Application of cubic B-spline quasi-interpolation for solving time-fractional partial differential equation

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Abstract   The purpose of this paper is to present a numerical scheme for solving time-fractional partial differential equation based on cubic B-spline quasi-interpolation. For this purpose, first we will approximate the time-fractional derivative by Laplace transform method and then by using of cubic B-spline quasi-interpolation, the spatial derivatives are approximated. Moreover, the stability of this method is studied. Finally, European call and put options are priced and we will show that the results are good agreement with the other methods. The main advantage of the resulting scheme is that the algorithm is very simple, so it is very easy to implement.

1. INTRODUCTION

In this paper we will consider the following time-fractional partial differential equation

$$\begin{cases} \dfrac{\partial^\alpha U(x,\tau)}{\partial \tau^\alpha} = p\dfrac{\partial^2 U(x,\tau)}{\partial x^2} + q\dfrac{\partial U(x,\tau)}{\partial x} - rU(x,\tau), \\ U(a,\tau) = f(\tau), \quad U(b,\tau) = g(\tau), \quad U(x,0) = h(x), \end{cases} \tag{1.1}$$

for all $(x,\tau) \in (a,b) \times (0,T)$, where $0 < \alpha \le 1$ and p, q and r are nonnegative. The fractional derivative operator is the Caputo-type fractional derivative defined as

$$\frac{\partial^\alpha U(x,\tau)}{\partial \tau^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_a^\tau (\tau - \xi)^{-\alpha} U'(x,\xi) d\xi.$$

Various phenomena in many different fields of sciences are represented by fractional partial differential equations. So, solving this type of equations commonly becomes an important issue. In the field of analysis, existence, and uniqueness of solutions of the fractional partial differential equation have been studied in [1, 13, 17] and [24]. Also, the authors of [8, 9] and [14, 15] derived semi-analytical and analytical methods for this type of equations. In the numerical domain, different schemes to numerically solve the fractional partial differential equation have been proposed. For example, in [10], the fractional problem by discretization and an interpolation method has been solved. In [19], a special splitting iteration method (CSCS) is utilized for solving fractional differential equation. In [11], a numerical method based on a Galerkin method for the time-fractional diffusion equation has been proposed. In [20], a Laplace method for solving the Caputo-type fractional partial differential equation has been used. The authors of [23] implemented a characteristic difference method for the fractional convection-diffusion equation. Also in [4], a fourth order accurate scheme for the space fractional-diffusion equation is proposed.

Interpolation is a very powerful tool for the approximation smooth functions. However, it needs to solve a vast system of linear equations with perhaps ill condition. Thus, quasi-interpolation method is proposed. This method yields solutions directly without solving any system of linear equations. Other main advantages are uncomplicated computation, almost optimal approximations for having a fairly small infinity norm, uniform boundedness freely of the degree or of the partition and so on [2, 16, 21, 22].

One of the most important fractional partial differential equations that is used for option pricing in financial markets is the time-fractional Black-Scholes (t-fB-S) equation. In [5] by considering the price change of the underlying fractal transmission system, a t-fB-S model is derived. In [25], an implicit numerical scheme with a temporal accuracy of order $2 - \alpha$ and spatial accuracy of the second order is constructed to approximate the t-fB-S model. A further work on the same model is done in [6]. They construct a three-point compact finite difference scheme on a non-uniform mesh for the t-fB-S equation.

In this paper, we work with the time-fractional differential equation with Caputo-type fractional derivative. Solving this type of equation needs, employing the values of all prior time steps. This requires a larger size of memory to store the necessary data when computing [18]. For solving this problem, we proposed a combinational numerical method based on the Laplace transform to approximate the time-fractional derivative and quasi-interpolation with cubic B-spline. Also, we prove the stability of the proposed method in details. Finally, European call and put options will be priced by these techniques.

We organize our paper as follows: In Section 2, by using of cubic B-spline quasi-interpolation, the spatial derivatives are approximated. Section 3 is devoted to constructing and analyzing the numerical technique and the stability of this method is studied in Section 4. We reported numerical simulations in Section 5. The brief conclusion is given in Section 6.

2. Cubic B-spline quasi-interpolants

For $I = [a, b]$, let $X_n = \{x_j, j = 0, \ldots, n\}$ where $x_j = a + jh$, $j = 0, \ldots, n$, $h = x_j - x_{j-1}$ with $x_0 = a$ and denote by $S_3(I, X_n)$ the space of splines of degree 3 and class C^2 defined on X_n. Let the B-spline basis of this space is $\{B_j^\nu, \ j \in J, \nu \in \mathbb{N}\}$, with $J = \{1, 2, \ldots, n+3\}$.

For $j \in J$, B_j^ν can be computed from the de Boor−Cox formula [7]

$$B_j^\nu(x) = \frac{x - x_{j-1}}{x_{\nu+j-1} - x_{j-1}} B_j^{\nu-1}(x) + \frac{x_{j+\nu} - x}{x_{\nu+j} - x_j} B_{j+1}^{\nu-1}(x),$$

with the anchor for the recursion being given by

$$B_j^0(x) = \left\{ \begin{array}{ll} 1, & x_{j-1} \le x < x_j, \\ 0, & \text{else}. \end{array} \right.$$

The formula of $\frac{d}{dx} B_j^\nu$ and $\frac{d^2}{dx^2} B_j^\nu$ of B-spline are as follows:

$$\frac{d}{dx} B_j^\nu(x) = \frac{\nu}{x_{\nu+j-1} - x_{j-1}} B_j^\nu(x) - \frac{\nu}{x_{\nu+j} - x_j} B_{j+1}^\nu(x)$$

and

$$\frac{d^2}{dx^2} B_j^\nu(x) = \frac{\nu}{x_{\nu+j-1} - x_{j-1}} \frac{d}{dx} B_j^\nu(x) - \frac{\nu}{x_{\nu+j} - x_j} \frac{d}{dx} B_{j+1}^\nu(x).$$

In this work, we use the cubic B-spline $\left(B_j^3(x)\right)$. In the following, we denote $B_j^3(x)$ by $B_j(x)$ for simplifying.

With these notations, the support of $B_j(x)$ is $[x_{j-4}, x_j]$. As usual, we add multiple nodes at the endpoints: $a = x_{-3} = x_{-2} = \ldots = x_0$ and $b = x_n = x_{n+1} = \ldots = x_{n+3}$.

A cubic B-spline quasi-interpolation has the form [21, 22]

$$Q_3 f = \sum_{j \in J} \mu_j B_j.$$

We insist that Q_3 for polynomials of total degree at most 3 is exact. So, for smooth functions, the approximation order of Q_3 is $O(h^4)$.

The coefficients of this approximation (μ_j) are linear combination of separate values of f on the set X_n:

$$\mu_1(f) = f_0,$$
$$\mu_2(f) = \frac{1}{18}(7f_0 + 18f_1 - 9f_2 + 2f_3),$$
$$\mu_j(f) = \frac{1}{6}(-f_{j-3} + 8f_{j-2} - f_{j-1}), \qquad j = 3, \ldots, n+1$$
$$\mu_{n+2}(f) = \frac{1}{18}(2f_{n-3} - 9f_{n-2} + 18f_{n-1} + 7f_n),$$
$$\mu_{n+3}(f) = f_n.$$

We approximate derivatives of f by derivatives of $Q_3 f(x)$ up to order 2. For this purpose, we evaluate the value of the derivatives of f at x_i by $(Q_3 f)' = \sum_{j=1}^{n+3} \mu_j(f) B_j'$ and $(Q_3 f)'' = \sum_{j=1}^{n+3} \mu_j(f) B_j''$. For evaluating the derivatives of f by derivatives of $Q_3 f$, we use the derivation matrices as follows:

By setting $\mathbf{y} \in \mathbb{R}^{n-1}$, $\mathbf{y}' \in \mathbb{R}^{n-1}$ and $\mathbf{y}'' \in \mathbb{R}^{n-1}$, $1 \le j \le n-1$ for the vectors

with components $y_j = f(x_j)$, $y'_j = (Q_3 f)'(x_j)$ and $y''_j = (Q_3 f)''(x_j)$ respectively, the derivation matrices $\mathcal{D}_3 and \overline{\mathcal{D}}_3 \in \mathbb{R}^{(n-1)\times(n-1)}$ are obtained:

$$\mathbf{y}' = \mathcal{D}_3 \mathbf{y} \tag{2.1}$$

where

$$\mathcal{D}_3 = \frac{1}{h} \begin{pmatrix}
-1/3 & -1/2 & 1 & -1/6 & 0 & 0 & \ldots & 0 & 0 \\
1/12 & -2/3 & 0 & 2/3 & -1/12 & 0 & \ldots & 0 & 0 \\
0 & 1/12 & -2/3 & 0 & 2/3 & -1/12 & \ldots & 0 & 0 \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
0 & 0 & \ldots & 1/12 & -2/3 & 0 & 2/3 & -1/12 & 0 \\
0 & 0 & \ldots & 0 & 1/12 & -2/3 & 0 & 2/3 & -1/12 \\
0 & 0 & \ldots & 0 & 0 & 1/6 & -1 & 1/2 & 1/3
\end{pmatrix}$$

and

$$\mathbf{y}'' = \overline{\mathcal{D}}_3 \mathbf{y} \tag{2.2}$$

where

$$\overline{\mathcal{D}}_3 = \frac{1}{h^2} \begin{pmatrix}
1 & -2 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
-1/6 & 5/3 & -3 & 5/3 & -1/6 & 0 & \ldots & 0 & 0 \\
0 & -1/6 & 5/3 & -3 & 5/3 & -1/6 & \ldots & 0 & 0 \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
0 & 0 & \ldots & -1/6 & 5/3 & -3 & 5/3 & -1/6 & 0 \\
0 & 0 & \ldots & 0 & -1/6 & 5/3 & -3 & 5/3 & -1/6 \\
0 & 0 & \ldots & 0 & 0 & 0 & 1 & -2 & 1
\end{pmatrix}.$$

3. SOLUTION METHODOLOGY

At the beginning, we approximate the Caputo-type time-fractional derivative by using the Laplace transform method as presented in [20]

$$\begin{aligned}
L\left\{ \frac{\partial^\alpha U(x,\tau)}{\partial \tau^\alpha} \right\} &= s^\alpha \hat{U}(x,s) - s^{\alpha-1} \hat{U}(x,0) \tag{3.1} \\
&= s^\alpha [\hat{U}(x,s) - s^{-1} \hat{U}(x,0)],
\end{aligned}$$

where $\hat{U}(x,s)$ is the Laplace transform of $U(x,\tau)$. Since $0 < \alpha < 1$, we linearize the term s^α as:

$$s^\alpha \approx \alpha s^1 + (1-\alpha)s^0 = \alpha s + (1-\alpha),$$

and then replace it into Eq. (3.1). This gives

$$\begin{aligned}
L\left\{ \frac{\partial^\alpha U(x,\tau)}{\partial \tau^\alpha} \right\} &\approx [\alpha s + (1-\alpha)][\hat{U}(x,s) - s^{-1}\hat{U}(x,0)] \\
&= \alpha s[\hat{U}(x,s) - s^{-1}\hat{U}(x,0)] + (1-\alpha)[\hat{U}(x,s) - s^{-1}\hat{U}(x,0)].
\end{aligned}$$

Thus, the inverse Laplace transform gives

$$\frac{\partial^\alpha U(x,\tau)}{\partial \tau^\alpha} \approx \alpha \frac{\partial U(x,\tau)}{\partial \tau} + (1-\alpha)[U(x,\tau) - U(x,0)].$$

By substituting the above formula into Eq. (1.1), the following equation is obtained:

$$\frac{\partial U}{\partial \tau} = \frac{p}{\alpha}\frac{\partial^2 U}{\partial x^2} + \frac{q}{\alpha}\frac{\partial U}{\partial x} + \left(1 - \frac{r}{\alpha} - \frac{1}{\alpha}\right)U + \left(\frac{1}{\alpha} - 1\right)U(x,0).$$

In order to elucidate the solution procedure of the cubic B-spline quasi-interpolant, we consider the following general differential equation:

$$\frac{\partial U(x,\tau)}{\partial \tau} = \mathcal{L}U(x,\tau), \tag{3.2}$$

where $\mathcal{L}$ is the linear operator as follows:

$$\mathcal{L}U = \frac{p}{\alpha}\frac{\partial^2 U}{\partial x^2} + \frac{q}{\alpha}\frac{\partial U}{\partial x} + \left(1 - \frac{r}{\alpha} - \frac{1}{\alpha}\right)U + \left(\frac{1}{\alpha} - 1\right)U(x,0).$$

By discretizing the Eq. (3.2) in time using the Crank-Nicolson method with mesh-length $\delta\tau = \frac{T}{N}$,

$$\frac{U_i^{m+1} - U_i^m}{\delta\tau} \cong \frac{1}{2}\left(\mathcal{L}U_i^{m+1} + \mathcal{L}U_i^m\right).$$

We can get

$$u_i^{m+1} - \frac{\delta\tau}{2}\mathcal{L}u_i^{m+1} = u_i^m + \frac{\delta\tau}{2}\mathcal{L}u_i^m, \tag{3.3}$$

where $u_i^m \approx U(x_i,\tau_m)$, $\tau_m = m\delta\tau$ and

$$\mathcal{L}u_i^m = \frac{p}{\alpha}\left(\frac{\partial^2 u}{\partial x^2}\right)_i^m + \frac{q}{\alpha}\left(\frac{\partial u}{\partial x}\right)_i^m + \left(1 - \frac{r}{\alpha} - \frac{1}{\alpha}\right)u_i^m + \left(\frac{1}{\alpha} - 1\right)u(x_i,0).$$

Let $A = (a_{ij})_{(n-1)\times(n-1)}$ and $U^m = [u_1^m, \ldots, u_{n-1}^m]^T$. Then by applying the Eq.s (2.1) and (2.2) for the first and second order spatial derivatives in $\mathcal{L}u_i^m$ and $\mathcal{L}u_i^{m+1}$, the Eq. (3.3) is reduced to the system in the form of matrix notations:

$$(I - A)U^{m+1} = (I + A)U^m + \delta\tau B + C \tag{3.4}$$

where I is the $(n-1)\times(n-1)$ identity matrix, the vector B is defined as follows:

$$B = \left[\left(\frac{1}{\alpha} - 1\right)u_i^0, \ldots, \left(\frac{1}{\alpha} - 1\right)u_i^0\right]^T,$$

boundary conditions are absorbed in the definition of vector C as:

$$C = \left[(-6a - 4b)(u_0^m + u_0^{m+1}), (a + b)(u_0^m + u_0^{m+1}), 0, \ldots, 0,\right.$$
$$\left.(a - b)(u_N^m + u_N^{m+1}), (-6a + 4b)(u_N^m + u_N^{m+1})\right]^T$$

and the matrix $A = (a_{ij})$ has entries

$$a_{i,j} = \begin{cases} a + b, & i = j + 2,\ i = 3, \ldots, n - 2, \\ c - d, & i = j + 1,\ i = 2, \ldots, n - 2, \\ e, & i = j,\ i = 2, \ldots, n - 2, \\ c + d, & i = j - 1,\ i = 2, \ldots, n - 2, \\ a - b, & i = j - 2,\ i = 2, \ldots, n - 3, \end{cases}$$

where

$$a = -\frac{\delta\tau p}{12\alpha h^2}, \qquad b = \frac{\delta\tau q}{24\alpha h}, \qquad c = \frac{5\delta\tau p}{6\alpha h^2}, \qquad d = \frac{\delta\tau q}{3\alpha h}$$

and

$$e = \frac{\delta\tau}{2}\left(1 - \frac{r}{\alpha} - \frac{1}{\alpha}\right) - \frac{3\delta\tau p}{2\alpha h^2}.$$

Also,

$$a_{1,1} = \frac{\delta\tau}{2}\left(1 - \frac{r}{\alpha} - \frac{1}{\alpha}\right) - \frac{\delta\tau p}{\alpha h^2} - \frac{\delta\tau q}{4\alpha h},$$

$$a_{1,2} = \frac{\delta\tau p}{2\alpha h^2} + \frac{\delta\tau q}{2\alpha h},$$

$$a_{1,3} = -\frac{\delta\tau q}{12\alpha h},$$

$$a_{n-1,n-3} = \frac{\delta\tau q}{12\alpha h},$$

$$a_{n-1,n-2} = \frac{\delta\tau p}{2\alpha h^2} - \frac{\delta\tau q}{2\alpha h},$$

$$a_{n-1,n-1} = \frac{\delta\tau}{2}\left(1 - \frac{r}{\alpha} - \frac{1}{\alpha}\right) - \frac{\delta\tau p}{\alpha h^2} - \frac{\delta\tau q}{4\alpha h}.$$

4. Stability analysis

Lemma 4.1. *Let $hq \le p \le \frac{3}{2}h^2(1 + r - \alpha)$, then the system of finite difference Eq. (3.4) is stable.*

Proof. We first show that under the above condition, the (complex-valued) eigenvalues of the matrix A have negative real parts. For i from 3 to $n - 3$, since $0 < \alpha \le 1$ and coefficients of the Eq. (1.1) are positive, so we have $e \le 0$. Since $q \le \frac{1}{h}p$, therefore $a + b \le 0$ and $c - d \ge 0$. Also, it is clearly that $c + d \ge 0$ and $a - b \le 0$. According to the Gershgorin circle theorem [12], the eigenvalues of the matrix A are in the disks centered at each diagonal entry $a_{i,i} = e \le 0$, with radius

$$r_i = \sum_{j=3,j\ne i}^{n-3} |a_{i,j}| = -2a + 2c = \frac{11\delta\tau p}{6\alpha h^2}.$$

If $p \le \frac{3}{2}h^2(1 + r - \alpha)$, then $\frac{\delta\tau p}{3\alpha h^2} \le \frac{-\delta\tau}{2}\left(1 - \frac{r}{\alpha} - \frac{1}{\alpha}\right)$. So we have

$$\frac{11\delta\tau p}{6\alpha h^2} \le \frac{3\delta\tau p}{2\alpha h^2} - \frac{\delta\tau}{2}\left(1 - \frac{r}{\alpha} - \frac{1}{\alpha}\right).$$

This means $r_i \le |a_{i,i}|, i = 3, \ldots, n - 3$.

For $i = 1$, we have $a_{1,1} \le 0$, $a_{1,2} \ge 0$ and $a_{1,3} \le 0$. Since $hq \le p \le \frac{3}{2}h^2(1 + r - \alpha)$, then we conclude $r_1 = |a_{1,2}| + |a_{1,3}| \le |a_{1,1}|$.

For $i = 2$, we must show

$$
\begin{aligned}
r_2 &= |a_{2,1}| + |a_{2,3}| + |a_{2,4}| \\
&= |c - d| + |c + d| + |a - b| \\
&\leq |e|.
\end{aligned}
\tag{4.1}
$$

According to the Lemma conditions, the above is satisfied.

For $i = n - 2$, we must show

$$
\begin{aligned}
r_{n-2} &= |a_{n-2,n-4}| + |a_{n-2,n-3}| + |a_{n-2,n-1}| \\
&= |a + b| + |c - d| + |c + d| \\
&\leq |e|,
\end{aligned}
\tag{4.2}
$$

by using the Lemma assumptions, the above is satisfied.

For $i = n - 1$, we have $a_{n-1,n-1} \leq 0$, $a_{n-1,n-2} \geq 0$ and $a_{n-1,n-3} \geq 0$. This implies $r_{n-1} = |a_{n-1,n-3}| + |a_{n-1,n-2}| \leq |a_{n-1,n-1}|$.

These Gershgorin circle theorem disks are within the left half of the complex plane. Therefore the eigenvalues of matrix A have negative real-parts.

Next λ is an eigenvalue of matrix A if and only if $(1 - \lambda)$ is an eigenvalue of the matrix $(I - A)$, if and only if $(1 + \lambda)/(1 - \lambda)$ is an eigenvalue of the matrix $(I - A)^{-1}(I + A)$. By the first part of this statement, we conclude that all the eigenvalues of the matrix $(I - A)$ have a magnitude larger than 1, and thus this matrix is invertible. Furthermore, by knowing that the real part of λ is negative, we get $|\frac{1+\lambda}{1-\lambda}| < 1$. Therefore, the spectral radius of the system matrix $(I - A)^{-1}(I + A)$ is less than one. Thus the system of Eq. (3.4) under the above condition is stable.

5. NUMERICAL RESULTS

In this section, in order to evaluate the accuracy of the method, we consider the time-fractional equation with a known analytical solution. Furthermore, we price European call and put options governed by a t-fB-S model.

Example 1. Consider the following time-fractional model

$$
\begin{aligned}
&\frac{\partial^\alpha U}{\partial t^\alpha} = \frac{\partial^2 U}{\partial x^2} + f(x,t), \qquad (x,t) \in (0,1) \times (0,T], \\
&U(x,0) = 0, \quad 0 < x < 1, \\
&U(0,t) = 0, \quad 0 < t \leq T, \\
&U(1,t) = 0, \quad 0 < t \leq T,
\end{aligned}
\tag{5.1}
$$

with the source term

$$
f(x,t) = \frac{2}{\Gamma(3 - \alpha)} t^{2-\alpha} \sin(2\pi x) + 4\pi^2 t^2 \sin(2\pi x).
$$

The exact solution of the above equation is $U(x,t) = t^2 \sin(2\pi x)$. The results of solving (5.1) using the presented method in the special points of x for $\alpha = 0.5$ at $t = 1$ is shown in Table 1. In Table 2, we show norm infinity of error compared with fully discrete direct discontinuous Galerkin method [11] for various values of h with $\alpha = 0.9$ up to $t = 1$. Comparison between the presented method and the methods of [11] shows the efficiency of the new method.

x_i	$\delta t = 0.02,$ $h = 0.01$	$\delta t = 0.01,$ $h = 0.005$	$\delta t = 0.01,$ $h = 0.0025$
0.1	2.25×10^{-4}	4.18×10^{-5}	6.16×10^{-6}
0.2	3.65×10^{-4}	6.77×10^{-5}	9.96×10^{-6}
0.3	3.64×10^{-4}	6.77×10^{-5}	9.96×10^{-6}
0.4	2.25×10^{-4}	4.18×10^{-5}	6.16×10^{-6}
0.5	2.35×10^{-15}	5.04×10^{-15}	5.97×10^{-14}
0.6	2.25×10^{-4}	4.18×10^{-5}	6.16×10^{-6}
0.7	3.64×10^{-4}	6.77×10^{-5}	9.96×10^{-6}
0.8	3.65×10^{-4}	6.77×10^{-5}	9.96×10^{-6}
0.9	2.25×10^{-4}	4.18×10^{-5}	6.16×10^{-6}

TABLE 1. Error of solving (5.1) using the presented method for $\alpha = 0.5$ at $t = 1$.

TABLE 2. Error of solving (5.1) using the presented method for $\alpha = 0.9$.

	$L^\infty - error$	
h	$[11](\delta t = \frac{1}{12000})$	our method $(\delta t = \frac{1}{50})$
1/12	4.30×10^{-2}	1.59×10^{-2}
1/16	2.46×10^{-2}	1.05×10^{-2}
1/20	1.58×10^{-2}	7.12×10^{-3}
1/24	1.10×10^{-2}	5.06×10^{-3}

Example 2. As discussed in [25], if the price change of the option in the financial market is considered as a fractal transmission system, V(s,t) should satisfy the following fractional partial differential equation system:

$$\begin{cases} \dfrac{\partial^\alpha V}{\partial t^\alpha} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 V}{\partial s^2} + rs\dfrac{\partial V}{\partial s} - rV = 0, \\ V(0,t) = \bar{p}(t), \quad V(\infty,t) = \bar{q}(t), \quad V(s,T) = \bar{\pi}(s), \end{cases} \tag{5.2}$$

for all $(s,t) \in (0,\infty) \times (0,T)$, where $V(s,t)$ is the European option price at asset price s and at time t, r is the risk-free interest rate, σ is the volatility and T is the expiry time.

To solve the fB-S equation numerically, it is necessary to truncate the original unbounded domain into a finite interval (L_1, L_2). Also, we let $t = T - \tau$, $s = e^x$ and
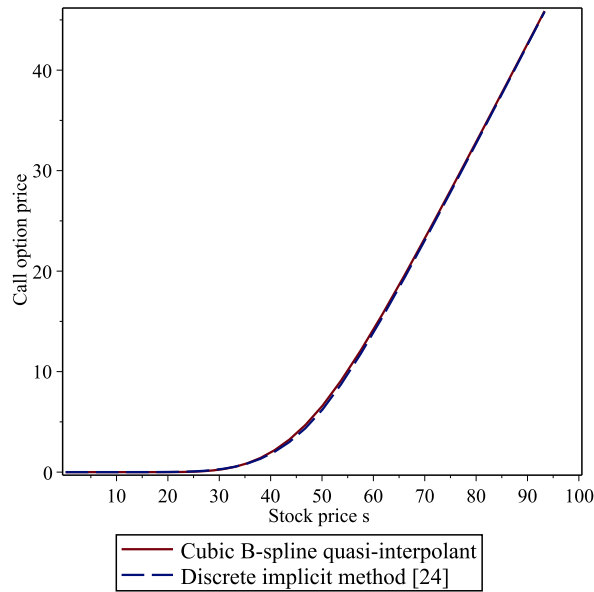
$V(s,t) = U(x,\tau)$, for $0 < \alpha < 1$, we have

$$
\begin{aligned}
\frac{\partial^\alpha V}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{V(s,t') - V(s,T)}{(t'-t)^\alpha} dt' \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{-d}{d\tau} \int_{T-\tau}^T \frac{V(s,t') - V(s,T)}{(t' - (T-\tau))^\alpha} dt' \\
&= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \int_0^\tau \frac{V(s,T-\eta) - V(s,T)}{(\tau - \eta)^\alpha} d\eta \\
&= \frac{-1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \int_0^\tau \frac{U(x,\eta) - U(x,0)}{(\tau - \eta)^\alpha} d\eta \\
&= -\frac{\partial^\alpha U}{\partial \tau^\alpha}.
\end{aligned}
\tag{5.3}
$$

Now, substituting (5.3) into (5.2) and performing some usual calculations, we obtain the following system

$$
\begin{cases}
\dfrac{\partial^\alpha U}{\partial \tau^\alpha} = \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 U}{\partial x^2} + \left(r - \dfrac{1}{2}\sigma^2\right) \dfrac{\partial U}{\partial x} - rU, \\
U(L_1, \tau) = p(\tau), \quad U(L_2, \tau) = q(\tau), \quad U(x,0) = \pi(x),
\end{cases}
\tag{5.4}
$$

for all $(x,\tau) \in (L_1, L_2) \times (0, T)$.

FIGURE 1. European call option value at $t = 0$

European call option governed by (5.4) with $r = 0.05$, $\sigma = 0.25$, $\alpha = 0.7$, $L_1 = \ln 0.1$, $L_2 = \ln 100$, $T = 1$ and $K = 50$. The terminal and boundary conditions are:

$$\pi(x) = \max(e^x - K, 0), \quad p(\tau) = 0, \quad q(\tau) = L_2 - Ke^{-r(\tau)}.$$

For the European put option, the market parameters are the same as above and the terminal and boundary conditions are:

$$\pi(x) = \max(K - e^x, 0), \quad p(\tau) = Ke^{-r(\tau)}, \quad q(\tau) = 0.$$

Note that after solving Eq. (5.4) with the developed method, we use $V(s, t) = U(x, \tau)$ to obtain the option value at $t = 0$. Also, these problems are solved in [25] by using of the discrete implicit method. For purpose of validation, we compared the solution of ours method with the method of [25] with $\alpha = 0.7$. This comparison is shown in Figs. 1 and 2, for call option price and put option price, respectively. Table 3 shows the call and put option price in different values of stock price. Excellent agreement of the new technique in comparison with the established method is found.

Remark 5.1. It should be noted that the condition of the Lemma 4.1 is one-sided. As the numerical results show, our problem is stable without the assumptions of the Lemma 4.1.

FIGURE 2. European put option value at $t = 0$
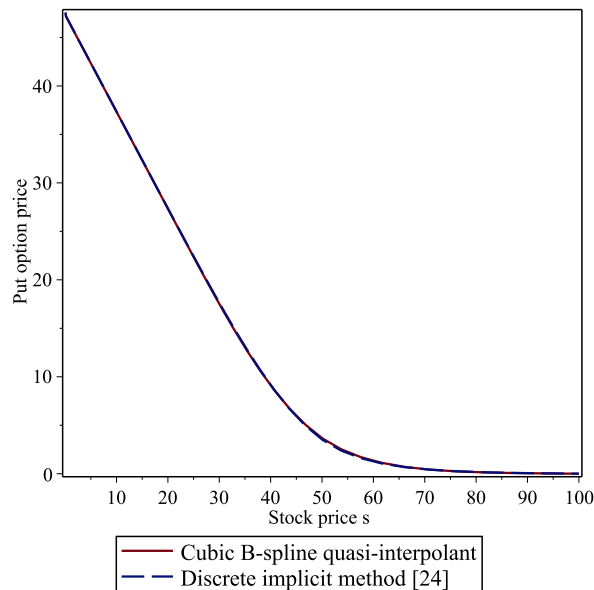
Note that for $\alpha = 1$, the problem (5.2) becomes the standard B-S model. The best way for testing the proposed new technique is to calculate our solution at $\alpha = 1$. The comparison between our solution and the standard B-S solution are plotted in Figs. 3 and 4.
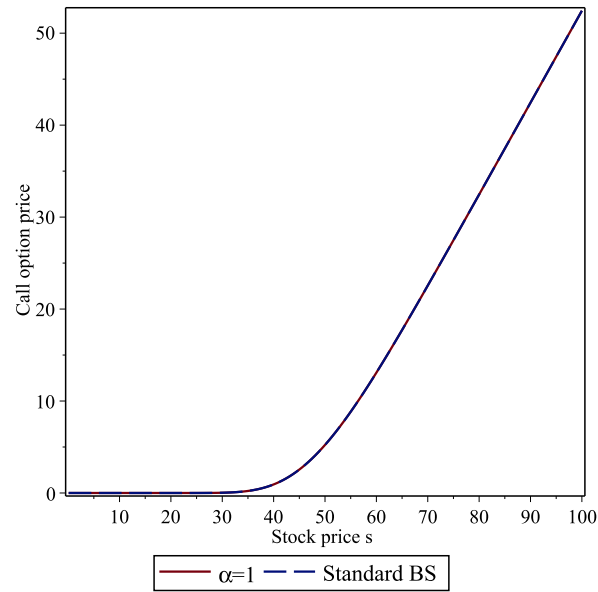
FIGURE 3. European call option value at $t = 0$

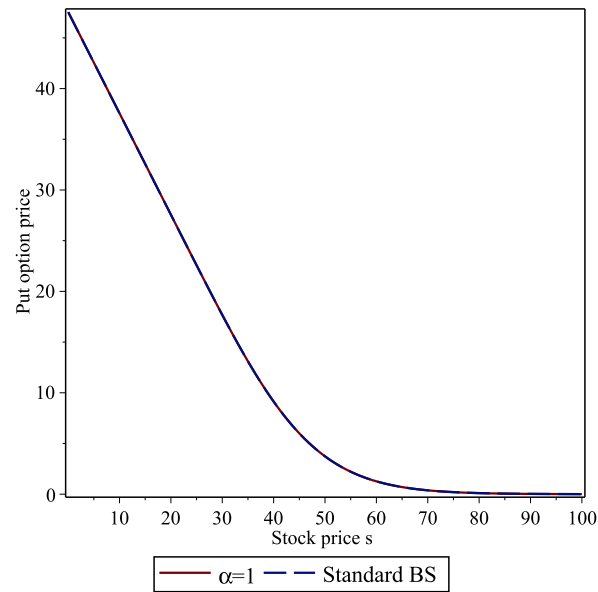FIGURE 4. European put option value at $t = 0$

TABLE 3. European call and put option price in different values of stock price.

	call option		put option	
s	[25]	our method	[25]	our method
10	1.64×10^{-5}	-3.16×10^{-8}	37.35	37.34
20	1.17×10^{-2}	3.17×10^{-3}	27.37	27.34
30	0.275	0.242	17.63	17.52
40	1.81	1.94	9.16	9.12
50	6.17	6.54	3.53	3.67
60	13.87	14.18	1.25	1.33
70	23.05	23.25	0.46	0.47
80	32.70	32.85	0.17	0.16
90	42.53	42.62	0.06	0.05

6. Conclusion

In this paper, we presented a combinational numerical scheme for solving time-fractional partial differential equation. We use the Laplace transform for time-fractional derivative and cubic B-spline for spatial derivatives. We also prove that this numerical scheme is conditionally stable. Finally, by using the proposed numerical technique, European call and put options are priced. The main advantage of this method is that the algorithm is very simple, so the implementation of it is very easy.

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