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A Computational Method for Solving the Lane-Emden Initial Value Problems

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Abstract In this work, we propose an efficient numerical algorithm based upon compact finite difference to solve Lane-Emden equations which are nonlinear ordinary differential equations. The presented method reduces the solution of Lane-Emden equations to the solution of a nonlinear system of equations. The numerical experiments show the accuracy and efficiency of this method.

Keywords. Lane-Emden equation, singular IVPs, Compact finite difference.

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1. Introduction

Consider the following Lane-Emden equation:

$$y''(x) + \frac{k}{x}y'(x) + f(x, y(x)) = h(x),\ \ k, x > 0, \tag{1.1}$$

with initial conditions:

$$y(0) = \alpha, y'(0) = \beta, \tag{1.2}$$

where $f(x, y)$ and $h(x)$ are some given continuous real valued functions. It is well known that an analytic solution of Lane-Emden type equation in the neighborhood of the singular point $x = 0$ is always possible[5].

The Eq. (1.1) can be used for several problems in mathematical physics and astrophysics. For instance, the theories of stellar structure, thermionic currents and also the thermal behaviour of a spherical cloud of gas, isothermal gas sphere can be formulated as Eq. (1.1) [4, 5]. In special form, for $f(x, y) = y^m$ and $h(x) = 0$, which known as standard Lane-Emden (or Lane-Emden of the first kind), this equation occurs in astrophysics to model the gravitational potential of polytropic fluids in a self-gravitating star[15]. Also, this form can be used to model the temperature of a spherical cloud of gas under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [5]. The Lane-Emden equation of the second kind, in which $f(x, y) = e^y$ and $h(x) = 0$, is used to formulate a thermal explosion in a cylindrical vessel or a rectangular slab[7, 27].

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Recently, many analytical and numerical methods have been proposed to solve Lane-Emden Eq. (1.1). Parand et al. [18, 19, 20, 21, 22, 23] solved Lane-Emden equation through different methods. In [20] a method is proposed based on Bessel orthogonal functions collocation method for the first and the second kind of Lane-Emden equation. Also, based on a modified generalized Laguerre functions Lagrangian method, Parand et al. [22] proposed a Lagrangian method for the first kind of Lane-Emden equation. Parand and Hashemi [19] applied a meshless method based on radial basis function differential quadrature(RBF-DQ) method to solve some well-known classes of Lane-Emden type equations. Also, in [18] a Hermit functions collocation (HFC) method is employed. In [23], a pseudospectral method based on rational Legendre functions is applied to solve Lane-Emden equations and in [21], Parand and Pirkhedri proposed Sinc-collocation method for solving standard Lane-Emden equation with initial conditions $y(0) = 1,\ y'(0) = 0$. In [25], Shiralashetti and Kumbinarasaiah proposed a method using Legendre, Hermite and Laguerre wavelets.

In [3], authors used iterative methods based on the Newton-Raphson-Kantorovich approximation and in [16], Pandey and Kumar introduced a numerical method using Bernstein operational matrix of differentiation, for special initial conditions $y(0) = a$ and $y'(0) = 0$. In [12], a Haar wavelet quasi-linearization approach for the first and the second kind of Lane-Emden equation is studied. Marzban et al. [13] applied hybrid functions to find out the numerical solution of Eq. (1.1) with initial condition $y(0) = a,\ y'(0) = 0$, for some particular nonlinear cases. With these initial conditions, Eq. (1.1) is solved in [17], using Legendre operational matrix of differentiation. Legendre spectral method has been used for Lane-Emden equation in [1, 14] for $f(x, y) = g(y),\ h(x) = 0$ and $f(x, y) = p(x)g(y),\ y(0) = \alpha,\ y'(0) = 0$ respectively.

Kazemi nasab et al. [11] suggested a numerical method based upon hybrid of Chebyshev wavelets and finite difference(CWFD) methods for the case where $f(x, y) = p(x)q(y)$ and expanded this technique for fractional Lane-Emden type equation in [10]. Saadatmandi et al. [24] proposed two computational schemes based on collocation method with operational matrices of orthonormal Bernstein polynomials for fractional Lane-Emden type equations.

By using variation iterative method, in [6] a numerical method has been proposed. Wazwaz [28, 29] has proposed a general way to find out exact and series solutions, by employing the Adomian decomposition method. Yousefi [26] obtained a numerical solution of Lane-Emden equations based on the Legendre wavelets method. In [9] Karimi Vanani and Aminataei constructed an approximate polynomial solution by using of integral operator for $0 \leq x \leq 1$.

In this work, a simple and accurate numerical technique for solving Eq. (1.1) will be constructed. In this technique, we use two compact finite difference schemes for the first and the second derivatives, and solution of Eq. (1.1) reduced to solution of a nonlinear system of equations.

The structure of the paper is as follows: In section 2, we review a simple compact finite difference and apply this for Lane-Emden Eq. (1.1). In section 3, the proposed approach is applied for some examples and a comparison is made with existing results in the literature. A very high level of accuracy shows the ability of our method for this problem. Finally, a conclusion is drawn in Section 4.

2. The proposed method

To produce a compact finite difference scheme, first of all, the domain $[0, a]$ is divided into N equal subinterval of width $h = \frac{a}{N}$. The grid points are shown by $x_i = ih,\ i = 0, \cdots, N$ and $y_i \approx y(x_i)$. By a compact finite difference formula presented in [31, 32], we have

$$\begin{cases} 4y'_1 + y'_2 = \frac{1}{h}[-\frac{11}{12}y_0 - 4y_1 + 6y_2 - \frac{4}{3}y_3 + \frac{1}{4}y_4], \\ y'_{i-1} + 4y'_i + y'_{i+1} = \frac{3}{h}(y_{i+1} - y_{i-1}),\ i = 1 \cdots, N-1, \\ y'_{N-2} + 4y'_{N-1} = \frac{1}{h}[-\frac{1}{4}y_{N-4} + \frac{4}{3}y_{N-3} - 6y_{N-2} + 4y_{N-1} + \frac{11}{12}y_N]. \end{cases} \tag{2.1}$$

The matrix form for Eq. (2.1) is $A_1 Y' = B_1 Y$, where

$$A_1 = \begin{pmatrix} 0 & 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \cdots & \cdots & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \cdots & 1 & 4 & 1 \\ 0 & \cdots & \cdots & 1 & 4 & 0 \end{pmatrix}_{(N+1)\times(N+1)},$$

$$B_1 = \frac{1}{h}\begin{pmatrix} -\frac{11}{12} & -4 & 6 & -\frac{4}{3} & \frac{1}{4} & 0 & \cdots & 0 \\ -3 & 0 & 3 & 0 & 0 & & \vdots & 0 \\ 0 & -3 & 0 & 3 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & & \cdots & 0 & 3 & 0 & -3 \\ 0 & \cdots & 0 & -\frac{1}{4} & \frac{4}{3} & -6 & 4 & \frac{11}{12} \end{pmatrix}_{(N+1)\times(N+1)}.$$

and

$$Y = [y_0, y_1, \cdots, y_N]^T,\ Y' = [y'_0, \cdots, y'_N]^T.$$

Lemma 2.1. *The coefficient matrix A_1 is invertible.*

Proof. Let's expand A_1 along the first column, therefore

$$det(A_1) \qquad = -det\begin{pmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 4 & 1 \\ \cdots & \cdots & 1 & 4 & 0 \end{pmatrix}_{N\times N}.$$

Now, by expanding along the last column, we have

$$det(A_1) = \pm det \begin{pmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 4 & 1 \\ 0 & \cdots & 0 & 1 & 4 \end{pmatrix}_{(N-1)\times(N-1)} \neq 0\ .$$

□

Similarly, for the second order derivative, we have

$$\begin{cases} 14y_1'' - 5y_2'' + 4y_3'' - y_4'' = \frac{12}{h^2}(y_0 - 2y_1 + y_2), \\ y_{i-1}'' + 10y_i'' + y_{i+1}'' = \frac{12}{h^2}(y_{i-1} - 2y_i + y_{i+1}),\ i = 1, \cdots, N-1, \\ -y_{N-4}'' + 4y_{N-3}'' - 5y_{N-2}'' + 14y_{N-1}' = \frac{12}{h^2}(y_{N-2} - 2y_{N-1} + y_N). \end{cases} \tag{2.2}$$

The truncation error for Eq. (2.1) and Eq. (2.2) is $O(h^4)$. In matrix form, Eq. (2.2) is written as:

$$A_2 Y'' = B_2 Y\ ,$$

where

$$A_2 = \begin{pmatrix} 0 & 14 & -5 & 4 & -1 & 0 & \cdots & 0 \\ 1 & 10 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 10 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ 0 & \cdots & & \cdots & 0 & 1 & 10 & 1 \\ 0 & \cdots & 0 & -1 & 4 & -5 & 14 & 0 \end{pmatrix}_{(N+1)\times(N+1)}$$

$$B_2 = \frac{12}{h^2} \begin{pmatrix} 1 & -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 & 1 \end{pmatrix}_{(N+1)\times(N+1)}$$

and

$$Y'' = [y_0'', \cdots, y_N'']^T.$$

It is easy to show that the coefficient matrix A_2 is invertible.
According to above,

$$Y' = A_1^{-1} B_1 Y,\ Y'' = A_2^{-1} B_2 Y, \tag{2.3}$$

or

$$y_i' = \sum_{j=0}^{N} c_{ij} y_j,\ y_i'' = \sum_{j=0}^{N} d_{ij} y_j,\ i = 0, 1, \cdots, N, \tag{2.4}$$

Table 1. The maximum absolute errors for $m = 0,\ 1,\ 5$ and $x \in [0,1]$for Example 3.1 .

	$h = 0.2$	$h = 0.1$
$m = 0$	$9.0e(-20)$	$2.0e(-19)$
$m = 1$	$8.93e(-7)$	$2.18e(-8)$
$m = 5$	$4.10e(-5)$	$7.29e(-7)$

where $C = (c)_{ij} = A_1^{-1}B_1$ and $D = (d)_{ij} = A_2^{-1}B_2$.
Due to the initial condition $y_0' = \beta$, we can have

$$\sum_{j=0}^{N} c_{0j}u_j = \beta, \tag{2.5}$$

as the first equation. Other equations can be written as

$$\sum_{j=0}^{N} d_{ij}y_j + \frac{\alpha}{x_i}\sum_{j=0}^{N} c_{ij}y_j + f_i = h_i,\ i = 1, \cdots, N-1, \tag{2.6}$$

where $f_i = f(x_i, y_i)$ and $h_i = h(x_i)$. Eqs. (2.5) and (2.6) form a nonlinear system, therefore by solving this system, we obtain the $y_j,\ j = 0, 1, \cdots, N$.

3. Numerical experiment

In this section, we apply the proposed method, to solve the Lane-Emden Eq. (1.1) with initial condition Eq. (1.2). Eq. (2.5) and Eq. (2.6) are solved by "fsolve" command in Maple17 with 20 digits.

3.1. **The standard Lane-Emden equation.**

Example 3.1. Consider the following Lane-Emden equation,

$$y''(x) + \frac{2}{x}y'(x) + y^m(x) = 0, x > 0, \tag{3.1}$$

with conditions $y(0) = 1$ and $y'(0) = 0$. The physically interesting range for m is $0 \leq m \leq 5$. For $m = 0, 1,\ 5$, the exact solutions for Eq. (3.1) are respectively

$$y(x) = 1 - \frac{1}{3!}x^2,\ y(x) = \frac{\sin(x)}{x}, y(x) = \big(1 + \frac{x^2}{3}\big)^{-\frac{1}{2}}.$$

Table 1 shows the maximum absolute errors for the interval $[0, 1]$.
For other values of m there is no exact analytical solution, therefore we construct the residual value as

$$Res(i) = |y_i'' + \frac{k}{x_i}y_i' + f_i - h_i|\ ,$$

which y_i'' and y_i' can been computed from Eq. (2.3). Tables 2-6 show some y_is for $m = 1.5, 2, 2.5, 3,$ and 4, respectively and compare with some well-known methods in other articles.

TABLE 2. Comparison of $y(x)$ values of standard Lane-Emden equation for $m = 1.5,\ a = 3.7,\ h = 0.1$(Example 3.1).

x_i	Horedt[8]	RBF-DQ[19]	Presented method	Res
0.1	$9.983346e(-1)$	$9.983345826e(-1)$	$9.9833453708e(-1)$	$6.21e(-18)$
0.5	$9.591039e(-1)$	$9.591038569e(-1)$	$9.5910384304e(-1)$	$2.82e(-18)$
1	$8.451698e(-1)$	$8.451697549e(-1)$	$8.4516967833e(-1)$	$2.20e(-19)$
3	$1.588576e(-1)$	$1.588576082e(-1)$	$1.5885754780e(-1)$	$7.34e(-19)$
3.6	$1.109099e(-2)$	$1.109099415e(-2)$	$1.1090950970e(-2)$	$1.80e(-20)$

TABLE 3. Comparison of $y(x)$ values of standard Lane-Emden equation for $m = 2,\ a = 4.4,\ h = 0.1$(Example 3.1) .

x_i	RBF-DQ[19]	Presented method	Res
0.1	$9.98334998e(-1)$	$9.98334903e(-1)$	$9.34e(-18)$
0.5	$9.59352716e(-1)$	$9.59352687e(-1)$	$8.58e(-18)$
3	$2.41824083e(-1)$	$2.41824085e(-1)$	$7.47e(-19)$
4.3	$6.81094327e(-3)$	$6.81106949e(-3)$	$1.76e(-19)$

TABLE 4. Comparison of $y(x)$ values of standard Lane-Emden equation for $m = 2.5,\ a = 5.4.\ h = 0.1$(Example 3.1) .

x_i	RBF-DQ[19]	Presented method	Res
0.1	$9.98335414e(-1)$	$9.98335253e(-1)$	$1.83e(-18)$
0.5	$9.59597754e(-1)$	$9.59597706e(-1)$	$1.46e(-17)$
1	$8.51944199e(-1)$	$8.51944009e(-1)$	$4.98e(-18)$
4	$1.37680733e(-1)$	$1.37680875e(-1)$	$8.01e(-19)$
5	$2.90191866e(-2)$	$2.90193596e(-2)$	$1.30e(-19)$
5.3	$4.25954353e(-3)$	$4.25972208e(-3)$	$1.35e(-20)$

TABLE 5. Comparison of $y(x)$ values of standard Lane-Emden equation for $m = 3,\ a = 6.9,\ h = 0.1$(Example 3.1).

x_i	RBF-DQ[19]	CWFD[11]	Presented method	Res
0.1	$9.983358e(-1)$	$9.983358e(-1)$	$9.983356e(-1)$	$4.30e(-18)$
0.5	$9.598391e(-1)$	$9.598391e(-1)$	$9.598390e(-1)$	$6.40e(-19)$
1	$8.550575e(-1)$	$8.550576e(-1)$	$8.550573e(-1)$	$1.36e(-17)$
5	$1.108199e(-2)$	$1.108198e(-1)$	$1.108200e(-1)$	$1.30e(-19)$
6	$4.373798e(-2)$	$4.373798e(-2)$	$4.373818e(-2)$	$4.08e(-19)$
6.8	$4.167789e(-3)$	$4.167789e(-3)$	$4.167985e(-3)$	$9.36e(-20)$

3.2. The second kind of Lane-Emden equation.

Example 3.2. Consider the following second kind of Lane-Emden equation

$$y''(x) + \frac{2}{x}y'(x) + e^{y(x)} = 0,\ x > 0; \tag{3.2}$$

TABLE 6. Comparison of $y(x)$ values of standard Lane-Emden equation for $m = 4,\ a = 15,\ h = 0.1$, for Example 3.1 .

x_i	Horedt[8]	RBF-DQ[19]	CWFD[11]	Presented method	Res
0.1	$9.983367e(-1)$	$9.983366e(-1)$	$9.983367e(-1)$	$9.983362e(-1)$	$2.48e(-18)$
0.2	$9.933862e(-1)$	$9.933862e(-1)$	$9.933862e(-1)$	$9.933861e(-1)$	$7.49e(-18)$
0.5	$9.603109e(-1)$	$9.603109e(-1)$	$9.603109e(-1)$	$9.603108e(-1)$	$1.88e(-18)$
1	$8.608138e(-1)$	$8.608144e(-1)$	$8.608138e(-1)$	$8.608135e(-1)$	$1.59e(-17)$
5	$2.359227e(-1)$	$2.352433e(-1)$	$2.359227e(-1)$	$2.359229e(-1)$	$1.08e(-18)$
10	$5.967274e(-2)$	$5.965197(-2)$	$5.967274e(-2)$	$5.967291e(-2)$	$1.27e(-19)$
14	$8.330527e(-3)$	$8.330447e(-3)$	$8.330527e(-3)$	$8.330677e(-3)$	$1.02e(-19)$
14.9	$5.764189e(-4)$	$5.763524e(-4)$	$5.76419e(-4)$	$5.765661e(-4)$	$1.81e(-20)$

TABLE 7. Comparison of $y(x)$ values obtained by presented method (for $a = 2.6,\ h = 0.1$) and some other results, for Example 3.2.

x_i	RBF-DQ[19]	ADM[29]	Presented method	Res
0.1	$-1.66583e(-3)$	$-1.665834e(-3)$	$-1.666587e(-3)$	$6.0e(-20)$
0.2	$-6.65336e(-3)$	$-6.653367e(-3)$	$-6.653373e(-3)$	$3.0e(-20)$
0.5	$-4.115395e(-2)$	$-4.115396e(-2)$	$-4.115397e(-2)$	$5.3e(-19)$
1	$-1.588277e(-1)$	$-1.588273e(-1)$	$-1.588277e(-1)$	$1.7e(-19)$
1.5	$-3.380194e(-1)$	$-3.380131e(-1)$	$-3.380195e(-1)$	$1.31e(-18)$
2	$-5.598230e(-1)$	$-5.599627e(-1)$	$-5.598231e(-1)$	$1.08e(-18)$
2.5	$-8.063409e(-1)$	$-8.100197e(-1)$	$-8.063410e(-1)$	$2.68e(-18)$

with conditions $y(0) = y'(0) = 0$. This equation has been solved in some literature. For instance Wazwaz in [29], by using ADM and series expansion obtained the following approximate solution

$$y(x) \simeq -\frac{1}{6}x^2 + \frac{1}{5\times 4!}x^4 - \frac{8}{21\times 6!}x^6 + \frac{122}{81\times 8!}x^8 - \frac{61\times 67}{495\times 10!}x^{10}.$$

Also, Bessel orthogonal functions collocation method[20], Lagrangian method [22], RBF-DQ method [19], HFC method [18], iterative method [3] and some other techniques are applied for Eq. (3.2). A comparison of y_i's obtained by the presented method in this work and some results are shown in Table 7.

3.3. **Other examples.**

Example 3.3. Consider the following nonlinear Lane-Emden equation given by[19, 18, 29]:

$$y''(x) + \frac{2}{x}y'(x) + sin(y(x)) = 0,\ x > 0 \qquad (3.3)$$

TABLE 8. Comparison of y_i, between presented method (for $a = 2.1,\ h = 0.1$) and solutions given by [18, 19, 29], for Example 3.3.

x_i	HFC[18]	RBF-DQ[19]	ADM[29]	Presented method	Res
0.1	0.99860514	0.9985979	0.9985979	0.9985979371	$9.97e(-18)$
0.2	0.99440627	0.99439626	0.9943962	0.9943962666	$9.36e(-18)$
0.5	0.96518817	0.96517778	0.9651777	0.9651777832	$5.00e(-18)$
1	0.86368813	0.86368112	0.8636811	0.863681139	$8.17e(-18)$
1.5	0.70505241	0.70504523	0.7050419	0.7050452474	$4.69e(-18)$
2	0.50646876	0.50646363	0.5063720	0.5064636131	$7.50e(-19)$

with initial conditions $y(0) = 1,\ y'(0) = 0$. By using ADM in [29], Wazwaz computed the following approximate solution

$$\begin{aligned} y(x) &\simeq 1 - \frac{1}{6}kx^2 + \frac{1}{120}klx^4 + k\left(\frac{1}{3024}k^2 - \frac{1}{5040}l^2\right)x^6 \\ &+kl\left(\frac{113}{3265920}k^2 + \frac{1}{362880}l^2\right)x^8 \\ &+k\left(\frac{1781}{898128000}k^2l^2 - \frac{1}{399168000}l^4 - \frac{19}{2395080}k^4\right)x^{10}, \end{aligned}$$

where $k = \sin(1)$ and $l = \cos(1)$. By solving Eq. (3.3), we obtain the shown results in Table 8. This table compares the y_i's obtained by the presented method in this work and some well known methods.

Example 3.4. Let $f(x, y) = 4(2e^{y(x)} + e^{\frac{y(x)}{2}}),\ h(x) = 0$ and $y(0) = y'(0) = 0$. Therefore the corresponding Lane-Emden equation has the following form

$$y''(x) + \frac{2}{x}y'(x) + 4(2e^{y(x)} + e^{\frac{y(x)}{2}}) = 0,\ y(0) = y'(0) = 0. \tag{3.4}$$

The exact solution for this problem is

$$y(x) = -2\ln(1 + x^2).$$

This problem has been solved by RBF-DQ method [19], HFC method [18], HPM method [30]. In this work, we applied the presented finite difference scheme for Eq. (3.4) and obtained the results shown in Table 9.

Example 3.5. Consider the following Lane-Emden equation

$$y''(x) + \frac{6}{x}y'(x) + 14y(x) = -4y(x)\ln(y),\ x > 0; \tag{3.5}$$

with the initial conditions $y(0) = 1,\ y'(0) = 0$. The exact solution for Eq(3.5) is

$$y(x) = e^{-x^2}.$$

From Table 10, it is observed that presented finite difference method is better than method proposed in [2].

TABLE 9. Numerical solution of the Lane-Emden equation and corresponding absolute error for $a = 10.1,\ h = 0.1$, for Example 3.4
.

x_i	RBF-DQ[19]	Presented method	Absolute error
0.1	−0.019900660	−0.019939474	$3.88e(-5)$
0.5	−0.44628712	−0.446297661	$1.06e(-5)$
1	−1.38629440	−1.38629665	$2.29e(-6)$
2	−3.2188869	−3.21886926	$6.57e(-6)$
3	−4.60506457	−4.60516394	$6.25e(-6)$
4	−5.6664689	−5.66642202	$4.66e(-6)$
5	−6.51642073	−6.51618993	$3.15e(-6)$
6	−7.22186619	−7.22183393	$1.89e(-6)$
7	−7.82405388	−7.82404511	$8.99e(-7)$
8	−8.34877271	−8.34877442	$1.17e(-7)$
9	−8.81343853	−8.81343899	$4.95e(-7)$
10	−9.23024103	−9.23024201	$9.74e(-7)$

TABLE 10. Comparison of absolute errors for presented method($a = 1,\ h = 0.1$) and method in [2], for Example 3.5.

x_i	Absolute error obtained by presented method	Absolute error obtained in [2]	Exact solution
0.1	$7.53e(-6)$	$4.89e(-5)$	0.99004983
0.2	$7.83e(-7)$	$6.84e(-6)$	0.96078944
0.3	$3.93e(-6)$	$8.03e(-7)$	0.91393118
0.4	$4.37e(-6)$	$8.38e(-6)$	0.85214379
0.5	$5.99e(-6)$	$1.28e(-5)$	0.77880078
0.6	$6.85e(-6)$	$5.32e(-5)$	0.69767632
0.7	$7.35e(-6)$	$2.07e(-4)$	0.61262639
0.8	$7.08e(-6)$	$2.94e(-4)$	0.52729242
0.9	$6.77e(-6)$	$1.42e(-3)$	0.44485806
1	$7.25e(-6)$	$3.07e(-3)$	0.36787944

Example 3.6. Consider the following Lane-Emden equation [9]:

$$y''(x) + \frac{2}{x}y'(x) + y(x) = 6 + 12x + x^2 + x^3,\ x \geq 0; \tag{3.6}$$

with initial condition $y(0) = y'(0) = 0$. The exact solution for this equation is $y(x) = x^2 + x^3$. Eq. (3.6) has been solved by HPM [30], ADM [28], RBF-DQ method [19]and HFC method [18]. We applied the presented method to solve this and in Table 11 compared our results with RBF-DQ method [19] and HFC method [18].

Table 11. Comparison of the absolute errors among the presented method ($a = 10, h = 0.1$) and [18, 19], for Example 3.6 .

x_i	our method	RBF-DQ[19]	HFC[18]
0.1	$5.00e(-20)$	$1.36e(-11)$	$1.82e(-6)$
0.5	$7.10e(-19)$	$2.39e(-9)$	$1.41e(-6)$
1	$2.60e(-18)$	$3.23e(-8)$	$1.25e(-6)$
2	$4.00e(-18)$	$2.49e(-6)$	$6.93e(-7)$
3	$6.20e(-17)$	$8.51e(-5)$	$7.58e(-8)$
4	$2.76e(-16)$	$4.86e(-5)$	$3.07e(-7)$
5	$6.30e(-16)$	$1.78e(-4)$	$3.21e(-7)$
6	$1.08e(-15)$	$7.51e(-4)$	$9.74e(-8)$
7	$7.50e(-16)$	$2.91e(-4)$	$2.05e(-7)$
8	$3.30e(-16)$	$1.27e(-5)$	$7.36e(-7)$
9	$3.44e(-15)$	$3.26e(-7)$	$4.61e(-6)$
10	$4.80e(-15)$	$1.52e(-10)$	$1.24e(-5)$

4. Conclusion

The main focus of this paper was to find a simple numerical algorithm for the Lane-Emden type initial value problem which occurs in some problem in mathematical physics and astrophysics. Based upon a compact finite difference scheme, we received an accurate and efficient method to approximate the solution of Lane-Emden equation.This numerical solution was obtained by solving a nonlinear system of equations. The results confirmed that the proposed method is accurate and efficient.

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