Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 8, No. 4, 2020, pp. 762-769
DOI:10.22034/cmde.2020.34001.1564

Isospectral sixth order Sturm-Liouville eigenvalue problems

Hanif Mirzaei
Faculty of Basic Sciences, Sahand University of Technology, Tabriz, Iran.
E-mail: h_mirzaei@sut.ac.ir

Abstract In this research, we introduce an approach to find a family of sixth order Sturm-Liouville problems having the same spectrum. Using Darboux Lemma and the fact that any second order Sturm-Liouville problem with the Dirichlet boundary conditions is equivalent to a sixth order Sturm-Liouville problem, the considered problems are formulated.

Keywords. Spectrum, Eigenvalue, Equivalence relation, Darboux lemma.

2010 Mathematics Subject Classification. 34B24, 34L05.

1. Introduction

Sturm-Liouville problems (SLP) are eigenvalue problems in one dimension and arise in different fields such as quantum mechanics, vibrating rods and beams, theory of hydrodynamic and magnetic hydrodynamic stability [5, 7, 21]. Also, these problems can be arisen in solving partial differential equations by separation of variables method. The SLP appeared in hydrodynamic and magnetic hydrodynamic are of higher order. Also, the higher order SLP appear in quantum mechanic, where certain partial differential eigenvalue problem can be transformed to a system of ordinary differential eigenvalue problem [7, 14]. Sturm-Liouville equations of order 2, 4 and 6 in canonical form are as follows:

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (a,b), \tag{1.1}$$

$$y^{(4)}(x) + (q_1(x)y'(x))' + q_2(x)y(x) = \lambda y(x), \quad x \in (a,b), \tag{1.2}$$

$$-y^{(6)}(x) + (q_1(x)y''(x))'' + (q_2(x)y'(x))' + q_3(x)y(x) = \lambda y(x), x \in (a,b). \tag{1.3}$$

Sixth order equations arise in study of circular structures [11]. For Sturm-Liouville equation of order $2n$, $2n$ boundary conditions at the end points a and b are given. The exact form of the boundary conditions will be given in section 2. Sturm-Liouville equation together with the corresponding boundary conditions is called a Sturm-Liouville problem. The value of λ that for which the SLP has a nontrivial solution is called an eigenvalue; and the corresponding nontrivial solution is called an eigenfunction. If the interval (a,b) is finite; and the coefficients $q_i(x)$ are in $L^1(a,b)$, then the SLP

Received: 22 June 2019 ; Accepted: 25 August 2019.

with the self-adjoint boundary conditions has real eigenvalues and can be ordered as follows:

$$\lambda_1 \leq \lambda_2 \leq \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty, \tag{1.4}$$

see [7, 8]. The eigenvalues of second order SLP with the Dirichlet boundary conditions are simple [1, 12]. In SLP of order $2n$, each eigenvalue has multiplicity at most n, i.e., for each eigenvalue λ there are at most n linearly independent solutions [7]. The set of all eigenvalues is called the spectrum. Two SLP that have the same spectrum are called isospectral. Sturm-Liouville problems have been studied by many researchers. Spectral properties and related inverse problems have been studied in [1, 3, 5, 17, 18, 19, 20]. In [7, 8, 9, 13, 14, 16, 23], the authors presented numerical methods to approximate the eigenvalues. Isospectral SLP of order 2, 4, and 6 have been studied in [5, 6, 10, 15, 22], but isospectral SLP of order six is not well studied comparing to second and fourth order problems. In section 2, first we define an equivalence relation between second and sixth order SLP, then we find a family of sixth order SLP equivalent to a second order problem. Also, some properties of the eigenvalues of fourth and sixth order SLP will be presented. In section 3, using the obtained equivalence relation and Darboux Lemma we find a family of sixth order SLP which are isospectral to a given one.

2. Equivalence relation and some eigenvalue properties

In this section, we define an equivalence relation between second and sixth order SLP. Also, we obtain some properties of the eigenvalues.

Definition 2.1. Two SLP of order two and six are said to be equivalent iff the following statements are equivalent:

(i) (λ, y) is an eigenpair of second order SLP,
(ii) (λ^3, y) is an eigenpair of sixth order SLP.

The following theorem shows that if (λ, y) is an eigenpair of second order SLP, then (λ^3, y) is an eigenpair of a sixth order SLP.

Theorem 2.2. *Suppose that $q(x) \in C^4[0,1]$ and (λ, y) is an arbitrary eigenpair of the second order SLP*

$$\begin{cases} y''(x) + (\lambda - q(x))y(x) = 0, \quad x \in (0,1), \\ y(0) = 0, \ y(1) = 0, \end{cases} \tag{2.1}$$

then, there exists a SLP of order six such that (λ^3, y) is an eigenpair of it.

Proof. Let (λ, y) be an eigenpair of problem (2.1). Since $q(x) \in C^4[0,1]$, the equation (2.1) implies that $y(x) \in C^6[0,1]$. Differentiating twice from (2.1) and substituting $y'' = (q(x) - \lambda)y$ we obtain

$$L^{4,q}y := y^{(4)}(x) - 2(q(x)y'(x))' + (q^2(x) - q''(x))y(x) = \lambda^2 y(x). \tag{2.2}$$

Taking limit from equation (2.1) as x tends to 0 and 1, respectively, and applying boundary conditions (2.1), we obtain

$$y(0) = y''(0) = 0, \quad y(1) = y''(1) = 0. \tag{2.3}$$

Thus (λ^2, y) is an eigenpair of a fourth order SLP. Differentiating twice from the equation (2.2) and substituting $y'' = (q(x)-\lambda)y$, then using (2.2) in resulting equation we obtain

$$\begin{aligned} L^{6,q}y := & - y^{(6)}(x) + 3[q(x)y''(x)]'' - [(3q^2(x) - 4q''(x))y'(x)]' \\ & + [q^{(4)} + q^3 - 2q'^2 - 3qq'']y(x) = \lambda^3 y(x). \end{aligned} \tag{2.4}$$

Taking limit from equation (2.2) as x tends to 0 and 1, respectively, and using the boundary conditions (2.3), we obtain

$$y^{(4)}(0) - 2q'(0)y'(0) = 0, \quad y^{(4)}(1) - 2q'(1)y'(1) = 0. \tag{2.5}$$

Thus (λ^3, y) is an eigenpair of sixth order SLP (2.4) with the boundary conditions (2.3) and (2.5). □

In the following lemma, we prove that the sixth order SLP obtained in Theorem 2.2 is self-adjoint.

Lemma 2.3. *The SLP (2.4) with the boundary conditions (2.3) and (2.5) is self-adjoint.*

Proof. We should prove that for any arbitrary functions u and v satisfying in the boundary conditions (2.3) and (2.5), we have

$$\langle L^{6,q}u, v\rangle = \langle u, L^{6,q}v\rangle,$$

where the inner product is defined as $\langle u, v\rangle = \int_0^1 u\bar{v}dx$. Applying the integrating by parts for integrals in $\langle L^{6,q}u, v\rangle$ and using the boundary conditions on u and v we find,

$$\begin{aligned} \langle L^{6,q}u, v\rangle = & [v'(1)u^{(4)}(1) - u'(1)v^{(4)}(1)] + [u'(0)v^{(4)}(0) - v'(0)u^{(4)}(0)] \\ & + \langle u, L^{6,q}v\rangle. \end{aligned} \tag{2.6}$$

From boundary conditions (2.5), we conclude that the bracketed terms in the equation (2.6) are zero. Thus the problem is self-adjoint. □

Lemma 2.3, shows that the eigenvalues of the sixth order SLP (2.4) with the boundary conditions (2.3) and (2.5) are real and can be ordered as (1.4). In [15], it is proved that the fourth order problem (2.2) with boundary conditions (2.3) and second order problem (2.1) are equivalent in the sense of Definition 2.1. In the following theorem, we find some eigenvalue properties of fourth order SLP (2.2) that we will need in the rest of the paper.

Theorem 2.4. *For fourth order SLP of the form (2.2), the following statements hold:*

(i). *The eigenvalues are nonnegative,*

(ii). *If $\lambda = 0$ is an eigenvalue, then it is simple and $\lambda = 0$ is an eigenvalue of second order SLP (2.1) with the same eigenfunction of the problem (2.2).*

Proof. Part (i). We prove by contradiction. Suppose that the operator $L^{4,q}$ has a negative eigenvalue $-\lambda^2$. Thus there exists an eigenfunction $y(x)$, such that

$$L^{4,q}y(x) = -\lambda^2 y(x).$$

We can factorize, this equation as follows

$$(L^{4,q} + \lambda^2)y(x) = (D^2 - q + \lambda i)(D^2 - q - \lambda i)y(x) = 0,$$

where i is imaginary number and D denotes differential operator with respect to x. Let $\varphi(x) = (D^2 - q - \lambda i)y(x)$. It is obvious that $\varphi(x)$ satisfies in the boundary conditions (2.1). If $\varphi(x) \equiv 0$, then $(\lambda i, y(x))$ is an eigenpair of (2.1). If $\varphi(x) \neq 0$, then $(-\lambda i, \varphi(x))$ is an eigenpair of (2.1). In both cases, we have a contradiction since the eigenvalues of SLP (2.1) are real. Thus the eigenvalues of SLP (2.2) are nonnegative. Part (ii). Suppose that $(0, y)$ is an eigenpair of SLP (2.2), we have

$$(D^2 - q)(D^2 - q)y(x) = 0.$$

Let $\varphi(x) = (D^2 - q)y(x)$, we claim that $\varphi(x)$ is equal to zero. Suppose that $\varphi(x) \neq 0$, we have

$$y''(x) - q(x)y(x) = \varphi(x), \tag{2.7}$$

and

$$\varphi''(x) - q(x)\varphi(x) = 0. \tag{2.8}$$

Multiplying equation (2.7) by $\varphi(x)$ and equation (2.8) by $y(x)$, subtracting the resulting equations, then integrating from 0 to 1, we obtain:

$$\int_0^1 \varphi^2(x)dx = 0.$$

This is a contradiction, thus $\varphi(x) = 0$ and $(0, y)$ is an eigenpair of second order SLP (2.1). Now, we prove the simplicity of the eigenvalue $\lambda = 0$. Suppose that y_1 and y_2 are two linearly independent functions such that $(0, y_1)$ and $(0, y_2)$ are eigenpairs of SLP (2.2). By the above results, $(0, y_1)$ and $(0, y_2)$ are eigenpairs of the second order SLP (2.1), too. This is a contradiction with simplicity of the eigenvalues of second order SLP. Thus, $\lambda = 0$ can not be the eigenvalue of fourth order SLP with the multiplicity two. $\square$

The following theorem together with Theorem 2.2 show that sixth order problem (2.4) and second order problem (2.1) are equivalent in the sense of Definition 2.1.

Theorem 2.5. *For sixth order SLP (2.4) with boundary conditions (2.3) and (2.5) the following statements hold:*

- (i) *If (λ^3, y) is an arbitrary eigenpair of sixth order SLP (2.4) with the boundary conditions (2.3) and (2.5), then (λ, y) is an eigenpair of second order SLP (2.1),*
- (ii) *The eigenvalues are simple.*

Proof. Part (i). By Lemma 2.3, this problem is self-adjoint. Thus the eigenvalues are real and can be denoted by λ^3. Suppose that $\lambda \neq 0$ and (λ^3, y) is an eigenpair. Thus $y(x)$ satisfies in equation (2.4). This equation can be factorized as follows:

$$\begin{aligned}\{D^4 - 2D[(q(x) + \frac{\lambda}{2})D] + (q^2(x) - q''(x) + \lambda q(x) + \lambda^2)\}\\ \{D^2 + \lambda - q(x)\}y(x) = 0.\end{aligned} \tag{2.9}$$

Let $\varphi(x) = \{D^2 + \lambda - q(x)\}y(x)$. We claim that $\varphi(x) = 0$. Suppose that $\varphi(x) \neq 0$. Since $y(x)$ satisfies in the boundary conditions (2.3) and (2.5), thus $\varphi(x)$ satisfies in the boundary conditions (2.3). Equation (2.9) is a fourth order Sturm-Liouville equation in terms of $\varphi(x)$ and can be written as follows:

$$\begin{aligned}\{D^4 - 2D[(q(x) + \frac{\lambda}{2})D] + [(q(x) + \frac{\lambda}{2})^2 - (q(x) + \frac{\lambda}{2})'']\}\varphi(x)\\ = -\frac{3}{4}\lambda^2\varphi(x).\end{aligned} \tag{2.10}$$

Equation (2.10) shows that $-\frac{3}{4}\lambda^2$ is an eigenvalue of a fourth order SLP of the form (2.2), in which the function $q(x)$ is replaced by $q(x) + \frac{\lambda}{2}$. This is a contradiction with part (i) of Theorem 2.4. Thus $\varphi(x) = y''(x) + (\lambda - q(x))y(x) = 0$, and (λ, y) is an eigenpair of SLP (2.1). Now, we prove part (i) for the case $\lambda = 0$. Suppose that $(0, y)$ is an eigenpair of (2.4). Similar to the case $\lambda \neq 0$ by substituting $\lambda = 0$ in equation (2.10), we obtain:

$$\{D^4 - 2D[q(x)D] + [q^2(x) - q''(x)]\}\varphi(x) = 0. \tag{2.11}$$

Thus $(0, \varphi(x))$ is an eigenpair of fourth order SLP and by part (ii) of Theorem 2.4, it is also eigenpair of second order SLP (2.1). Analogous to the proof of Theorem 2.4, we have

$$\int_0^1 \varphi^2(x)dx = 0.$$

Thus $\varphi(x) = 0$ and $(0, y)$ is an eigenpair of (2.1).
Proof of part (ii). By part (i), if λ^3 is an eigenvalue of sixth order SLP with multiplicity $k > 1$, then λ is an eigenvalue of second order problem with the same multiplicity $k > 1$. This is a contradiction with the simplicity of the eigenvalues of the second order SLP. Thus the eigenvalues are simple. □

Theorems 2.2 and 2.5 show that the SLP (2.1) and sixth order SLP (2.4) are equivalent in the sense of Definition 2.1.

3. Isospectral sixth order Sturm-Liouville problems

In this section, by using the equivalence relation obtained in section 2 and Darboux Lemma, we find a family of sixth order SLP which are isospectral to a given one.
If A and B are two linear operators, then the operators AB and BA have the same eigenvalues except perhaps for zero eigenvalue. Thus for finding isospectral operators, we can factorize the given operator as a product of two operators. Then by reversing the factors, we obtain a new operator which is isospectral to the initial one. This idea is applied for second order SLP (2.1) and obtained a family of isospectral problems. This method is known as Darboux Lemma:

Lemma 3.1. *(Darboux Lemma)[5, 15] Suppose that (λ_n, g_n) is an arbitrary eigenpair of the problem:*

$$\begin{cases} y'' + (\lambda - \hat{q})y = 0, \\ y(0) = 0, \quad y(1) = 0. \end{cases} \tag{3.1}$$

Then problem (3.1) *and problem*

$$\begin{cases} y'' + (\lambda - q_{n,\alpha}(x))y = 0, \\ y(0) = 0, \quad y(1) = 0. \end{cases} \tag{3.2}$$

have the same eigenvalues, where

$$q_{n,\alpha}(x) = \hat{q}(x) - 2(ln(1 + \alpha \int_0^x g_n^2(t)dt))'', \quad n = 1, 2, \cdots, \tag{3.3}$$

the eigenfunctions $\{g_n\}_{n=1}^{\infty}$ *are orthonormal and* $\alpha > -1$ *is an arbitrary real number.*

This idea is applied for fourth and sixth order SLP [2, 4, 22]. But unlike second order problems, the factorization of fourth and sixth order equations lead to a system of nonlinear ordinary differential equation. In general, we can not solve this system analytically. In [4, 22], for different cases the nonlinear system is solved by using Lie symmetry method and obtained isospectral problems. In relation (2.9), the sixth order equation is factorized as a product of two operators of second and fourth order. Simple calculations show that by reversing the factors, we don't obtain new sixth order SLP. Thus we can not obtain isospectral problems by this idea. In the following Theorem using Darboux Lemma and equivalence relation obtained in the previous section we find a family of sixth order problems of the form (2.4) which are isospectral.

Theorem 3.2. *Let* $q(x) \in C^4[0,1]$ *be given, then sixth order SLP (2.4) with the boundary conditions (2.3) and (2.5) is isospectral to the following problems:*

$$\begin{cases} -u^{(6)}(x) + 3[q_{n,\alpha}(x)u''(x)]'' - [(3q_{n,\alpha}^2(x) - 4q''_{n,\alpha}(x))u'(x)]' \\ +[q_{n,\alpha}^{(4)} + q_{n,\alpha}^3 - 2q'^2_{n,\alpha} - 3q_{n,\alpha}q''_{n,\alpha}]u(x) = \lambda^3 u(x), \\ u(0) = u''(0) = u^{(4)}(0) - 2q'_{n,\alpha}(0)u'(0) = 0, \\ u(1) = u''(1) = u^{(4)}(1) - 2q'_{n,\alpha}(1)u'(1) = 0, \end{cases} \tag{3.4}$$

where $q_{n,\alpha}$ *is given by (3.3).*

Proof. From Lemma 2.3, both of the problems (2.4) and (3.4) are self-adjoint. Thus they have real eigenvalues which can be denoted by λ^3. It should be proved that if λ^3 is an arbitrary eigenvalue of problem (2.4) then, λ^3 is an eigenvalue of the problem (3.4) and vice versa. Suppose that (λ^3, y) is an arbitrary eigenpair of the problem (2.4). Theorem 2.5 implies that (λ, y) is an eigenpair of (2.1). Using Darboux Lemma, we conclude that there exists an eigenfunction $u(x)$ such that (λ, u) is an eigenpair of the problem

$$\begin{cases} u''(x) + (\lambda - q_{n,\alpha}(x))u(x) = 0, \quad x \in (0,1), \\ u(0) = 0, \quad u(1) = 0. \end{cases} \tag{3.5}$$

Applying Theorem 2.2 for problem (3.5) we find that (λ^3, u) is an eigenpair of the problem (3.4). Similarly, we can prove that if (λ^3, u) is an arbitrary eigenpair of the problem (3.4) then (λ^3, y) is an eigenpair of the problem (2.4) with boundary conditions (2.3) and (2.5). Thus these problems are isospectral. □

Note that the family of isospectral problems obtained in Theorem 3.2 are different from the results in [4]. We apply this method in the following example.

Example 3.3. Consider the following SLP of order six:

$$\begin{cases} y^{(6)}(x) = -\lambda^3 y(x), \quad 0 < x < 1, \\ y(0) = y''(0) = y^{(4)}(0) = 0, \\ y(1) = y''(1) = y^{(4)}(1) = 0. \end{cases} \tag{3.6}$$

Using Theorems 2.2 and 2.5, this problem is equivalent to the following second order problem:

$$\begin{cases} y''(x) = -\lambda y(x), \quad 0 < x < 1, \\ y(0) = 0, \\ y(1) = 0. \end{cases} \tag{3.7}$$

The eigenvalues of the problem (3.7) are $\lambda_n = n^2\pi^2$ and the corresponding orthogonal eigenfunctions are $g_n(x) = \sqrt{2}\sin(n\pi x)$. By Darboux Lemma, the problem (3.7) is isospectral to the problem

$$\begin{cases} u''(x) + (\lambda - q_{n,\alpha}(x))u(x) = 0, \quad 0 < x < 1, \\ u(0) = 0, \quad u(1) = 0, \end{cases} \tag{3.8}$$

where

$$q_{n,\alpha} = 4\alpha \frac{\alpha - \alpha\cos(2n\pi x) - n\pi(1+\alpha x)\sin(2n\pi x)}{(1 + \alpha x - \frac{\alpha}{2n\pi}\sin(2n\pi x))^2}. \tag{3.9}$$

Using Theorems 2.2 and 2.5, problem (3.8) is equivalent to the problem

$$\begin{cases} u^{(6)}(x) - 3[q_{n,\alpha}(x)u''(x)]'' + [(3q_{n,\alpha}^2(x) - 4q_{n,\alpha}''(x))u'(x)]' \\ \qquad -[q_{n,\alpha}^{(4)} + q_{n,\alpha}^3 - 2{q'}_{n,\alpha}^{2} - 3q_{n,\alpha}q_{n,\alpha}'']u(x) = -\lambda^3 u(x), \\ u(0) = u''(0) = u^{(4)}(0) + 8\alpha n^2\pi^2 u'(0) = 0, \\ u(1) = u''(1) = u^{(4)}(1) + \frac{8\alpha n^2\pi^2\cos^2(1)}{1+\alpha}u'(1) = 0. \end{cases} \tag{3.10}$$

Using Theorem 3.2, for any $n \in \mathbb{N}$ and real number $\alpha > -1$, problem (3.10) defines a family of two parameters sixth order SLP which are isospectral to the problem (3.6). Note that for $\alpha = 0$, the problem (3.10) reduces to the problem (3.6).

Remark 3.4. Continuing the process of Theorem 2.2, we can find Sturm-Liouville problems of order $8, 10, \cdots$, which are non self-adjoint. Note that in this paper we need the problem to be self-adjoint. For example, the problem of order 8 is self-adjoint if $q'(0) = q'(1) = 0$. But in Darboux Lemma we have $q'_{n,\alpha}(0) = -4\alpha {g'_n}^2(0) \neq 0$ and $q'_{n,\alpha}(1) = \frac{-4\alpha {g'_n}^2(1)}{1+\alpha} \neq 0$. Thus we can not obtain isospectral problems of order eight using this method.

4. Conclusion

In this paper, we have introduced an equivalence relation between second and sixth order SLP and proved that the second order SLP with the Dirichlet boundary conditions is equivalent to a class of sixth order SLP. Some properties of the eigenvalues of fourth and sixth order SLP are investigated. The isospectral problems are obtained by using the equivalence relation and Darbox Lemma.

References

[1] G. Freiling and V. A. Yurko, *Inverse Sturm-Liouville problems and their applications*, NOVA Science publications, New York, 2001.

[2] K. Ghanbari, *On the Isospectral beams*, Electronic Journal of Differential Equations, Conference *12* (2005), 57-64.

[3] K. Ghanbari, H. Mirzaei, and G. L. M. Gladwell, *Reconstructing of a Rod using one spectrum and minimal mass condition*, Inverse problem in science and engineering, *22*(2) (2013), 325–333.

[4] K. Ghanbari and H. Mirzaei, *On the Isospectral Sixth Order Sturm-Liouville Equation*, Journal of Lie Theory, *23*(4) (2013), 921–935.

[5] G. M. L. Gladwell, *Inverse problem in vibration*, Kluwer academic publishers, New York, 2004.

[6] G. M. L. Gladwell and A. Morassi, *A family of isospectral Euler-Bernoulli beams*, Inverse problems, *26*(3) (2010), 1–12.

[7] L. Greenberg and M. Marletta, *Numerical methods for higher order Sturm-Liouville problems*, Journal of Computational and Applied Mathematics, *125*(1-2) (2000), 367–383.

[8] L. Greenberg, *A prufer method for calculating eigenvalues of selfadjoint systems of ordinary differential equations, Parts 1 and 2*, Technical Report, University of Maryland, 1999.

[9] L. Greenberg and M. Marletta, *Oscillation theory and numerical solution of sixth order Sturm-Liouville problems*, SIAM J. Numer. Anal, *35*(5) (1998), 2070–2098.

[10] T. Gülsen and E. S. Panahkov, *On the isospectrality of the scalar energy-dependent Schrödinger problems*, Turkish Journal of Mathematics, *42*(1) (2018), 139–154.

[11] R .H. Gutierrez and P. A. Laura, *Vibrations of non-uniform rings studied by means of the differential quadrature method*, J. Sound Vib, *185*(3) (1995), 507–513.

[12] A. Kirsch, *An Introduction to the Mathematical Theory of Inverse Problems*, Springer-Verlag, New York, 1996.

[13] V. Ledoux, *Study of Special Algorithms for solving Sturm-Liouville and Schrodinger Equations*. PhD thesis, Ghent University, 2007.

[14] M. Marletta, *Numerical solution of eigenvalue problems for Hamiltonian systems*, Adv. Comput. Math. *2*(2) (1994), 155–184.

[15] H. Mirzaei, *A family of isospectral fourth order Sturm-Liouville problems and equivalent beam equations*, Mathematical Communications, *23*(1) (2018), 15–27.

[16] H. Mirzaei, *Computing the eigenvalues of fourth order Sturm-Liouville problems with Lie Group method*, Journal of Numerical Analysis and Optimizations, *7*(1) (2017), 1–12.

[17] S. Mosazadeh, *A new approach to uniqueness for inverse Sturm–Liouville problems on finite intervals*, Turkish Journal of Mathematics, *41*(5) (2017), 1224–1234.

[18] A. Neamaty and S. H. Akbarpoor, *Numerical solution of inverse nodal problem with an eigenvalue in the boundary condition*, Inverse Problems in Science and Engineering, *25*(7) (2017), 978–994.

[19] M. Shahriari, A. A. Jodayree, and G. Teschl, *Uniqueness for Inverse Sturm-Liouville Problems with a Finite Number of Transmission Conditions*, Journal of Mathematical Analysis and Applications, *395*(1) (2012), 19–29.

[20] M. Shahriari, *Inverse Sturm-Liouville problems with transmission and spectral parameter boundary conditions*, Comput. Methods Differ. Equ, *2*(3) (2014), 123–139.

[21] K. V. Singh, *The transcendental eigenvalue problem and its application in system identification*, PhD thesis, Louisiana State University, 2003.

[22] C. S. Wafo, *Isospectral Euler-Bernoulli beams via factorization and the Lie method*, International Journal of Non-Linear Mechanics, *44*(4) (2009), 396–403.

[23] U. Yucel, *Numerical approximations of Sturm–Liouville eigenvalues using Chebyshev polynomial expansions method*, Cogent Mathematics, *2*(1) (2015), 1–13.