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On the new extensions to the Benjamin-Ono equation

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Abstract In this paper, we analytically study the newly developed (2+1)-dimensional Benjamin-Ono equation by Wazwaz and propose its (3+1)-dimensional version. For this purpose, we successfully employed the modified extended tanh expansion method to construct certain hyperbolic, periodic and complex solitary wave structures simulated with the aid of symbolic computation using Mathematica. Also, we have depicted graphically the constructed solutions.

Keywords. Solitary waves; Benjamin-Ono equations; Extended tanh method.

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1. Introduction

Nonlinear partial differential equations are important equations used in modeling many phenomena in science and engineering applications. In particular, the class of evolution equations plays a fundamental role in fluid dynamical problems including the Korteweg-de-Vries equation, Burgers equation, Boussinesq-Burgers equation and Benjamin-Ono equation among others. Of interest is the Benjamin-Ono equation which describes the long internal gravity waves in deep stratified fluids:

$$u_{tt} + \alpha(u^2)_{xx} + \beta u_{xxxx} = 0, \tag{1.1}$$

where α and β are non zero constants. Furthermore, the literature is rich in different studies regarding Benjamin-Ono equation. We mention among them: the existence of multi-soliton solution by [6, 15], existence of periodic solutions by Ambrose and Wilkening [3], and solution stability by Angulo et al. [5] and certain discrete solutions by Tutiya et al. [28]. See also [3, 9, 13, 17, 26, 27] and references therein for other articles.

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However, Wazwaz [29] recently developed the (2+1)-dimensional version of the Benjamin-Ono equation given in (1.1), namely

$$u_{tt} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \gamma u_{yyyy} = 0, \tag{1.2}$$

where α, β and γ are non zero constants and γ never zero. In [29], Wazwaz presented certain exponential, hyperbolic and period solutions using the Hirota bilinear and ansatze methods to acquire solutions with Painlevé integrability check.

Further, we proposed in this paper the (3+1)-dimensional Benjamin-Ono equation in the same manner as in (1.2) that reads

$$u_{tt} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \gamma u_{yyyy} + \delta u_{zzzz} = 0, \tag{1.3}$$

where α, β, γ and δ are non zero constants. Again, new solitary wave solutions for the (2+1) and (3+1)-dimensional Benjamin-Ono equations given in Eqs. (1.2) and (1.3) will be investigated using the modified extended tanh expansion method [18, 21, 22]. See also [1, 2, 7, 8, 10, 11, 12, 14, 16, 19, 20, 23, 24, 25, 30] for other related methods. The rest of the paper is organized as follows: Section 2 gives the analysis of the method. In Section 3, we presented various new exact solutions for the two equations and present some graphical illustrations in Section 4 and conclusion in Sections 5, respectively.

2. Analysis of the Method

We present the modified extended tanh expansion method in this section. We suppose that nonlinear partial differential equation takes the form

$$F(u, u_x, u_t, u_{xx}, u_{tt}, u_{xxx}, ...) = 0. \tag{2.1}$$

Employing the transformation

$$u(x,t) = f(\xi), \quad \xi = ax - ct - x_0,$$

where a and c are nonzero constants and x_0 is arbitrary constant; (2.1) converts to nonlinear ordinary differential equations of the form

$$H(f', f'', f''', ...) = 0, \tag{2.2}$$

where the derivatives are with respect to ξ. It is assumed that the solution of (2.2) takes the following form by the modified extended tanh expansion method [18, 21] given by the finite series:

$$f(\xi) = A_0 + \sum_{n=1}^{n=N} \left(A_n \Phi^n(\xi) + \frac{B_n}{\Phi^n(\xi)} \right), \tag{2.3}$$

where $A_0, A_n, B_n, ...; n = 1, 2, ...N$ (not all zero) are to be computed; where N is positive integer that is determined by balancing the highest order derivative with the highest nonlinear terms in the equation; $\Phi(\xi)$ satisfies the Riccati differential equation:

$$\Phi'(\xi) = w + \Phi^2(\xi), \tag{2.4}$$

where w is a constant. Further, the Riccati differential equation in (2.4) has solutions of the form:

(i) if $w < 0$, then

$$\Phi(\xi) = -\sqrt{-w}\tanh(\sqrt{-w}\xi),$$
$$\Phi(\xi) = -\sqrt{-w}\coth(\sqrt{-w}\xi),$$

(ii) if $w = 0$, then

$$\Phi(\xi) = -\frac{1}{\xi},$$

(iii) if $w > 0$, then

$$\Phi(\xi) = \sqrt{w}\tanh(\sqrt{w}\xi),$$
$$\Phi(\xi) = -\sqrt{w}\coth(\sqrt{w}\xi).$$

Substituting (2.3) and its necessary derivatives into (2.2) gives a polynomial in $\Phi(\xi)$. Collecting coefficients of the obtained polynomials and subsequently setting each one to zero, we will get a set of over-determined algebraic equations for $A_0, A_n, B_n, ...; (n = 1, 2, ...)$, and w with the help of the Mathematica. Finally, solving the obtained algebraic equations and coupling with the above solutions of Riccati equation into (2.4), we obtain the solution of (2.1).

3. Applications

In this section, cetain new solitary wave solutions to this new equations are preseneted using the modified extended tanh method presented above.

3.1. **The (2+1)-dimensional Benjamin-Ono equation.**

In this subsection, we examine the (2+1)-dimensional Benjamin-Ono equation given by from (1.2)

$$u_{tt} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \gamma u_{yyyy} = 0, \tag{3.1}$$

where $u(x, y, t)$ is a sufficiently smooth function and α, β and γ are nonzero parameters. To construct some solitary wave solutions, we substitute

$$u(x, y, t) = f(\xi), \quad \xi = ax + by - ct - x_0, \tag{3.2}$$

into (3.1) where,

$$u_{tt} = c^2 f'(\xi), \;\; u_x = af'(\xi), \;\; u_{xx} = a^2 f''(\xi),$$

$$u_{xxxx} = a^4 f''''(\xi), \;\; u_{yyyy} = b^4 f''''(\xi),$$

and get the following nonlinear ordinary differential equation

$$c^2 f'' + \alpha a^2 (f^2)'' + \beta a^4 f'''' + \gamma b^4 f'''' = 0. \tag{3.3}$$

Integrating (3.3) once with respect to ξ, yields

$$c^2 f + \alpha a^2 (f^2) + \beta a^4 f'' + \gamma b^4 f'' = 0, \tag{3.4}$$

where the integration constants are assumed zero for simplicity. Balancing f^2 and f'' in (3.4), we get $2N = N + 2$, so $N = 2$. This gives a truncated series from (2.3) as follows

$$f(\xi) = A_0 + A_1\Phi(\xi) + A_2\Phi^2(\xi) + \frac{B_1}{\Phi(\xi)} + \frac{B_2}{\Phi(\xi)^2}. \tag{3.5}$$

Substituting (3.5) into (3.4) and equating the coefficient of each power of $\Phi(\xi)$ to zero. We get a system of algebraic equations given below

$$\begin{gathered}
6a^4w^2\beta B_2 + 6b^4w^2\gamma B_2 + a^2\alpha B_2^2 = 0,\\
2a^4w^2\beta B_1 + 2b^4w^2\gamma B_1 + 2a^2\alpha B_1B_2 = 0,\\
a^2\alpha B_1^2 + c^2B_2 + 8a^4w\beta B_2 + 8b^4w\gamma B_2 + 2a^2\alpha A_0B_2 = 0,\\
c^2A_0+a^2\alpha A_0^2+2a^4w^2\beta A_2+2b^4w^2\gamma A_2+2a^2\alpha A_1B_1+2a^4\beta B_2+2b^4\gamma B_2+2a^2\alpha A_2B_2 = 0,\\
c^2B_1 + 2a^4w\beta B_1 + 2b^4w\gamma B_1 + 2a^2\alpha A_0B_1 + 2a^2\alpha A_1B_2 = 0,\\
c^2A_1 + 2a^4w\beta A_1 + 2b^4w\gamma A_1 + 2a^2\alpha A_0A_1 + 2a^2\alpha A_2B_1 = 0,\\
a^2\alpha A_1^2 + c^2A_2 + 8a^4w\beta A_2 + 8b^4w\gamma A_2 + 2a^2\alpha A_0A_2 = 0,\\
2a^4\beta A_1 + 2b^4\gamma A_1 + 2a^2\alpha A_1A_2 = 0,\\
6a^4\beta A_2 + 6b^4\gamma A_2 + a^2\alpha A_2^2 = 0.
\end{gathered}$$

Solving the above system, gives:

Case 1

$$A_0 = -\frac{3c^2}{2a^2\alpha}, A_1 = 0, B_1 = 0, B_2 = 0, A_2 = -\frac{3c^2}{2a^2w\alpha}, b = \mp\frac{\left(c^2 - 4a^4w\beta\right)^{1/4}}{\sqrt{2}w^{1/4}\gamma^{1/4}}.$$

Thus, the following solutions are formed:

$$u_{1,2}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{3c^2}{2a^2w\alpha}(\sqrt{w}\tanh(\sqrt{w}\xi))^2,\ if\ w > 0, \tag{3.6}$$

$$u_{3,4}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{3c^2}{2a^2w\alpha}(\sqrt{w}\coth(\sqrt{w}\xi))^2,\ if\ w > 0, \tag{3.7}$$

$$u_{5,6}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{3c^2}{2a^2w\alpha}(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2,\ if\ w < 0, \tag{3.8}$$

$$u_{7,8}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{3c^2}{2a^2w\alpha}(\sqrt{-w}\coth(\sqrt{-w}\xi))^2,\ if\ w < 0, \tag{3.9}$$

where,

$$\xi = ax + by - ct - x_0.$$

Case 2

$$A_0 = \frac{c^2}{2a^2\alpha}, A_1 = 0, B_1 = 0, B_2 = 0, A_2 = \frac{3c^2}{2a^2 w\alpha}, b = \mp\frac{\left(-c^2 - 4a^4 w\beta\right)^{1/4}}{\sqrt{2}w^{1/4}\gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$u_{9,10}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{3c^2}{2a^2 w\alpha}(\sqrt{w}\tanh(\sqrt{w}\xi))^2, \ if \ w > 0, \tag{3.10}$$

$$u_{11,12}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{3c^2}{2a^2 w\alpha}(\sqrt{w}\coth(\sqrt{w}\xi))^2, \ if \ w > 0, \tag{3.11}$$

$$u_{13,14}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{3c^2}{2a^2 w\alpha}(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2, \ if \ w < 0, \tag{3.12}$$

$$u_{15,16}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{3c^2}{2a^2 w\alpha}(\sqrt{-w}\coth(\sqrt{-w}\xi))^2, \ if \ w < 0, \tag{3.13}$$

where,

$$\xi = ax + by - ct - x_0.$$

Case 3

$$A_0 = \frac{c^2}{2a^2\alpha}, A_1 = 0, B_1 = 0, B_2 = 0, A_2 = \frac{3c^2}{2a^2 w\alpha}, b = \mp\frac{i\left(-c^2 - 4a^4 w\beta\right)^{1/4}}{\sqrt{2}w^{1/4}\gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$u_{17,18}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{3c^2}{2a^2 w\alpha}(\sqrt{w}\tanh(\sqrt{w}\xi))^2, \ if \ w > 0, \tag{3.14}$$

$$u_{19,20}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{3c^2}{2a^2 w\alpha}(\sqrt{w}\coth(\sqrt{w}\xi))^2, \ if \ w > 0, \tag{3.15}$$

$$u_{21,22}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{3c^2}{2a^2 w\alpha}(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2, \ if \ w < 0, \tag{3.16}$$

$$u_{23,24}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{3c^2}{2a^2 w\alpha}(\sqrt{-w}\coth(\sqrt{-w}\xi))^2, \ if \ w < 0, \tag{3.17}$$

where,

$$\xi = ax + by - ct - x_0.$$

Case 4

$$A_0 = \frac{-3c^2}{2a^2\alpha}, A_1 = 0, B_1 = 0, B_2 = 0, A_2 = \frac{-3c^2}{2a^2 w\alpha}, b = \mp\frac{i\left(c^2 - 4a^4 w\beta\right)^{1/4}}{\sqrt{2}w^{1/4}\gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$u_{25,26}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{3c^2}{2a^2w\alpha}(\sqrt{w}\tanh(\sqrt{w}\xi))^2,\ if\ w>0, \tag{3.18}$$

$$u_{27,28}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{3c^2}{2a^2w\alpha}(\sqrt{w}\coth(\sqrt{w}\xi))^2,\ if\ w>0, \tag{3.19}$$

$$u_{29,30}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{3c^2}{2a^2w\alpha}(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2,\ if\ w<0, \tag{3.20}$$

$$u_{31,32}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{3c^2}{2a^2w\alpha}(\sqrt{-w}\coth(\sqrt{-w}\xi))^2,\ if\ w<0, \tag{3.21}$$

where,

$$\xi = ax + by - ct - x_0.$$

Case 5

$$A_0 = -\frac{3c^2}{2a^2\alpha}, A_1 = 0, B_1 = 0, A_2 = 0, B_2 = -\frac{3wc^2}{2a^2\alpha}, b = \mp\frac{\left(c^2 - 4a^4w\beta\right)^{1/4}}{\sqrt{2}w^{1/4}\gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$u_{33,34}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{w}\tanh(\sqrt{w}\xi))^2},\ if\ w>0, \tag{3.22}$$

$$u_{35,36}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{w}\coth(\sqrt{w}\xi))^2},\ if\ w>0, \tag{3.23}$$

$$u_{37,38}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2},\ if\ w<0, \tag{3.24}$$

$$u_{39,40}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{-w}\coth(\sqrt{-w}\xi))^2},\ if\ w<0, \tag{3.25}$$

where,

$$\xi = ax + by - ct - x_0.$$

Case 6

$$A_0 = \frac{c^2}{2a^2\alpha}, A_1 = 0, B_1 = 0, A_2 = 0, B_2 = \frac{3wc^2}{2a^2\alpha}, b = \mp\frac{\left(-c^2 - 4a^4w\beta\right)^{1/4}}{\sqrt{2}w^{1/4}\gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$u_{41,42}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{w}\tanh(\sqrt{w}\xi))^2},\ if\ w>0, \tag{3.26}$$

$$u_{43,44}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{w}\coth(\sqrt{w}\xi))^2}, \ if \ w > 0, \tag{3.27}$$

$$u_{45,46}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{\frac{3wc^2}{2a^2w\alpha}}{(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2}, \ if \ w < 0, \tag{3.28}$$

$$u_{47,48}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{\frac{3wc^2}{2a^2w\alpha}}{(\sqrt{-w}\coth(\sqrt{-w}\xi))^2}, \ if \ w < 0, \tag{3.29}$$

where,

$$\xi = ax + by - ct - x_0.$$

Case 7

$$A_0 = \frac{c^2}{2a^2\alpha}, A_1 = 0, B_1 = 0, A_2 = 0, B_2 = \frac{3wc^2}{2a^2\alpha}, b = \mp\frac{i\left(-c^2 - 4a^4w\beta\right)^{1/4}}{\sqrt{2}w^{1/4}\gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$u_{49,50}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{w}\tanh(\sqrt{w}\xi))^2}, \ if \ w > 0, \tag{3.30}$$

$$u_{51,52}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{w}\coth(\sqrt{w}\xi))^2}, \ if \ w > 0, \tag{3.31}$$

$$u_{53,54}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2}, \ if \ w < 0, \tag{3.32}$$

$$u_{55,56}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{-w}\coth(\sqrt{-w}\xi))^2}, \ if \ w < 0, \tag{3.33}$$

where,

$$\xi = ax + by - ct - x_0.$$

Case 8

$$A_0 = \frac{-3c^2}{2a^2\alpha}, A_1 = 0, B_1 = 0, A_2 = 0, B_2 = \frac{-3wc^2}{2a^2\alpha}, b = \mp\frac{i\left(c^2 - 4a^4w\beta\right)^{1/4}}{\sqrt{2}w^{1/4}\gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$u_{57,58}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{w}\tanh(\sqrt{w}\xi))^2}, \ if \ w > 0, \tag{3.34}$$

$$u_{59,60}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{w}\coth(\sqrt{w}\xi))^2}, \ if \ w > 0, \tag{3.35}$$

$$u_{61,62}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2},\ if\ w<0, \tag{3.36}$$

$$u_{63,64}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{-w}\coth(\sqrt{-w}\xi))^2},\ if\ w<0, \tag{3.37}$$

where,

$$\xi = ax + by - ct - x_0.$$

Case 9

$$A_0 = -\frac{3c^2}{4a^2\alpha},\quad A_1 = 0,\quad B_1 = 0,\quad B_2 = -\frac{3c^2 w}{8a^2\alpha},\quad A_2 = -\frac{3c^2}{8a^2 w\alpha},$$

$$b = \mp\frac{\left(c^2 - 16a^4 w\beta\right)^{1/4}}{2w^{1/4}\gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$\begin{aligned} u_{65,66}(x,t) = &-\frac{3c^2}{4a^2\alpha} - \frac{3c^2}{8a^2 w\alpha}(\sqrt{w}\tanh(\sqrt{w}\xi))^2 \\ &-\frac{\frac{3c^2 w}{8a^2\alpha}}{(\sqrt{w}\tanh(\sqrt{w}\xi))^2},\ w>0, \end{aligned} \tag{3.38}$$

$$\begin{aligned} u_{67,68}(x,t) = &-\frac{3c^2}{4a^2\alpha} - \frac{3c^2}{8a^2 w\alpha}(\sqrt{w}\coth(\sqrt{w}\xi))^2 \\ &-\frac{\frac{3c^2 w}{8a^2\alpha}}{(\sqrt{w}\coth(\sqrt{w}\xi))^2},\ w>0, \end{aligned} \tag{3.39}$$

$$\begin{aligned} u_{69,70}(x,t) = &-\frac{3c^2}{4a^2\alpha} - \frac{3c^2}{8a^2 w\alpha}(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2 \\ &-\frac{\frac{3c^2 w}{8a^2\alpha}}{(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2},\ w<0, \end{aligned} \tag{3.40}$$

$$\begin{aligned} u_{71,72}(x,t) = &-\frac{3c^2}{4a^2\alpha} - \frac{3c^2}{8a^2 w\alpha}(\sqrt{-w}\coth(\sqrt{-w}\xi))^2 \\ &-\frac{\frac{3c^2 w}{8a^2\alpha}}{(\sqrt{-w}\coth(\sqrt{-w}\xi))^2},\ w<0, \end{aligned} \tag{3.41}$$

where,

$$\xi = ax + by - ct - x_0.$$

Case 10

$$A_0 = -\frac{c^2}{4a^2\alpha},\quad A_1 = 0,\quad B_1 = 0,\quad B_2 = \frac{3c^2 w}{8a^2\alpha},\quad A_2 = \frac{3c^2}{8a^2 w\alpha},$$

$$b = \mp \frac{\left(-c^2 - 16a^4 w\beta\right)^{1/4}}{2w^{1/4}\gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$\begin{aligned} u_{73,74}(x,t) = & -\frac{c^2}{4a^2\alpha} + \frac{3c^2}{8a^2 w\alpha}(\sqrt{w}\tanh(\sqrt{w}\xi))^2 \\ & + \frac{\frac{3c^2 w}{8a^2\alpha}}{(\sqrt{w}\tanh(\sqrt{w}\xi))^2}, \ w > 0, \end{aligned} \tag{3.42}$$

$$\begin{aligned} u_{75,76}(x,t) = & -\frac{c^2}{4a^2\alpha} + \frac{3c^2}{8a^2 w\alpha}(\sqrt{w}\coth(\sqrt{w}\xi))^2 \\ & + \frac{\frac{3c^2 w}{8a^2\alpha}}{(\sqrt{w}\coth(\sqrt{w}\xi))^2}, \ if \ w > 0, \end{aligned} \tag{3.43}$$

$$\begin{aligned} u_{77,78}(x,t) = & -\frac{c^2}{4a^2\alpha} + \frac{3c^2}{8a^2 w\alpha}(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2 \\ & + \frac{\frac{3c^2 w}{8a^2\alpha}}{(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2}, \ w < 0, \end{aligned} \tag{3.44}$$

$$\begin{aligned} u_{79,80}(x,t) = & -\frac{c^2}{4a^2\alpha} + \frac{3c^2}{8a^2 w\alpha}(\sqrt{-w}\coth(\sqrt{-w}\xi))^2 \\ & + \frac{\frac{3c^2 w}{8a^2\alpha}}{(\sqrt{-w}\coth(\sqrt{-w}\xi))^2}, \ w < 0, \end{aligned} \tag{3.45}$$

where,

$$\xi = ax + by - ct - x_0.$$

Case 11

$$A_0 = -\frac{3c^2}{4a^2\alpha}, \quad A_1 = 0, \quad B_1 = 0, \quad B_2 = -\frac{3c^2 w}{8a^2\alpha}, \quad A_2 = -\frac{3c^2}{8a^2 w\alpha},$$

$$b = \mp \frac{i\left(c^2 - 16a^4 w\beta\right)^{1/4}}{2w^{1/4}\gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$\begin{aligned} u_{81,82}(x,t) = & -\frac{3c^2}{4a^2\alpha} - \frac{3c^2}{8a^2 w\alpha}(\sqrt{w}\tanh(\sqrt{w}\xi))^2 \\ & - \frac{\frac{3c^2 w}{8a^2\alpha}}{(\sqrt{w}\tanh(\sqrt{w}\xi))^2}, \ w > 0, \end{aligned} \tag{3.46}$$

$$\begin{aligned} u_{83,84}(x,t) = & -\frac{3c^2}{4a^2\alpha} - \frac{3c^2}{8a^2 w\alpha}(\sqrt{w}\coth(\sqrt{w}\xi))^2 \\ & - \frac{\frac{3c^2 w}{8a^2\alpha}}{(\sqrt{w}\coth(\sqrt{w}\xi))^2}, \ w > 0, \end{aligned} \tag{3.47}$$

$$\begin{aligned} u_{85,86}(x,t) = &-\frac{3c^2}{4a^2\alpha} - \frac{3c^2}{8a^2w\alpha}(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2 \\ &-\frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2}, \ w<0, \end{aligned} \tag{3.48}$$

$$\begin{aligned} u_{87,88}(x,t) = &-\frac{3c^2}{4a^2\alpha} - \frac{3c^2}{8a^2w\alpha}(\sqrt{-w}\coth(\sqrt{-w}\xi))^2 \\ &-\frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{-w}\coth(\sqrt{-w}\xi))^2}, \ w<0, \end{aligned} \tag{3.49}$$

where,

$$\xi = ax + by - ct - x_0.$$

Case 12

$$A_0 = -\frac{c^2}{4a^2\alpha}, \quad A_1 = 0, \quad B_1 = 0, \quad B_2 = \frac{3c^2w}{8a^2\alpha}, \quad A_2 = \frac{3c^2}{8a^2w\alpha},$$

$$b = \mp\frac{i\left(-c^2 - 16a^4w\beta\right)^{1/4}}{2w^{1/4}\gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$\begin{aligned} u_{89,90}(x,t) = &-\frac{c^2}{4a^2\alpha} + \frac{3c^2}{8a^2w\alpha}(\sqrt{w}\tanh(\sqrt{w}\xi))^2 \\ &+\frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{w}\tanh(\sqrt{w}\xi))^2}, \ w>0, \end{aligned} \tag{3.50}$$

$$\begin{aligned} u_{91,92}(x,t) = &-\frac{c^2}{4a^2\alpha} + \frac{3c^2}{8a^2w\alpha}(\sqrt{w}\coth(\sqrt{w}\xi))^2 \\ &+\frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{w}\coth(\sqrt{w}\xi))^2}, \ if \ w>0, \end{aligned} \tag{3.51}$$

$$\begin{aligned} u_{93,94}(x,t) = &-\frac{c^2}{4a^2\alpha} + \frac{3c^2}{8a^2w\alpha}(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2 \\ &+\frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2}, \ w<0, \end{aligned} \tag{3.52}$$

$$\begin{aligned} u_{95,96}(x,t) = &-\frac{c^2}{4a^2\alpha} + \frac{3c^2}{8a^2w\alpha}(\sqrt{-w}\coth(\sqrt{-w}\xi))^2 \\ &+\frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{-w}\coth(\sqrt{-w}\xi))^2}, \ w<0, \end{aligned} \tag{3.53}$$

where,

$$\xi = ax + by - ct - x_0.$$

3.2. **The (3+1)-dimensional Benjamin-Ono equation.** In this subsection, we examine the (3+1)-dimensional Benjamin-Ono equation that reads

$$u_{tt}+\alpha(u^2)_{xx}+\beta u_{xxxx}+\gamma u_{yyyy}+\delta u_{zzzz}=0, \tag{3.54}$$

where $u(x,y,z,t)$ is a sufficiently smooth function and α,β,γ and δ are nonzero parameters. We make use of the following transformation solution

$$u(x,y,z,t)=f(\xi),\quad \xi=ax+by+dz-ct-x_0,$$

into (3.54) as in above. We also deduce the relations:

$$u_{tt}=c^2f'(\xi),\;\; u_x=af'(\xi),\;\; u_{xx}=a^2f''(\xi),$$
$$u_{xxxx}u=a^4f''''(\xi),\;\; u_{yyyy}=b^4f''''(\xi),\;\; u_{zzzz}=d^4f''''(\xi).$$

The above relations convert (3.54) into a nonlinear ordinary differential equation as follows

$$c^2f''+\alpha a^2(f^2)''+\beta a^4f''''+\gamma b^4f''''+\delta d^4f''''=0. \tag{3.55}$$

Integrating (3.55) once with respect to ξ, and setting the integration constant zero we obtain:

$$c^2f+\alpha a^2(f^2)+\beta a^4f''+\gamma b^4f''+\delta d^4f''=0. \tag{3.56}$$

Balancing f^2 and f'' in (3.56) gives $2N=N+2$, or so $N=2$. This offers a truncated series as the following form

$$f(\xi)=A_0+A_1\Phi(\xi)+A_2\Phi^2(\xi)+\frac{B_1}{\Phi(\xi)}+\frac{B_2}{\Phi(\xi)^2}. \tag{3.57}$$

Substituting (3.57) into (3.56) and equating the coefficient of each power of $\Phi(\xi)$ to zero we get the following system of algebraic equations

$$c^2A_0+a^2\alpha A_0^2+2a^4w^2\beta A_2+2b^4w^2\gamma A_2+2d^4w^2\delta A_2+2a^2\alpha A_1B_1+$$
$$2a^4\beta B_2+2b^4\gamma B_2+2d^4\delta B_2+2a^2\alpha A_2B_2=0,$$
$$6a^4w^2\beta B_2+6b^4w^2\gamma B_2+6d^4w^2\delta B_2+a^2\alpha B_2^2=0,$$
$$2a^4w^2\beta B_1+2b^4w^2\gamma B_1+2d^4w^2\delta B_1+2a^2\alpha B_1B_2=0,$$
$$a^2\alpha B_1^2+c^2B_2+8a^4w\beta B_2+8b^4w\gamma B_2+8d^4w\delta B_2+2a^2\alpha A_0B_2=0,$$
$$c^2B_1+2a^4w\beta B_1+2b^4w\gamma B_1+2d^4w\delta B_1+2a^2\alpha A_0B_1+2a^2\alpha A_1B_2=0,$$
$$c^2A_1+2a^4w\beta A_1+2b^4w\gamma A_1+2d^4w\delta A_1+2a^2\alpha A_0A_1+2a^2\alpha A_2B_1=0,$$
$$a^2\alpha A_1^2+c^2A_2+8a^4w\beta A_2+8b^4w\gamma A_2+8d^4w\delta A_2+2a^2\alpha A_0A_2=0,$$
$$2a^4\beta A_1+2b^4\gamma A_1+2d^4\delta A_1+2a^2\alpha A_1A_2=0,$$
$$6a^4\beta A_2+6b^4\gamma A_2+6d^4\delta A_2+a^2\alpha A_2^2=0.$$

Solving the above system, yields

Case 1

$$A_0=-\frac{3c^2}{2a^2\alpha},A_1=0,B_1=0,B_2=0,A_2=-\frac{3c^2}{2a^2w\alpha},$$

$$b = \mp \frac{\left(c^2 - 4a^4 w\beta - 4d^4 w\delta\right)^{1/4}}{\sqrt{2} w^{1/4} \gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$u_{1,2}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{3c^2}{2a^2 w\alpha}(\sqrt{w}\tanh(\sqrt{w}\xi))^2, \ if \ w > 0, \tag{3.58}$$

$$u_{3,4}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{3c^2}{2a^2 w\alpha}(\sqrt{w}\coth(\sqrt{w}\xi))^2, \ if \ w > 0, \tag{3.59}$$

$$u_{5,6}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{3c^2}{2a^2 w\alpha}(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2, \ if \ w < 0, \tag{3.60}$$

$$u_{7,8}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{3c^2}{2a^2 w\alpha}(\sqrt{-w}\coth(\sqrt{-w}\xi))^2, \ if \ w < 0, \tag{3.61}$$

where,

$$\xi = ax + by + dz - ct - x_0.$$

Case 2

$$A_0 = \frac{c^2}{2a^2\alpha}, A_1 = 0, B_1 = 0, B_2 = 0, A_2 = \frac{3c^2}{2a^2 w\alpha},$$

$$b = \mp \frac{\left(-c^2 - 4a^4 w\beta - 4d^4 w\delta\right)^{1/4}}{\sqrt{2} w^{1/4} \gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$u_{9,10}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{3c^2}{2a^2 w\alpha}(\sqrt{w}\tanh(\sqrt{w}\xi))^2, \ if \ w > 0, \tag{3.62}$$

$$u_{11,12}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{3c^2}{2a^2 w\alpha}(\sqrt{w}\coth(\sqrt{w}\xi))^2, \ if \ w > 0, \tag{3.63}$$

$$u_{13,14}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{3c^2}{2a^2 w\alpha}(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2, \ if \ w < 0, \tag{3.64}$$

$$u_{15,16}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{3c^2}{2a^2 w\alpha}(\sqrt{-w}\coth(\sqrt{-w}\xi))^2, \ if \ w < 0, \tag{3.65}$$

where,

$$\xi = ax + by + dz - ct - x_0.$$

Case 3

$$A_0 = \frac{c^2}{2a^2\alpha}, A_1 = 0, B_1 = 0, B_2 = 0, A_2 = \frac{3c^2}{2a^2 w\alpha},$$

$$b = \mp \frac{i\left(-c^2 - 4a^4 w\beta - 4d^4 w\delta\right)^{1/4}}{\sqrt{2}w^{1/4}\gamma^{1/4}}.$$

Hence, the solution is formed as:

$$u_{17,18}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{3c^2}{2a^2 w\alpha}(\sqrt{w}\tanh(\sqrt{w}\xi))^2, \ if \ w > 0, \tag{3.66}$$

$$u_{19,20}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{3c^2}{2a^2 w\alpha}(\sqrt{w}\coth(\sqrt{w}\xi))^2, \ if \ w > 0, \tag{3.67}$$

$$u_{21,22}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{3c^2}{2a^2 w\alpha}(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2, \ if \ w < 0, \tag{3.68}$$

$$u_{23,24}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{3c^2}{2a^2 w\alpha}(\sqrt{-w}\coth(\sqrt{-w}\xi))^2, \ if \ w < 0, \tag{3.69}$$

where,

$$\xi = ax + by + dz - ct - x_0.$$

Case 4

$$A_0 = \frac{-3c^2}{2a^2\alpha}, A_1 = 0, B_1 = 0, B_2 = 0, A_2 = \frac{-3c^2}{2a^2 w\alpha},$$

$$b = \mp \frac{i\left(c^2 - 4a^4 w\beta - 4d^4 w\delta\right)^{1/4}}{\sqrt{2}w^{1/4}\gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$u_{25,26}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{3c^2}{2a^2 w\alpha}(\sqrt{w}\tanh(\sqrt{w}\xi))^2, \ if \ w > 0, \tag{3.70}$$

$$u_{27,28}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{3c^2}{2a^2 w\alpha}(\sqrt{w}\coth(\sqrt{w}\xi))^2, \ if \ w > 0, \tag{3.71}$$

$$u_{29,30}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{3c^2}{2a^2 w\alpha}(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2, \ if \ w < 0, \tag{3.72}$$

$$u_{31,32}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{3c^2}{2a^2 w\alpha}(\sqrt{-w}\coth(\sqrt{-w}\xi))^2, \ if \ w < 0, \tag{3.73}$$

where,

$$\xi = ax + by + dz - ct - x_0.$$

Case 5

$$A_0 = -\frac{3c^2}{2a^2\alpha}, A_1 = 0, B_1 = 0, A_2 = 0, B_2 = -\frac{3wc^2}{2a^2\alpha},$$

$$b = \mp \frac{\left(c^2 - 4a^4 w\beta - 4d^4 w\delta\right)^{1/4}}{\sqrt{2} w^{1/4} \gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$u_{33,34}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{w}\tanh(\sqrt{w}\xi))^2},\ if\ w > 0, \tag{3.74}$$

$$u_{35,36}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{w}\coth(\sqrt{w}\xi))^2},\ if\ w > 0, \tag{3.75}$$

$$u_{37,38}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2},\ if\ w < 0, \tag{3.76}$$

$$u_{39,40}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{-w}\coth(\sqrt{-w}\xi))^2},\ if\ w < 0, \tag{3.77}$$

where,

$$\xi = ax + by + dz - ct - x_0.$$

Case 6

$$A_0 = \frac{c^2}{2a^2\alpha}, A_1 = 0, B_1 = 0, A_2 = 0, B_2 = \frac{3wc^2}{2a^2\alpha},$$

$$b = \mp \frac{\left(-c^2 - 4a^4 w\beta - 4d^4 w\delta\right)^{1/4}}{\sqrt{2} w^{1/4} \gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$u_{41,42}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{w}\tanh(\sqrt{w}\xi))^2},\ if\ w > 0, \tag{3.78}$$

$$u_{43,44}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{w}\coth(\sqrt{w}\xi))^2},\ if\ w > 0, \tag{3.79}$$

$$u_{45,46}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{\frac{3wc^2}{2a^2 w\alpha}}{(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2},\ if\ w < 0, \tag{3.80}$$

$$u_{47,48}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{\frac{3wc^2}{2a^2 w\alpha}}{(\sqrt{-w}\coth(\sqrt{-w}\xi))^2},\ if\ w < 0, \tag{3.81}$$

where,

$$\xi = ax + by + dz - ct - x_0.$$

Case 7

$$A_0 = \frac{c^2}{2a^2\alpha}, A_1 = 0, B_1 = 0, A_2 = 0, B_2 = \frac{3wc^2}{2a^2\alpha},$$

$$b = \mp\frac{i\left(-c^2 - 4a^4w\beta - 4d^4w\delta\right)^{1/4}}{\sqrt{2}w^{1/4}\gamma^{1/4}}.$$

Hence, the solution is formed as:

$$u_{49,50}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{w}\tanh(\sqrt{w}\xi))^2}, \ if \ w > 0, \tag{3.82}$$

$$u_{51,52}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{w}\coth(\sqrt{w}\xi))^2}, \ if \ w > 0, \tag{3.83}$$

$$u_{53,54}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2}, \ if \ w < 0, \tag{3.84}$$

$$u_{55,56}(x,t) = \frac{c^2}{2a^2\alpha} + \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{-w}\coth(\sqrt{-w}\xi))^2}, \ if \ w < 0, \tag{3.85}$$

where,

$$\xi = ax + by + dz - ct - x_0.$$

Case 8

$$A_0 = \frac{-3c^2}{2a^2\alpha}, A_1 = 0, B_1 = 0, A_2 = 0, B_2 = \frac{-3wc^2}{2a^2\alpha},$$

$$b = \mp\frac{i\left(c^2 - 4a^4w\beta - 4d^4w\delta\right)^{1/4}}{\sqrt{2}w^{1/4}\gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$u_{57,58}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{w}\tanh(\sqrt{w}\xi))^2}, \ if \ w > 0, \tag{3.86}$$

$$u_{59,60}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{w}\coth(\sqrt{w}\xi))^2}, \ if \ w > 0, \tag{3.87}$$

$$u_{61,62}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2}, \ if \ w < 0, \tag{3.88}$$

$$u_{63,64}(x,t) = -\frac{3c^2}{2a^2\alpha} - \frac{\frac{3wc^2}{2a^2\alpha}}{(\sqrt{-w}\coth(\sqrt{-w}\xi))^2}, \ if \ w < 0, \tag{3.89}$$

where,

$$\xi = ax + by + dz - ct - x_0.$$

Case 9

$$A_0 = -\frac{3c^2}{4a^2\alpha}, \quad A_1 = 0, \quad B_1 = 0, \quad B_2 = -\frac{3c^2 w}{8a^2\alpha}, \quad A_2 = -\frac{3c^2}{8a^2 w\alpha},$$

$$b = \mp\frac{\left(c^2 - 16a^4 w\beta - 16d^4 w\delta\right)^{1/4}}{2w^{1/4}\gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$u_{65,66}(x,t) = -\frac{3c^2}{4a^2\alpha} - \frac{3c^2}{8a^2 w\alpha}(\sqrt{w}\tanh(\sqrt{w}\xi))^2 - \frac{\frac{3c^2 w}{8a^2\alpha}}{(\sqrt{w}\tanh(\sqrt{w}\xi))^2}, \ w > 0, \tag{3.90}$$

$$u_{67,68}(x,t) = -\frac{3c^2}{4a^2\alpha} - \frac{3c^2}{8a^2 w\alpha}(\sqrt{w}\coth(\sqrt{w}\xi))^2 - \frac{\frac{3c^2 w}{8a^2\alpha}}{(\sqrt{w}\coth(\sqrt{w}\xi))^2}, \ w > 0, \tag{3.91}$$

$$u_{69,70}(x,t) = -\frac{3c^2}{4a^2\alpha} - \frac{3c^2}{8a^2 w\alpha}(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2 - \frac{\frac{3c^2 w}{8a^2\alpha}}{(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2}, \ w < 0, \tag{3.92}$$

$$u_{71,72}(x,t) = -\frac{3c^2}{4a^2\alpha} - \frac{3c^2}{8a^2 w\alpha}(\sqrt{-w}\coth(\sqrt{-w}\xi))^2 - \frac{\frac{3c^2 w}{8a^2\alpha}}{(\sqrt{-w}\coth(\sqrt{-w}\xi))^2}, \ w < 0, \tag{3.93}$$

where,

$$\xi = ax + by + dz - ct - x_0.$$

Case 10

$$A_0 = -\frac{c^2}{4a^2\alpha}, \quad A_1 = 0, \quad B_1 = 0, \quad B_2 = \frac{3c^2 w}{8a^2\alpha}, \quad A_2 = \frac{3c^2}{8a^2 w\alpha},$$

$$b = \mp\frac{\left(-c^2 - 16a^4 w\beta - 16d^4 w\delta\right)^{1/4}}{2w^{1/4}\gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$\begin{aligned} u_{73,74}(x,t) = & -\frac{c^2}{4a^2\alpha} + \frac{3c^2}{8a^2w\alpha}(\sqrt{w}\tanh(\sqrt{w}\xi))^2 \\ & + \frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{w}\tanh(\sqrt{w}\xi))^2}, \; w > 0, \end{aligned} \tag{3.94}$$

$$\begin{aligned} u_{75,76}(x,t) = & -\frac{c^2}{4a^2\alpha} + \frac{3c^2}{8a^2w\alpha}(\sqrt{w}\coth(\sqrt{w}\xi))^2 \\ & + \frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{w}\coth(\sqrt{w}\xi))^2}, \; if \; w > 0, \end{aligned} \tag{3.95}$$

$$\begin{aligned} u_{77,78}(x,t) = & -\frac{c^2}{4a^2\alpha} + \frac{3c^2}{8a^2w\alpha}(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2 \\ & + \frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2}, \; w < 0, \end{aligned} \tag{3.96}$$

$$\begin{aligned} u_{79,80}(x,t) = & -\frac{c^2}{4a^2\alpha} + \frac{3c^2}{8a^2w\alpha}(\sqrt{-w}\coth(\sqrt{-w}\xi))^2 \\ & + \frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{-w}\coth(\sqrt{-w}\xi))^2}, \; w < 0, \end{aligned} \tag{3.97}$$

where,

$$\xi = ax + by + dz - ct - x_0.$$

Case 11

$$A_0 = -\frac{3c^2}{4a^2\alpha}, \quad A_1 = 0, \quad B_1 = 0, \quad B_2 = -\frac{3c^2w}{8a^2\alpha}, \quad A_2 = -\frac{3c^2}{8a^2w\alpha},$$

$$b = \mp\frac{i\left(c^2 - 16a^4w\beta - 16d^4w\delta\right)^{1/4}}{2w^{1/4}\gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$\begin{aligned} u_{81,82}(x,t) = & -\frac{3c^2}{4a^2\alpha} - \frac{3c^2}{8a^2w\alpha}(\sqrt{w}\tanh(\sqrt{w}\xi))^2 \\ & - \frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{w}\tanh(\sqrt{w}\xi))^2}, \; w > 0, \end{aligned} \tag{3.98}$$

$$\begin{aligned} u_{83,84}(x,t) = & -\frac{3c^2}{4a^2\alpha} - \frac{3c^2}{8a^2w\alpha}(\sqrt{w}\coth(\sqrt{w}\xi))^2 \\ & - \frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{w}\coth(\sqrt{w}\xi))^2}, \; w > 0, \end{aligned} \tag{3.99}$$

$$u_{85,86}(x,t) = -\frac{3c^2}{4a^2\alpha} - \frac{3c^2}{8a^2w\alpha}(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2 - \frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2}, \ w<0, \tag{3.100}$$

$$u_{87,88}(x,t) = -\frac{3c^2}{4a^2\alpha} - \frac{3c^2}{8a^2w\alpha}(\sqrt{-w}\coth(\sqrt{-w}\xi))^2 - \frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{-w}\coth(\sqrt{-w}\xi))^2}, \ w<0, \tag{3.101}$$

where,

$$\xi = ax + by + dz - ct - x_0.$$

Case 12

$$A_0 = -\frac{c^2}{4a^2\alpha}, \quad A_1 = 0, \quad B_1 = 0, \quad B_2 = \frac{3c^2w}{8a^2\alpha}, \quad A_2 = \frac{3c^2}{8a^2w\alpha},$$

$$b = \mp\frac{i\left(-c^2 - 16a^4w\beta - 16d^4w\delta\right)^{1/4}}{2w^{1/4}\gamma^{1/4}}.$$

Hence, the following solutions are formed:

$$u_{89,90}(x,t) = -\frac{c^2}{4a^2\alpha} + \frac{3c^2}{8a^2w\alpha}(\sqrt{w}\tanh(\sqrt{w}\xi))^2 + \frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{w}\tanh(\sqrt{w}\xi))^2}, \ w>0, \tag{3.102}$$

$$u_{91,92}(x,t) = -\frac{c^2}{4a^2\alpha} + \frac{3c^2}{8a^2w\alpha}(\sqrt{w}\coth(\sqrt{w}\xi))^2 + \frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{w}\coth(\sqrt{w}\xi))^2}, \ if \ w>0, \tag{3.103}$$

$$u_{93,94}(x,t) = -\frac{c^2}{4a^2\alpha} + \frac{3c^2}{8a^2w\alpha}(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2 + \frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{-w}\tanh(\sqrt{-w}\xi))^2}, \ w<0, \tag{3.104}$$

$$u_{95,96}(x,t) = -\frac{c^2}{4a^2\alpha} + \frac{3c^2}{8a^2w\alpha}(\sqrt{-w}\coth(\sqrt{-w}\xi))^2 + \frac{\frac{3c^2w}{8a^2\alpha}}{(\sqrt{-w}\coth(\sqrt{-w}\xi))^2}, \ w<0, \tag{3.105}$$

where,

$$\xi = ax + by + dz - ct - x_0.$$

4. Some graphical illustrations

In this section, we give some graphical illustrations of the acquired solutions of our equations. The 2-dimensional and 3-dimensional plots of certain solutions are presented as follows:

Figure 1. Graphs of (3.15) for the (2+1)-dimensional Benjamin-Ono equation.

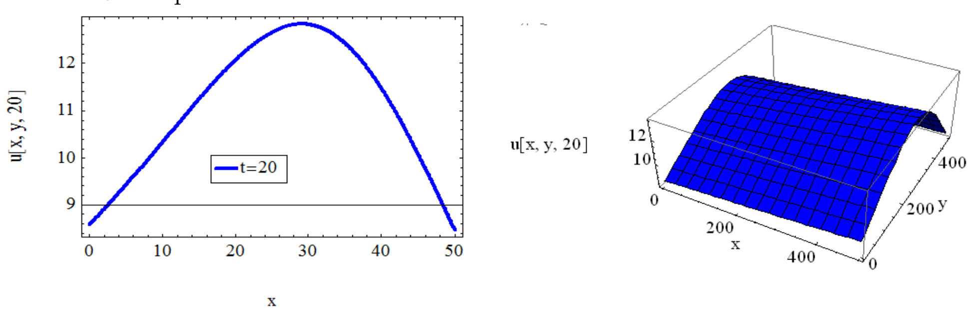
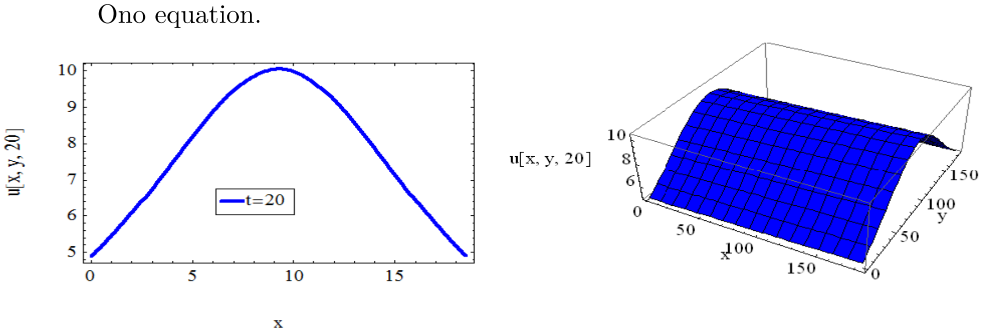


Figure 2. Graphs of (3.44) for the (2+1)-dimensional Benjamin-Ono equation.



5. Conclusion

The present paper analytically studies the newly developed (2+1)-dimensional Benjamin-Ono equation by Wazwaz and the new (3+1)-dimensional Benjamin-Ono equation proposed by the authors. The powerful modified extended tanh expansion method was successfully employed to acquire certain hyperbolic and periodic solitary wave structures with the help of the Mathematica software. This new equations will play vital role in fluid dynamics. We depicted the acquired solutions graphically to visualize the results.

Figure 3. Graph of (3.53) for the (2+1)-dimensional Benjamin-Ono equation.

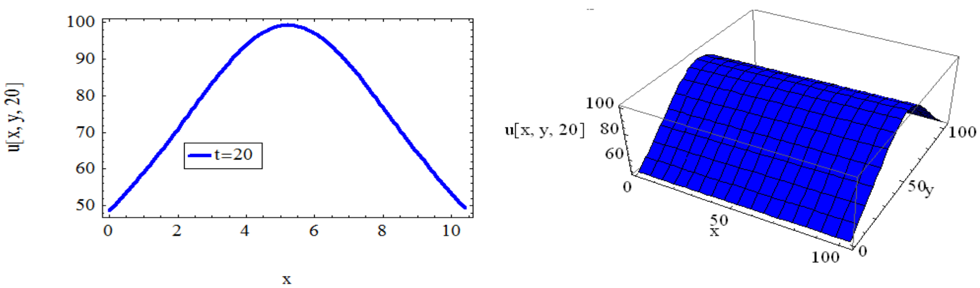


Figure 4. Graphs of (3.58) for the (3+1)-dimensional Benjamin-Ono equation.

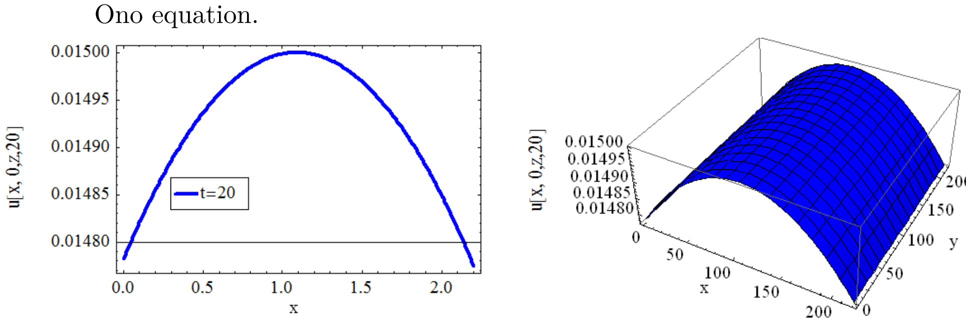
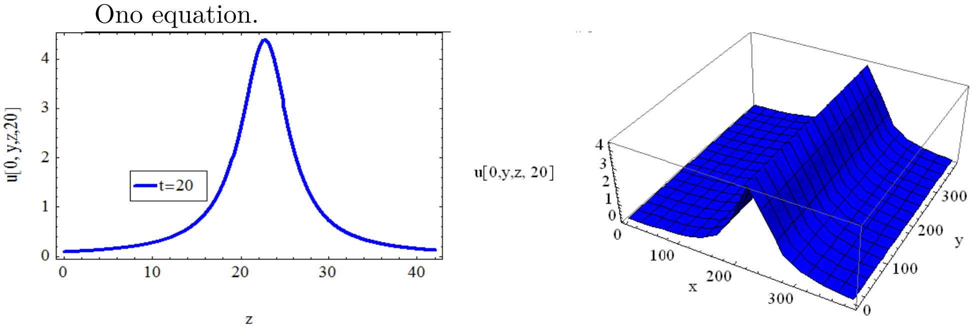


Figure 5. Graphs of (3.104) for the (3+1)-dimensional Benjamin-Ono equation.



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