = x_{i+1} - x_i$, $i = 0, ..., N-1$. Let u_i's are the values of a function $u(x)$ at each x_i. To interpolate u, we define a trigonometric spline function (TS function) $t(x)$ of class $c^2[a, b]$ as the solution of the following differential equation in each subinterval $I_i = [x_i, x_{i+1}]$,

$$t^{(4)}(x) + \lambda_i^2 t''(x) = 0; x \in (x_i, x_{i+1}), \tag{2.1}$$

with boundary conditions

$$\begin{aligned} t(x_i) &= u_i, \qquad t(x_{i+1}) = u_{i+1}, \\ t''(x_i) &= t''_i, \qquad t''(x_{i+1}) = t''_{i+1}, \end{aligned} \tag{2.2}$$

where $\lambda_i^{,}$ s are non negative real parameters and $t''_i, i = 0, ..., N$, next to be determined. Note that when $\lambda_i \to 0$, we have $t^4(x) \to 0$ and $t(x)$ becomes the cubic spline function. The solution of the boundary value problem "(2.1)-(2.2)" is

$$t(x) = A_i + B_i x + C_i \cos \lambda_i x + D_i \sin \lambda_i x. \tag{2.3}$$

This solution is based on trigonometric functions $\sin \lambda_i x$ and $\cos \lambda_i x$, therefore, we called it "TS spline function", which is different from trigonometric spline already exist in the literature. From the conditions "(2.2)", we can determine the coefficients of function $t(x)$ in "(2.2)" as follows

$$A_i = \frac{1}{h_i} x_{i+1}(u_i + \frac{t''_i}{\lambda_i^2}) - \frac{1}{h_i} x_i(u_{i+1} + \frac{t''_{i+1}}{\lambda_i^2}),$$

$$B_i = \frac{1}{h_i}(u_{i+1} - u_i) + \frac{1}{h_i \lambda_i^2}(t''_{i+1} - t''_i),$$

$$C_i = \frac{1}{s_i \lambda_i^2}(\sin(\lambda_i x_i) t''_{i+1} - \sin(\lambda_i x_{i+1}) t''_i),$$

$$D_i = \frac{1}{s_i \lambda_i^2}(\cos(\lambda_i x_{i+1}) t''_i - \cos(\lambda_i x_i) t''_{i+1}),$$

where $s_i = \sin(\lambda_i h_i)$.

To derive expressions for determining $t''_i, i = 0, ..., N$, we consider two end conditions $t'(a) = u'(a)$ and $t'(b) = u'(b)$. Then, we conclude that t''_i are the solutions of the following tridiagonal linear system

$$\begin{cases} p_0 t''_0 + q_0 t''_1 = u'(a) - (u_1 - u_0)/h_0, \\ q_{i-1} t''_{i-1} + (p_i + p_{i-1}) t''_i + q_i t''_{i+1} = \frac{u_i - u_{i-1}}{h_{i-1}} - \frac{u_{i+1} - u_i}{h_i}, i = 1, \cdots, N-1, \\ q_{N-1} t''_{N-1} + p_{N-1} t''_N = (u_N - u_{N-1})/h_{N-1} - u'(b), \end{cases} \tag{2.4}$$

where

$$p_i = \frac{\lambda_i h_i c_i - s_i}{h_i s_i \lambda_i^2}, \qquad q_i = \frac{s_i - h_i \lambda_i}{h_i s_i \lambda_i^2}, \qquad c_i = \cos(\lambda_i h_i), i = 0, ..., N-1.$$

Let $\theta_i = \lambda_i h_i$,where $0 < \theta_i < 1$, then, we can write

$$p_i = h_i \frac{\theta_i c_i - s_i}{\theta_i^2 s_i}, \tag{2.5}$$

$$q_i = h_i \frac{s_i - \theta_i}{\theta_i^2 s_i}. \tag{2.6}$$

From Taylor expansions we find

$$| \frac{q_i}{p_i} |= \frac{\theta_i - s_i}{s_i - \theta_i c_i} = \frac{\frac{1}{3!}\theta_i^3 - \frac{1}{5!}\theta_i^5 + \frac{1}{7!}\theta_i^7 \pm ...}{\frac{1}{3}\theta_i^3 - \frac{1}{30}\theta_i^5 + \frac{1}{840}\theta_i^7 \pm ...} = \frac{1}{2}(\frac{1 - \frac{1}{20}\theta_i^2 + \frac{1}{840}\theta_i^4 \pm ...}{1 - \frac{1}{10}\theta_i^2 + \frac{1}{280}\theta_i^4 \pm ...}).$$

Without loss of significance we can consider

$$| \frac{q_i}{p_i} |= \frac{1}{2}(\frac{1 - \frac{1}{20}\theta_i^2}{1 - \frac{1}{10}\theta_i^2}) \leq \frac{5}{9} < 1. \tag{2.7}$$

Also,

$$\begin{aligned} \frac{| q_{i-1} | + | q_i |}{| p_{i-1} + p_i |} &= \frac{h_{i-1}\theta_i^2 s_i(\theta_{i-1} - s_{i-1}) + h_i\theta_{i-1}^2 s_{i-1}(\theta_i - s_i)}{h_{i-1}\theta_i^2 s_i(s_{i-1} - \theta_{i-1}c_{i-1}) + h_i\theta_{i-1}^2 s_{i-1}(s_i - \theta_i c_i)} \\ &\leq max\{\frac{\theta_{i-1} - s_{i-1}}{s_{i-1} - \theta_{i-1}c_{i-1}}, \frac{\theta_i - s_i}{s_i - \theta_i c_i}\} \leq \frac{5}{9} < 1. \end{aligned} \tag{2.8}$$

Thus, by using inequalities "(2.7)" and "(2.8)" the coefficient matrix of the tridiagonal system "(2.4)" is strictly diagonally dominant so that t_i'' could be uniquely determined.

3. Convergence analysis of trigonometric spline

In this section we establish order of convergence for the TR spline function $t(x)$. Let $\sigma(x)$ be the cubic spline function which interpolates $u(x)$ at the same mesh points in the partition $\triangle$ of the interval [a,b] with similar end conditions, we have the following relations for σ'' at nodal points [2]

$$\begin{cases} 2\sigma_0'' + \sigma_1'' = \frac{6}{h_0}((u_1 - u_0)/h_0 - u'(a)), \\ \mu_i\sigma_{i-1}'' + 2\sigma_i'' + (1 - \mu_i)\sigma_{i+1}'' = \frac{6}{h_{i-1}+h_i}(\frac{u_{i+1}-u_i}{h_i} - \frac{u_i - u_{i-1}}{h_{i-1}}), \\ \sigma_{N-1}'' + 2\sigma_N'' = \frac{6}{h_{N-1}}(u'(b) - (u_N - u_{N-1})/h_{N-1}), \end{cases} \tag{3.1}$$

where $\mu_i = \frac{h_{i-1}}{h_{i-1}+h_i}$, $i = 1, \cdots, N-1$.
Let $\delta = \sigma - t$,from the first equations in the systems "(2.4)" and "(3.1)", we have

$$p_0 t_0'' + q_0 t_1'' = \frac{-h_0}{6}(2\sigma_0'' + \sigma_1''),$$

and for $i = 1, 2, ..., N-1$, we obtain

$$q_{i-1}t_{i-1}'' + (p_i + p_{i-1})t_i'' + q_i t_{i+1}'' = -\frac{h_{i-1} + h_i}{6}(\mu_i\sigma_{i-1}'' + 2\sigma_i'' + (1 - \mu_i)\sigma_{i+1}''),$$

and from the last equations in systems"(2.4)" and "(3.1)" we have

$$q_{N-1}t_{N-1}'' + p_{N-1}t_N'' = \frac{-h_{N-1}}{6}(\sigma_{N-1}'' + 2\sigma_N'').$$

So we can conclude the following system for δ

$$\begin{cases} \delta_0'' + \frac{q_0}{p_0}\delta_1'' = (\frac{h_0}{3p_0} + 1)\sigma_0'' + \frac{h_0/6+q_0}{p_0}\sigma_1'', \\ \\ \frac{q_{i-1}}{p_{i-1}+p_i}\delta_{i-1}'' + \delta_i'' + \frac{q_i}{p_{i-1}+p_i}\delta_{i+1}'' = \\ \frac{h_{i-1}/6+q_{i-1}}{p_{i-1}+p_i}\sigma_{i-1}'' + (\frac{h_{i-1}+h_i}{3(p_{i-1}+p_i)} + 1)\sigma_i'' + \frac{h_i/6+q_i}{p_{i-1}+p_i}\sigma_{i+1}'', \qquad i = 1, \cdots, N-1 \\ \\ \frac{q_{N-1}}{p_{N-1}}\delta_{N-1}'' + \delta_N'' = \frac{h_{N-1}/6+q_{N-1}}{p_{N-1}}\sigma_{N-1}'' + (\frac{h_{N-1}}{3p_{N-1}} + 1)\sigma_N'', \end{cases} \tag{3.2}$$

where $\delta_i'' = \sigma_i'' - t_i''$, $i = 0, 1, ..., N$.
Let E and S be the vectors with components δ_i'' and σ_i'' , $i = 1, ..., N$, respectively, then the system "(3.2)" can be written in matrix form as follows

$$(I + P)E = HS, \tag{3.3}$$

where

$$P = \begin{pmatrix} 0 & \frac{q_0}{p_0} & & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \frac{q_{i-1}}{p_{i-1}+p_i} & 0 & \frac{q_i}{p_{i-1}+p_i} & & \\ & & & & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & & \frac{q_{N-1}}{p_{N-1}} & 0 \end{pmatrix},$$

$$\text{H} = \begin{pmatrix} \frac{h_0}{3p_0} + 1 & \frac{r_0}{p_0} & & & & \\ \ddots & \ddots & \ddots & & & \\ & \frac{r_{i-1}}{p_{i-1}+p_i} & \frac{h_{i-1}+h_i}{3(p_{i-1}+p_i)} + 1 & \frac{r_i}{p_{i-1}+p_i} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{r_{N-1}}{p_{N-1}} & \frac{h_{N-1}}{3p_{N-1}} + 1 \end{pmatrix},$$

where $r_i = h_i/6 + q_i, i = 1, ..., N-1$.
From "(2.5)" and "(2.6)" and by Taylor expansions we find

$$\mid \frac{h_i}{3p_i} + 1 \mid = \mid \frac{\theta_i^2 s_i + 3(\theta_i c_i - s_i)}{3(\theta_i c_i - s_i)} \mid = \frac{1}{15}\theta_i^2 (\frac{1 - \frac{1}{14}\theta_i^2 + \frac{1}{494}\theta_i^4 \pm ...}{1 - \frac{1}{10}\theta_i^2 + \frac{1}{280}\theta_i^4 \pm ...}) \leq \frac{2}{27}\theta_i^2,$$

$$\mid \frac{h_{i-1} + h_i}{3(p_{i-1} + p_i)} + 1 \mid =$$

$$\mid \frac{h_{i-1}\theta_i^2 s_i(\theta_{i-1}^2 s_{i-1} + 3\theta_{i-1}c_{i-1} - 3s_{i-1}) + h_i\theta_{i-1}^2 s_{i-1}(\theta_i^2 s_i + 3\theta_i c_i - 3s_i)}{3h_{i-1}\theta_i^2 s_i(\theta_{i-1}c_{i-1} - s_{i-1}) + 3h_i\theta_{i-1}^2 s_{i-1}(\theta_i c_i - s_i)} \mid$$

$$\leq \frac{2}{27} max\{\theta_{i-1}^2, \theta_i^2\} \leq \frac{2}{27}\lambda_{max}^2 h^2,$$

where $\lambda_{max} = max_i\{\lambda_i\}$ and $h = max_i\{h_i\}$. Also

$$| \frac{r_i}{p_{i-1}+p_i} |= \frac{| h_i/6+q_i |}{| p_{i-1}+p_i |} =$$

$$\frac{| \theta_{i-1}^2 s_{i-1} h_i(\theta_i^2 s_i + 6(s_i-\theta_i)) |}{| 6h_{i-1}\theta_i^2 s_i(\theta_{i-1}c_{i-1}-s_{i-1}) + 6h_i\theta_{i-1}^2 s_{i-1}(\theta_i c_i - s_i) |} \leq$$

$$\frac{7}{120}\theta_i^2(\frac{1-\frac{3}{49}\theta_i^2 \pm ...}{1-\frac{1}{10}\theta_i^2 \pm ...}) \leq \frac{7}{108}\theta_i^2,$$

The above inequalities yield $\| H \|_\infty \leq \frac{11}{54}\lambda_{max}^2 h^2$. In the other hand "(2.7)" and "(2.8)" imply that $\| P \|_\infty \leq \frac{5}{9} < 1$, and also the matrix $(I+P)$ is diagonally dominant so that $(I+P)^{-1}$ exist. Then from "(3.3)" we have

$$\begin{aligned} \| E \|_\infty &\leq \| (I+P)^{-1} \|_\infty \| H \|_\infty \| S \|_\infty \\ &\leq \frac{\| H \| \| S \|}{1- \| P \|} \leq \frac{11}{24}\lambda_{max}^2 h^2 max_i | \sigma_i'' | . \end{aligned} \tag{3.4}$$

Theorem 3.1. *Let $\triangle$ be a given partition of the interval $[a,b]$ by the knots x_i, $i = 0,...,N$, with $h_i = x_{i+1}-x_i$. Let $t(x)$ be the TR spline function and $\sigma(x)$ be the cubic spline function which interpolate u at the knots x_i, then*

$$\| \sigma^{(j)}(x) - t^{(j)}(x) \| \leq \frac{5}{4}\lambda_{\max}^2 h^{4-j} \max_i | \sigma_i'' |, j=0,1,2. \tag{3.5}$$

where $h = max_i h_i$.

Proof. For $x \in [x_i, x_{i+1}]$, recalling definitions of functions $\sigma(x)$ and $t(x)$ gives

$$\sigma''(x) = \frac{x_{i+1}-x}{h_i}\sigma_i'' + \frac{x-x_i}{h_i}\sigma_{i+1}'',$$

$$t''(x) = \frac{1}{s_i}(\sin(\lambda_i x_{i+1})t_i'' - \sin(\lambda_i x_i)t_{i+1}'')\cos\lambda_i x +$$

$$\frac{1}{s_i}(\cos(\lambda_i x_i)t_{i+1}'' - \cos(\lambda_i x_{i+1})t_i'')\sin\lambda_i x.$$

Let $\delta = \sigma - t$, then

$$\begin{aligned} \delta''(x) =& [\frac{x_{i+1}-x}{h_i} - \frac{\sin\lambda_i(x_{i+1}-x)}{s_i}]\sigma_i'' + [\frac{x-x_i}{h_i} - \frac{\sin\lambda_i(x-x_i)}{s_i}]\sigma_{i+1}'' \\ &+ \frac{\sin\lambda_i(x_{i+1}-x)}{s_i}\delta_i'' + \frac{\sin\lambda_i(x-x_i)}{s_i}\delta_{i+1}''. \end{aligned} \tag{3.6}$$

From Taylor expansions of $\sin\lambda_i(x_{i+1}-x)$ and $\sin\lambda_i(x-x_i)$ and $sin\lambda_i h_i(s_i)$, we conclude that

$$
\begin{aligned}
&\mid \frac{x_{i+1}-x}{h_i} - \frac{\sin\lambda_i(x_{i+1}-x)}{s_i} \mid \leq \frac{1}{6}\lambda_i^2 h_i^2,\\
&\mid \frac{x-x_i}{h_i} - \frac{\sin\lambda_i(x-x_i)}{s_i} \mid \leq \frac{1}{6}\lambda_i^2 h_i^2,\\
&\mid \frac{\sin\lambda_i(x_{i+1}-x)}{s_i} \mid \leq 1, \quad \mid \frac{\sin\lambda_i(x-x_i)}{s_i} \mid \leq 1.
\end{aligned}
$$

Therefore, from "(3.6)" and using "(3.4)" we obtain

$$\parallel \delta''(x) \parallel \leq \frac{5}{4}\lambda_{\max}^2 h^2 \max_i \mid \sigma_i'' \mid . \tag{3.7}$$

In the other hand at each nodal point $x_i, i = 0, 1, ..., N$, we have $\delta(x_i) = \delta(x_{i+1}) = 0$, then, from Roll's theorem there exist $\zeta_i \in (x_i, x_{i+1})$ so that $\delta'(\zeta_i) = 0$ and for $x \in (x_i, x_{i+1})$ we have

$$\delta(x) = \int_{x_i}^{x} \delta'(t)\, dt, \qquad \delta'(x) = \int_{\zeta_i}^{x} \delta''(t)\, dt.$$

This together with "(3.7)"yields "(3.5)". □

So, from this and the convergence theorem for the cubic spline [29] we can conclude that

Corollary. Let $u(x) \in C^4[a,b]$ and $\triangle$ be a given partition of the interval $[a,b]$ by the knots x_i, $i = 0, ..., N$, with $h_i = x_{i+1} - x_i$. If $t(x)$ be the TR spline function which interpolates $u(x)$ at the knots x_i, then there are constants γ_j independent of h but depend on λ_{max} such that

$$\parallel u^{(j)} - t^{(j)} \parallel \leq \gamma_j h^{(4-j)}, j = 0, 1, 2. \tag{3.8}$$

4. Trigonometric basis functions

Consider a uniform partition of the interval [a,b] by the knots $x_i, i = 0, ..., N$, with $h = x_{i+1} - x_i$ and let $t(x)$ be the TS spline function which interpolates $u(x)$ as follows

$$t(x) = A + Bx + C\cos\lambda x + D\sin\lambda x,$$

with the following end conditions
$t(x_0) = u(x_0), t(x_N) = u(x_N), t'(x_0) = u'(x_0), t'(x_N) = u'(x_N)$,
where λ is uniform non negative parameter.

We define basis functions for the space of TR spline, by introducing three additional nodal points on each side of the partition. Thus the partition becomes

$$\triangle := \{x_{-3} < x_{-2} < x_{-1} < x_0 < ... < x_N < x_{N+1} < x_{N+2} < x_{N+3}\}.$$

Let $\beta_i(x)$ be TR spline functions centered at $x_i, i = -1, ..., N+1$ which have minimal support of four intervals in $[x_{i-2}, x_{i+2}]$, with knots at $x_{i-2} < x_{i-1} < x_i < x_{i+1} < x_{i+2}$. (as the same as cubic spline, each TR spline with support of less than four intervals, is the zero function). Therefore $\beta_i(x)$ and its first and second derivatives in the end points x_{i-2} and x_{i+2} must be equal to zero, that is

$$\begin{cases} \beta_i(x_{i-2}) = \beta_i'(x_{i-2}) = \beta_i''(x_{i-2}) = 0, \\ \beta_i(x_{i+2}) = \beta_i'(x_{i+2}) = \beta_i''(x_{i+2}) = 0. \end{cases} \tag{4.1}$$

For normalization we set

$$\beta_i(x_i) = 1, \qquad \beta_i'(x_i) = 0. \tag{4.2}$$

At first we derive $\beta_i(x)$ in the interval $[x_i, x_{i+2}]$ and then reflect the result about the line $x = x_i$. We consider

$$\beta_i(x) = \begin{cases} \bar{A}_1 + \bar{B}_1(x-x_i) + \bar{C}_1 \cos\lambda(x-x_i) + \bar{D}_1 \sin\lambda(x-x_i) & x \in [x_i, x_{i+1}], \\ \bar{A}_2 + \bar{B}_2(x-x_i) + \bar{C}_2 \cos\lambda(x-x_i) + \bar{D}_2 \sin\lambda(x-x_i) & x \in [x_{i+1}, x_{i+2}]. \end{cases} \tag{4.3}$$

From "(4.1)"-"(4.3)" and continuity of the function $\beta_i(x)$ and also its first and second derivatives we obtain the TR B-splines as follows

$$\beta_i(x) = \begin{cases} \bar{B}_2(x_{i-2}-x) - (\bar{B}_2/\lambda)\sin\lambda(x_{i-2}-x), & x \in [x_{i-2}, x_{i-1}], \\ \bar{A}_1 + \bar{B}_1(x_i-x) + \bar{C}_1\cos\lambda(x_i-x) + \bar{D}_1\sin\lambda(x_i-x), & x \in [x_{i-1}, x_i], \\ \bar{A}_1 + \bar{B}_1(x-x_i) + \bar{C}_1\cos\lambda(x-x_i) + \bar{D}_1\sin\lambda(x-x_i), & x \in [x_i, x_{i+1}], \\ \bar{B}_2(x-x_{i+2}) - (\bar{B}_2/\lambda)\sin\lambda(x-x_{i+2}), & x \in [x_{i+1}, x_{i+2}], \\ 0 & otherwise \end{cases} \tag{4.4}$$

where
$\bar{A}_1 = \frac{\lambda hc}{\lambda hc - s}, \bar{B}_1 = \frac{-\lambda(1+2c)}{2(\lambda hc-s)}, \bar{C}_1 = \frac{-c}{\lambda hc - s}, \bar{D}_1 = \frac{1+2c}{2(\lambda hc-s)},$
$\bar{B}_2 = \frac{\lambda}{2(\lambda hc-s)}, \quad c = \cos\lambda h, \quad s = \sin\lambda h.$
Also,

$$\beta_i'(x) = \begin{cases} -\bar{B}_2 + \bar{B}_2\cos\lambda(x_{i-2}-x), & x \in [x_{i-2}, x_{i-1}], \\ -\bar{B}_1 + \lambda\bar{C}_1\sin\lambda(x_i-x) - \lambda\bar{D}_1\cos\lambda(x_i-x), & x \in [x_{i-1}, x_i], \\ \bar{B}_1 - \lambda\bar{C}_1\sin\lambda(x-x_i) + \lambda\bar{D}_1\cos\lambda(x-x_i), & x \in [x_i, x_{i+1}], \\ \bar{B}_2 - \bar{B}_2\cos\lambda(x-x_{i+2}), & x \in [x_{i+1}, x_{i+2}], \end{cases}$$

and

$$\beta_i''(x) = \begin{cases} \lambda\bar{B}_2\sin\lambda(x_{i-2}-x), & x \in [x_{i-2}, x_{i-1}], \\ -\lambda^2\bar{C}_1\cos\lambda(x_i-x) - \lambda^2\bar{D}_1\sin\lambda(x_i-x), & x \in [x_{i-1}, x_i], \\ -\lambda^2\bar{C}_1\cos\lambda(x-x_i) - \lambda^2\bar{D}_1\sin\lambda(x-x_i), & x \in [x_i, x_{i+1}], \\ \lambda\bar{B}_2\sin\lambda(x-x_{i+2}), & x \in [x_{i+1}, x_{i+2}], \end{cases}$$

The values of $\beta_i(x)$ and their derivatives are tabulated in table 1. Note that if λ tends to zero we imply $\beta_i(x_{i+1}) \to \frac{1}{4}$, which is compatible with normal form of cubic spline limit.

TABLE 1. trigonometric B-splines values

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$\beta_i(x)$	0	$\frac{s-\lambda h}{2(\lambda hc-s)}$	1	$\frac{s-\lambda h}{2(\lambda hc-s)}$	0
$\beta_i'(x)$	0	$\frac{\lambda(c-1)}{2(\lambda hc-s)}$	0	$\frac{\lambda(1-c)}{2(\lambda hc-s)}$	0
$\beta_i''(x)$	0	$\frac{-\lambda^2 s}{2(\lambda hc-s)}$	$\frac{\lambda^2 s}{(\lambda hc-s)}$	$\frac{-\lambda^2 s}{2(\lambda hc-s)}$	0

Now we show that the set of functions $\beta_i(x), i = -1, ..., N+1$, form a basis for the space of TS splines over the region [a,b]. It can be easily seen that $\beta_i(x), i = -1, ..., N+1$, are linearly independent. If we suppose $t(x) = \sum_{i=-1}^{N+1} b_i\beta_i(x)$, then

$$t(x_j) = \sum_{i=-1}^{N+1} b_i\beta_i(x_j) = \frac{s-\lambda h}{2(\lambda hc-s)}b_{j-1} + b_j + \frac{s-\lambda h}{2(\lambda hc-s)}b_{j+1}$$

$$t'(x_j) = \sum_{i=-1}^{N+1} b_i\beta_i'(x_j) = \frac{\lambda(c-1)}{2(\lambda hc-s)}b_{j-1} + \frac{\lambda(1-c)}{2(\lambda hc-s)}b_{j+1}$$

From the slope end conditions we find the linear system as follows

$$\begin{cases} \frac{\lambda(c-1)}{2(\lambda hc-s)}b_{-1} + \frac{\lambda(1-c)}{2(\lambda hc-s)}b_1 = u'(x_0), \\ \frac{s-\lambda h}{2(\lambda hc-s)}b_{j-1} + b_j + \frac{s-\lambda h}{2(\lambda hc-s)}b_{j+1} = u(x_j), 0 \le j \le N \\ \frac{\lambda(c-1)}{2(\lambda hc-s)}b_{N-1} + \frac{\lambda(1-c)}{2(\lambda hc-s)}b_{N+1} = u'(x_N). \end{cases} \tag{4.5}$$

This tridiagonal system is diagonally dominant so that b_i's uniquely could be determined.

5. TRIGONOMETRIC B-SPLINE COLLOCATION METHOD FOR THE SOLUTION OF NONLINEAR KLEIN-GORDON EQUATION

In this section we propose the collocation method based on TS B-splines to solve the nonlinear Klein-Gordon equation in the following form

$$\frac{\partial^2 u}{\partial t^2} - \mu\frac{\partial^2 u}{\partial x^2} + \nu u + F(u) = g(x,t), \quad x \in [a,b], t \ge t_0, \tag{5.1}$$

with initial conditions

$$u(x,t_0) = \vartheta(x), u_t(x,t_0) = \omega(x), \tag{5.2}$$

And Dirichlet boundary conditions as follows

$$u(a,t) = \phi_1(t), u(b,t) = \phi_2(t), \tag{5.3}$$

where $\mu > 0$ and ν are known real constants and $F(u)$ is a non linear function such that $\partial F/\partial u \ge 0$. Let us consider a uniform mesh in the domain of the Eq."(5.1)",

$[a, b] * [t_0, t]$ with the grid points (x_i, t_l), where $x_i = a + ih$, $i = 0, 1, ..., N$. and $t_l = t_0 + lk$, $l = 0, 1, 2, ...$ and h, k are the space and time step sizes respectively. At first we discretize the second order time derivative term in Eq."(5.1)", using finite difference approximation with uniform step size k as follows

$$u_{tt}^n \simeq \frac{1}{k^2}(1 + \frac{1}{12}\delta_t^2)^{-1}\delta_t^2 u^n, \tag{5.4}$$

where $\delta_t^2 u^n = u^{n-1} - 2u^n + u^{n+1}$, $u^n = u(x, t_n), u^0 = u(x, t_0) = \vartheta(x)$.
Substituting "(5.4)" in Eq."(5.1)" we obtain

$$\delta_t^2 u^n + k^2(1 + \frac{1}{12}\delta_t^2)(-\mu u_{xx}^n + \nu u^n + F(u^n)) = k^2(1 + \frac{1}{12}\delta_t^2)g(x, t_n).$$

Thus we have

$$\frac{-\mu k^2}{12}u_{xx}^{n+1} + (1 + \frac{\nu k^2}{12})u^{n+1} + \frac{k^2}{12}F(u^{n+1}) = G(x, t_{n+1}), \tag{5.5}$$

where

$$\begin{aligned} G(x, t_{n+1}) = & \frac{k^2}{12}(g^{n-1} + 10g^n + g^{n+1}) + \frac{\mu k^2}{12}(10u_{xx}^n + u_{xx}^{n-1}) \\ & -\frac{\nu k^2}{12}(10u^n + u^{n-1}) + 2u^n - u^{n-1} - \frac{k^2}{12}(10F(u^n) + F(u^{n-1})), \end{aligned} \tag{5.6}$$

and $g^n = g(x, t_n)$.
Relation "(5.5)" is a three time level scheme, which requires two initial value at the zeroth and first time levels. Using Taylor expansion for u at $t = t_0$ we have

$$u^1 = u^0 + ku_t^0 + \frac{k^2}{2}u_{tt}^0 + \frac{k^3}{6}u_{ttt}^0 + \frac{k^4}{24}u_{tttt}^0 + O(k^5). \tag{5.7}$$

From initial conditions we have

$$i) u^0 = \vartheta(x), \qquad ii) u_t^0 = \omega(x). \tag{5.8}$$

Eq."(5.1)" can be written as follows

$$u_{tt} = \mu u_{xx} - \nu u - F(u) + g(x, t) \tag{5.9}$$

When t tends to t_0, from "(5.9)" and using "(5.8)" we have

$$u_{tt}^0 = \mu\vartheta''(x) - \nu\vartheta(x) - F(\vartheta(x)) + g(x, t_0) \tag{5.10}$$

Differentiating "(5.9)" with respect to t and using "(5.6)" we get

$$u_{ttt}^0 = \mu\omega''(x) - \nu\omega(x) - F_u^0.\omega(x) + g_t(x, t_0) \tag{5.11}$$

By differentiating "(5.9)" twice with respect to t and also twice with respect to x and using "(5.6)" we obtain

$$\begin{aligned} u_{tttt}^0 = & \mu^2\vartheta^4(x) - 2\mu\nu\vartheta''(x) + \nu^2\vartheta(x) - \mu F_{xx}^0 \\ & + \nu F(\vartheta(x)) - F_{tt}^0 + \mu g_{xx}(x, t_0) - \nu g(x, t_0) + g_{tt}(x, t_0). \end{aligned} \tag{5.12}$$

By substituting "(5.8)" and "(5.10)"- "(5.12)" in "(5.7)" we achieve the forth order approximation of u at $t = t_1$. Next we will approximate the solutions $u^n, (n \geq 2)$ using TR B-spline collocation method.

Let us consider $U(x,t)$ be the TR B-spline approximate solution of the boundary value problem "(5.1)"-"(5.3)" in the form

$$U(x,t) = \sum_{j=-1}^{N+1} \alpha_j(t)\beta_j(x), \tag{5.13}$$

where $\alpha_j(t)$ are unknown time dependent coefficients and $\beta_j(x)$ are TR B-spline functions defined in "(4.4)"
Assume $U^n = U(x,t_n)$ be the trigonometric approximation to the exact solution $u(x,t)$ at $t = t_n$, then corresponding to "(5.5)" we have

$$\frac{-\mu k^2}{12} U_{xx}^{n+1} + (1 + \frac{\nu k^2}{12}) U^{n+1} + \frac{k^2}{12} F(U^{n+1}) = G(x, t_{n+1}). \tag{5.14}$$

Substituting "(5.13)"in "(5.14)" at $x = x_i, i = 0, 1, ..., N$, we find

$$\frac{-\mu k^2}{12} \sum_{j=-1}^{N+1} \overline{\alpha_j} \beta_j''(x_i) + (1 + \frac{\nu k^2}{12}) \sum_{j=-1}^{N+1} \overline{\alpha_j} \beta_j(x_i) + \frac{k^2}{12} F(\sum_{j=-1}^{N+1} \overline{\alpha_j} \beta_j(x_i)) = \overline{G_i}, \tag{5.15}$$

where $\overline{\alpha_j} = \alpha_j(t_{n+1})$ and $\overline{G_i} = G(x_i, t_{n+1})$.

From the values of B-spline functions in table1, we obtain the following non linear system of equations

$$\begin{aligned} &\frac{\mu k^2 \lambda^2 s + (12 + \nu k^2)(s - \lambda h)}{24(\lambda hc - s)} \overline{\alpha_{i-1}} + (\frac{-\mu k^2 \lambda^2 s}{12(\lambda hc - s)} + 1 + \frac{\nu k^2}{12}) \overline{\alpha_i} \\ &+ \frac{\mu k^2 \lambda^2 s + (12 + \nu k^2)(s - \lambda h)}{24(\lambda hc - s)} \overline{\alpha_{i+1}} \\ &+ \frac{k^2}{12} F(\frac{s - \lambda h}{2(\lambda hc - s)} \overline{\alpha_{i-1}} + \overline{\alpha_i} + \frac{s - \lambda h}{2(\lambda hc - s)} \overline{\alpha_{i+1}}) = \overline{G_i}, \quad i = 0, ..., N. \end{aligned} \tag{5.16}$$

In order to find a unique solution for the above system we eliminate $\overline{\alpha_{-1}}$ and $\overline{\alpha_{N+1}}$ from boundary conditions as follows,
from "(5.13)" and using the first boundary condition at $x = a$ we have

$$U(a, t_{n+1}) = \frac{s - \lambda h}{2(\lambda hc - s)} \overline{\alpha_{-1}} + \overline{\alpha_0} + \frac{s - \lambda h}{2(\lambda hc - s)} \overline{\alpha_1} = \phi_1(t_{n+1}). \tag{5.17}$$

Eliminating $\overline{\alpha_{-1}}$ from Eq. (37) for $i = 0$ and Eq. (38) implies

$$\begin{aligned} &\frac{\mu k^2 \lambda^3 sh(1 - c)}{12(s - \lambda h)(\lambda hc - s)} \overline{\alpha_0} + \frac{k^2}{12} F(\phi_1(t_{n+1})) = \\ &\overline{G_0} - (\frac{\mu k^2 \lambda^2 s}{12(s - \lambda h)} + 1 + \frac{\nu k^2}{12}) \phi_1(t_{n+1}). \end{aligned} \tag{5.18}$$

Similarly from the second boundary condition at $x = b$ we have

$$U(b, t_{n+1}) = \frac{s - \lambda h}{2(\lambda hc - s)} \overline{\alpha_{N-1}} + \overline{\alpha_N} + \frac{s - \lambda h}{2(\lambda hc - s)} \overline{\alpha_{N+1}} = \phi_2(t_{n+1}). \tag{5.19}$$

After eliminating $\overline{\alpha_{N+1}}$ from "(5.16)" and "(5.19)" for $i = N$, we obtain

$$\frac{\mu k^2\lambda^3 sh(1-c)}{12(s-\lambda h)(\lambda hc - s)}\overline{\alpha_N} + \frac{k^2}{12}F(\phi_2(t_{n+1})) = \overline{G_N} - (\frac{\mu k^2\lambda^2 s}{12(s-\lambda h)} + 1 + \frac{\nu k^2}{12})\phi_2(t_{n+1}). \tag{5.20}$$

Equations "(5.18)" and "(5.20)" together with "(5.16)" for i=1,...,N-1, form a nonlinear $(N+1)*(N+1)$ system as follows

$$A\overline{\alpha} + \frac{k^2}{12}\overline{F} = \overline{G}, \tag{5.21}$$

where

$$A = \begin{pmatrix} a_0 & 0 & 0 & & \\ b & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & b & a & b \\ & & 0 & 0 & a_0 \end{pmatrix}, \qquad \overline{\alpha} = \begin{pmatrix} \overline{\alpha_0} \\ \overline{\alpha_1} \\ \vdots \\ \overline{\alpha_N} \end{pmatrix}, \tag{5.22}$$

$$\overline{F} = \begin{pmatrix} F(\phi_1(t_{n+1})) \\ \overline{F_1} \\ \vdots \\ \overline{F_{N-1}} \\ F(\phi_2(t_{n+1})) \end{pmatrix}, \qquad \overline{G} = \begin{pmatrix} \overline{G_0} - b_0\phi_1(t_{n+1}) \\ \overline{G_1} \\ \vdots \\ \overline{G_{N-1}} \\ \overline{G_N} - b_0\phi_2(t_{n+1}) \end{pmatrix}, \tag{5.23}$$

and

$$a_0 = \frac{\mu k^2\lambda^3 sh(1-c)}{12(s-\lambda h)(\lambda hc - s)}, \qquad b_0 = \frac{\mu k^2\lambda^2 s}{12(\lambda hc - s)} + 1 + \frac{\nu k^2}{12},$$

$$a = \frac{-\mu k^2\lambda^2 s}{12(\lambda hc - s)} + 1 + \frac{\nu k^2}{12}, \qquad b = \frac{\mu k^2\lambda^2 s}{24(\lambda hc - s)} + (1 + \frac{\nu k^2}{12})\frac{s-\lambda h}{2(\lambda hc - s)},$$

$$\overline{F_i} = F(\frac{s-\lambda h}{2(\lambda hc - s)}\overline{\alpha_{i-1}} + \overline{\alpha_i} + \frac{s-\lambda h}{2(\lambda hc - s)}\overline{\alpha_{i-1}}), i = 1, ..N-1. \tag{5.24}$$

6. Convergence of the method for solution of the nonlinear Klein-Gordon equation

We denote the exact solution of nonlinear problem "(5.1)"-"(5.3)" at $t = t_{n+1}$, by $\bar{u}(x)$. Let $\bar{U}(x) = \sum_{j=-1}^{N+1}\bar{\alpha}_j\beta_j(x)$ be the unique TR spline approximation to $\bar{u}(x)$, and also $\hat{U}(x) = \sum_{j=-1}^{N+1}\hat{\alpha}_j\beta_j(x)$ be the computed TR spline which interpolate $\bar{u}(x)$. To prove that $\bar{U}(x)$ converges to $\bar{u}(x)$ uniformly, at first we estimate a bound for the error $\| \bar{U}(x) - \hat{U}(x) \|$. For this we need to prove the following lemma

Lemma 6.1. *The TS B-splines $\beta_{-1}, \beta_0, ..., \beta_{N+1}$ satisfy the inequality*

$$\sum_{i=-1}^{N+1} \mid \beta_i(x) \mid \leq \frac{23}{9}. \tag{6.1}$$

Proof. From definitions of the TS B-splines in table.1, at each nodal point $x = x_i$ we have

$$\begin{aligned} \sum_{i=-1}^{N+1} \mid \beta_i(x) \mid &= \mid \beta_{i-1}(x) \mid + \mid \beta_i(x) \mid + \mid \beta_{i+1}(x) \mid \\ &= \mid \frac{s-\lambda h}{2(\lambda hc - s)} \mid +1+ \mid \frac{s-\lambda h}{2(\lambda hc - s)} \mid . \end{aligned} \tag{6.2}$$

Using "(2.7)" but in uniform mesh we conclude that

$$\mid \frac{s-\lambda h}{\lambda hc - s} \mid \leq \frac{5}{9}.$$

Substituting the above relation in (47) gives

$$\sum_{i=-1}^{N+1} \mid \beta_i(x) \mid \leq 1 + \frac{5}{9} < \frac{23}{9}.$$

Also for the point x in each subinterval $[x_{i-1}, x_i]$ we have

$$\begin{aligned} \sum_{i=-1}^{N+1} \mid \beta_i(x) \mid &= \mid \beta_{i-2}(x) \mid + \mid \beta_{i-1}(x) \mid + \mid \beta_i(x) \mid + \mid \beta_{i+1}(x) \mid \\ &= \mid \frac{s-\lambda h}{2(\lambda hc - s)} \mid +1+1+ \mid \frac{s-\lambda h}{2(\lambda hc - s)} \mid \leq \frac{23}{9}. \end{aligned}$$

Which proves lemma. □

Corresponding to equation system "(5.21)" for the approximate solution $\bar{U}$ we have the following system for the computed solution $\hat{U}$

$$A\hat{\alpha} + \frac{k^2}{12}\hat{F} = \hat{G}, \tag{6.3}$$

where

$$\hat{F} = \begin{pmatrix} F(\phi_1(t_{n+1})) \\ \hat{F_1} \\ \vdots \\ \hat{F_{N-1}} \\ F(\phi_2(t_{n+1})) \end{pmatrix}, \hat{\alpha} = \begin{pmatrix} \hat{\alpha_0} \\ \hat{\alpha_1} \\ \vdots \\ \\ \hat{\alpha_N} \end{pmatrix}, \qquad \hat{G} = \begin{pmatrix} \hat{G_0} - b_0\phi_1(t_{n+1}) \\ \hat{G_1} \\ \vdots \\ \hat{G_{N-1}} \\ \hat{G_N} - b_0\phi_2(t_{n+1}) \end{pmatrix}, \tag{6.4}$$

and

$$\hat{F_i} = F(\frac{s-\lambda h}{2(\lambda hc - s)}\hat{\alpha_{i-1}} + \hat{\alpha_i} + \frac{s-\lambda h}{2(\lambda hc - s)}\hat{\alpha_{i-1}}), i = 1, ...N-1. \tag{6.5}$$

Subtracting "(5.21)" from "(6.3)" we get

$$A(\hat{\alpha}-\bar{\alpha})+\frac{k^2}{12}(\hat{F}-\bar{F})=\hat{G}-\bar{G}. \tag{6.6}$$

From definitions "(5.24)" and "(6.5)" for the components of $\bar{F}$ and $\hat{F}$, the mean value theorem implies that

$$(\hat{F}-\bar{F})=\frac{\partial F}{\partial u}(\eta)J(\hat{\alpha}-\bar{\alpha}), \tag{6.7}$$

where $\eta \in (0,1)$ and J is the following $(N+1)*(N+1)$ matrix

$$J=\begin{pmatrix} 0 & 0 & & & \\ \frac{s-\lambda h}{2(\lambda hc-s)} & 1 & \frac{s-\lambda h}{2(\lambda hc-s)} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{s-\lambda h}{2(\lambda hc-s)} & 1 & \frac{s-\lambda h}{2(\lambda hc-s)} \\ & & & 0 & 0 \end{pmatrix}$$

Substituting "(6.7)" into "(6.6)" yields

$$D(\hat{\alpha}-\bar{\alpha})=\hat{G}-\bar{G}, \tag{6.8}$$

where $D=A+\frac{k^2}{12}\frac{\partial F}{\partial u}(\eta)J$.
For sufficiently small value of h we obtain

$$\begin{aligned}&\mid d_{ii}\mid -(\mid d_{i,i-1}\mid + \mid d_{i,i+1}\mid) \geq \\ &\mid a+\frac{k^2}{12}\frac{\partial F}{\partial u}(\eta)\mid -2\mid b+\frac{k^2}{12}\frac{s-\lambda h}{2(\lambda hc-s)}\frac{\partial F}{\partial u}(\eta)\mid \geq \kappa_0>0.\end{aligned} \tag{6.9}$$

Hence the tridiagonal coefficient matrix D is nonsingular and from "(6.8)" we get

$$\hat{\alpha}-\bar{\alpha}=D^{-1}(\hat{G}-\bar{G}). \tag{6.10}$$

Also from Varah [30] we have

$$\parallel D^{-1}\parallel_{\infty} \leq 1/(min(\mid d_{ii}\mid -\sum_{i\neq j}\mid d_{ij}\mid).$$

Therefore, from "(6.9)" we can conclude that

$$\parallel D^{-1}\parallel_{\infty} \leq \frac{1}{\kappa_0}, \tag{6.11}$$

and then, "(6.10)" implies that

$$\parallel \hat{\alpha}-\bar{\alpha}\parallel \leq \frac{1}{\kappa_0}\parallel \hat{G}-\bar{G}\parallel . \tag{6.12}$$

In the other hand from "(5.15)" and definition of $\bar{U}(x)$ we find that

$$\overline{G_i}=\frac{-\mu k^2}{12}\overline{U}''(x_i)+(1+\frac{\nu k^2}{12})\overline{U}(x_i)+\frac{k^2}{12}F(\overline{U}(x_i)),$$

Then,

$$| \hat{G}_i - \overline{G_i} |\leq \frac{\mu k^2}{12} | \hat{U}''(x_i) - \overline{U}''(x_i) | +(1+\frac{\nu k^2}{12}) | \hat{U}(x_i) - \overline{U}(x_i) | +$$

$$\frac{k^2}{12} | F(\hat{U}(x_i)) - F(\overline{U}(x_i)) | . \tag{6.13}$$

From a theorem in analysis [23] we have

$$| F(\hat{U}(x_i)) - F(\overline{U}(x_i)) |\leq M_1 | \hat{U}(x_i) - \overline{U}(x_i) |,$$

where $\| F'(U) \|\leq M_1$.
Hence, from "(6.13)" and using "(3.8)", the corollary of theorem 2, we obtain

$$\| \hat{G}_i - \overline{G_i} \|\leq Mh^2, \tag{6.14}$$

where $M = \frac{\mu k^2}{12}\gamma_2 + (1+\frac{\nu k^2}{12})\gamma_0 h^2 + \frac{k^2}{12}M_1\gamma_0 h^2$. Substituting "(6.14)" in "(6.12)" yields

$$\| \hat{\alpha} - \bar{\alpha} \|\leq \frac{M}{\kappa_0}h^2. \tag{6.15}$$

Theorem 6.2. *Suppose that $\bar{u}(x)$ and $\bar{U}(x)$ be the exact and TR collocation approximate solution of equation "(5.5)", respectively. Then*

$$\| \bar{U}(x) - \bar{u}(x) \|\leq \gamma h^2, \tag{6.16}$$

for h sufficiently small.

Proof. Since $\bar{U}(x) - \hat{U}(x) = \sum_{i=-1}^{N+1}(\bar{\alpha}_i(x) - \hat{\alpha}_i(x))\beta_i(x)$
Then

$$\| \bar{U}(x) - \hat{U}(x) \|\leq\| (\bar{\alpha} - \hat{\alpha}) \| \sum_{i=-1}^{N+1} | \beta_i(x) |$$

From "(6.1)" and "(6.15)" we imply that

$$\| \bar{U}(x) - \hat{U}(x) \|\leq \frac{23}{9}\frac{M}{\kappa_0}h^2.$$

In the other hand, from (16) we get

$$\| \hat{U}(x) - \bar{u}(x) \|\leq \gamma_0 h^4,$$

Thus

$$\| \bar{U}(x) - \bar{u}(x) \|\leq\| \bar{U}(x) - \hat{U}(x) \| + \| \hat{U}(x) - \bar{u}(x) \|\leq \gamma h^2,$$

where $\gamma = \frac{23}{9}\frac{M}{\kappa_0} + \gamma_0 h^2$. □

Table 2. Errors in the numerical solution for Example 1

t	0.02	0.10	0.50	1.00
L_∞(Our method)	3.19e-10	2.97e-7	3.11e-5	2.02e-4
L_∞(method [9])	8.4e-7	5.4e-5	1.2e-3	4.3e-3

7. Numerical results

In this section the presented method applied to five test examples to demonstrate viability and efficiency of the proposed method. In the first three examples the computed L_∞ , L_2 and Root-Mean-Square (RMS) errors of the solutions at grid points are tabulated in tables and also compared with the results of existing numerical methods. in example 4 the exact solution is unknown therefore the computed solutions of our method is compared with the other existing results. In the last example the present method is applied for solving hyperbolic equation and our results are compared with the results in [1]. The graph of the solutions in second and third examples are given in Figures 1 - 2.

Example 1. Consider the linear Klein-Gordon equation

$$u_{tt} - u_{xx} - 2u = -2sin(x)sin(t), \quad 0 < x < \frac{\pi}{2}, t > 0,$$

subject to the following initial and boundary conditions

$$u(x,0) = 0, \quad u_t(x,0) = sin(x), \quad 0 \leq x \leq \frac{\pi}{2},$$

$$u(0,t) = 0, \quad u(\frac{\pi}{2},t) = sin(t), \quad t \geq 0.$$

The exact solution of this problem is given by

$$u(x,t) = sin(x)sin(t).$$

The numerical solution of this problem is obtained by our method with $h = \pi/10$, $k = 0.01$ and $\lambda = 0.9$. The L_∞ errors of the method at different time levels are tabulated in table2 and compared with results by the method in [9].

Example 2. Consider the nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + \frac{\pi^2}{4}u + u^2 = x^2 sin^2(\frac{\pi}{2}t), \quad -1 < x < 1, t > 0,$$

subject to the following initial and boundary conditions

$$u(x,0) = 0, \quad u_t(x,0) = \frac{\pi}{2}x, \quad -1 \leq x \leq 1,$$

$$u(-1,t) = -sin(\frac{\pi}{2}t), u(1,t) = sin(\frac{\pi}{2}t), \quad t \geq 0.$$

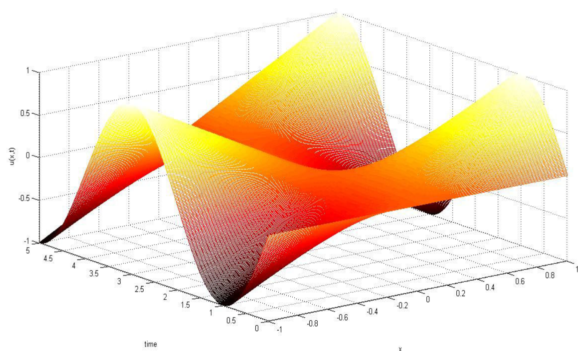
The exact solution is given by

$$u(x,t) = xsin(\frac{\pi}{2}t).$$

TABLE 3. Errors in the numerical solution for Example 2

t	1	2	3	4	5
Our method					
L_∞-error	1.03e-10	2.86e-10	2.51e-10	4.20e-10	4.09e-10
L_2-error	6.12e-10	1.52e-9	1.42e-9	2.56e-9	2.62e-9
RMS-error	6.09e-11	1.51e-10	1.41e-10	2.54e-10	2.61e-10
method [20]					
L_∞-error	3.97e-6	1.51e-6	2.14e-6	1.86e-6	5.08e-6
L_2-error	2.71e-5	8.97e-6	1.49e-5	1.05e-5	3.36e-5
RMS-error	2.69e-6	8.93e-7	1.48e-6	1.05e-6	3.34e-6

FIGURE 1. Space-time graph of the solution up to t=5 with h=.02, k=.01, for Example 2.



We applied our method to solve this problem, with $h = 0.02$, $k = 0.01$ and $\lambda = 0.9$. The L_∞, L_2 and RMS errors at grid points are tabulated in table 3. The comparison shows that our method is more accurate than the method in [20]. The space-time graph of the numerical solutions is illustrated in Figure 1.

Example 3. Consider the nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + u^2 = -xcos(t) + x^2cos^2(t), \qquad -1 < x < 1, t > 0,$$

subject to the following initial and boundary conditions

$$u(x,0) = x, u_t(x,0) = 0, \qquad -1 \leq x \leq 1,$$
$$u(-1,t) = -cos(t), u(1,t) = cos(t), \qquad t \geq 0.$$

The exact solution is given by

$$u(x,t) = xcos(t).$$

The numerical solution of this problem is obtained by our method. The L_∞, L_2 and RMS errors at grid points, using $h = 0.02$ $k = 0.0001$ and $\lambda = 0.6$, are tabulated in

FIGURE 2. Space-time graph of the solution up to t=5 with h=0.04, k=0.0001, for Example 3.

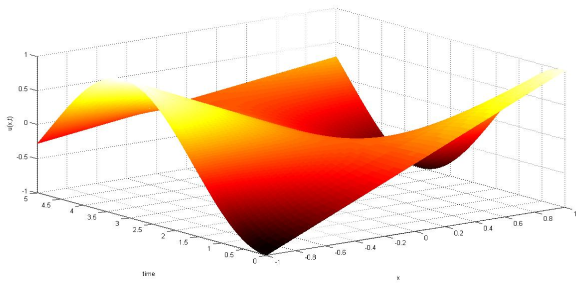


TABLE 4. Errors in the numerical solution for Example 3 (h=0.02)

t	1	3	5	7	10
Our method					
L_∞-error	1.60e-11	1.34e-11	2.90e-11	3.92e-11	8.23e-11
L_2-error	6.25e-11	5.64e-11	1.03e-10	1.84e-10	2.88e-10
RMS-error	6.22e-12	5.62e-12	1.02e-11	1.83e-11	2.87e-11
method [20]					
L_∞-error	1.03e-9	1.00e-9	2.56e-10	1.13e-9	9.46e-10
L_2-error	7.01e-9	6.59e-9	1.29e-9	7.47e-9	5.84e-9
RMS-error	6.97e-10	6.55e-10	1.28e-10	7.44e-10	5.81e-10
method [5]					
L_∞-error	1.25e-5	1.55e-5	3.37e-5	3.77e-5	1.30e-5
L_2-error	6.54e-5	1.17e-4	2.20e-4	2.58e-4	7.98e-5
RMS-error	6.50e-6	1.16e-5	2.19e-5	2.57e-5	7.94e-6

table 4 and are compared with the results in [20] and [5]. The space-time graph of the computed solutions is illustrated in Figure 2.

Example 4. Consider the nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + u^2 = 0, 0 < x < 1, t > 0,$$

subject to the following initial and boundary conditions

$$u(x,0) = 1 + sin(x), u_t(x,0) = 0, \quad 0 \le x \le 1,$$

$$u(0,t) = 1 - \frac{t^2}{2}, u(1,t) = \frac{221}{120} - \frac{305}{144}t^2 + \frac{103}{144}t^4, \quad t \ge 0.$$

The exact solution of this problem is unknown. the numerical solutions by our method using $h = 0.1$, $k = 0.01$ and $\lambda = 0.9$ are tabulated in table 5 at three time levels and are compared with solutions by methods in [20],[10],[34].

Table 5. Computed solutions for Example 4

x	0.1	0.2	0.4	0.6	0.8	1.0
t=0.1						
Our method	1.09329	1.19050	1.37784	1.54957	1.69872	1.82056
method [20]	1.09329	1.19050	1.37784	1.54962	1.69904	1.82056
method [10]	1.09333	1.19060	1.37807	1.55000	1.69908	1.82120
method [34]	1.09329	1.19050	1.37784	1.54962	1.69908	1.82038
t=0.2						
Our method	1.07373	1.16613	1.34340	1.50516	1.64498	1.75809
method [20]	1.07373	1.16613	1.34343	1.50506	1.64511	1.75809
method [10]	1.07372	1.16613	1.34342	1.50505	1.64467	1.75808
method[34]	1.07372	1.16613	1.34343	1.50507	1.64499	1.75806
t=0.3						
Our method	1.04132	1.12598	1.28712	1.43198	1.55749	1.65684
method [20]	1.04132	1.12597	1.28709	1.43244	1.55736	1.65684
method [10]	1.04131	1.12597	1.28708	1.43244	1.55706	1.65683
method [34]	1.04132	1.12597	1.28708	1.43249	1.55721	1.65720

Table 6. Errors in the solution for Example 5 ($t = 1, h = 0.01$) at certain nods

x	0.1	0.2	0.3	0.4	0.5
Absolute error [1]	1.11e-3	1.02e-3	9.54e-4	8.99e-4	8.56e-4
Our absolute error	3.8e-5	7.2e-5	9.9e-5	1.1e-4	1.2e-4

Example 5. Consider the one dimensional wave equation

$$u_{tt} - u_{xx} = (\pi^2 + \frac{1}{4})e^{(\frac{-t}{2})}sin(\pi x), \qquad 0 < x < 1, t > 0,$$

subject to the following initial and boundary conditions

$$u(x,0) = sin(\pi x), u_t(x,0) = \frac{-1}{2}sin(\pi x), \quad 0 \le x \le 1,$$
$$u(0,t) = 0, u(1,t) = 0, \quad t \ge 0.$$

The exact solution is given by

$$u(x,t) = e^{(\frac{-t}{2})}sin(\pi x).$$

In table 6 the absolute errors for this problem obtained by our method at different nodes with $h = 0.01$, $k = 0.0001$ and $\lambda = 0.7$ and compared with the results in [1]. The L_∞ errors at grid points, using $h = 0.25$, $k = 0.0001$ and $\lambda = 0.7$, are tabulated in table 7 and are compared with the results in [4] and [24],[1].

Table 7. L_∞ errors for Example 5 ($h = 0.025$)

t=1	method[4]	method[24]	method[1]	Our method
L_∞-error	1.50e-2	3.60e-3	2.45e-3	1.22e-4

Table 8. L_∞ errors and order of convergence for Example 5 ($t = 1$)

h	L_∞ in[1]	Conv. Order	L_∞ Our method	Conv. Order
$h = \frac{1}{4}$	0.03440552	3.8	0.04504420	2.009
$h = \frac{1}{8}$	0.00899331	2.2	0.01119384	2.003
$h = \frac{1}{12}$	0.00414931	1.7	0.00496842	2.001
$h = \frac{1}{16}$	0.00244299	1.5	0.007900906	2.001
$h = \frac{1}{20}$	0.00165117		0.001787368	

8. conclusion

Using collocation method based on the introduced trigonometric B-spline function, we have proposed numerical solution of the nonlinear Klein-Gordon equation. This approach reduces the problem to a nonlinear system of equations which can be solved numerically at each time level. The second order of convergence of our developed method has been proved. The implementation of the method for different values of parameter, shows that the best choose is $0 < \lambda < 1$. To discuss the accuracy of the method at long time levels, not only the l_∞ errors, but also l_2 errors and RMS-errors have been computed. The numerical results tabulated in the tables indicated that our method perform good accuracy and in comparison with the existing methods in the literature is more accurate.

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