



Some results on reflected forward-backward stochastic differential equations

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Abstract This paper is concerned with the reflected forward-backward stochastic differential equations with continuous monotone coefficients. Using the continuity approach, we prove that there exists at least one solution for the reflected forward-backward stochastic differential equations. The distinct character of our result is that the coefficient of the reflected forward SDEs contains the solution variable of the reflected BSDEs.

Keywords. Forward-backward stochastic differential equations, Increasing processes, Monotonicity condition.

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1. INTRODUCTION

In this paper, we study the existence of solutions for some reflected forward-backward stochastic differential equations.

In the early 1990s, the theory of backward and forward-backward stochastic differential equations (BSDEs and FBSDEs, for short, respectively) emerged as a major tool in the fields of mathematical finance and stochastic optimizations. In [6], it is shown that the value processes of the optimal stopping problem can be presented as solutions of reflected BSDEs. Solutions of classic BSDEs, in finance, can be considered as the recursive utility of an investor, which means that the decision of investors will be affected by his wealth.

The study of FBSDEs was started in the early 1990s. Since the discussion by Antonelli [1, 2] about the existence of local solution for FBSDEs, quite a few authors have contributed to the solvability of FBSDEs with a finite time horizon. Antonelli [1, 2] also constructed a counterexample showing that for coupled FBSDEs, large time duration might lead to non-solvability.

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Non-linear backward stochastic differential equations were first studied in [12], who proved the existence and uniqueness of the adapted strong solution, under smooth square integrability assumptions on the coefficient and the terminal condition, plus that the coefficient $h(t, \omega, y, z)$ is (t, ω) -uniformly Lipschitz in (y, z) . El Karoui et al. introduced the notion of reflected BSDE (RBSDE in short) [6], with one continuous lower barrier. Following this paper, Cvitanic and Karatzas [3], introduced the notion of reflected BSDE with two continuous barriers. Among the BSDEs, El Karoui et al. [6] introduced a special class of reflected BSDEs, which is a BSDE with a solution that is forced to stay above a lower barrier. Later, Michael Kohlmann [8] studied the relationships between adjoints of stochastic control problems with the derivative of the value function, and the latter with the value function of a stopping problem. These results were applied to the pricing of contingent claims.

Lepeltier and San Martin [9] relaxed the condition on the barriers, and then, utilizing a penalization method, proved the existence of a result without any assumption other than the square integrability one on the coefficients. Later, Lepeltier and Xu studied the case when the barriers are right continuous and left limit (RCLL in short), and proved the existence and uniqueness of a strong solution in both the Picard iteration and penalization method. Peng and Xu [13] considered the most general case when barriers are just L^2 -processes by using the penalization method and studied a special penalized BSDE, which is penalized with two barriers at the same time. They proved that the solutions of these equations converge to the solution of reflected BSDE. Peng et al. [14] developed a parallel method of reflected BSDEs on option pricing. This method is based on block allocation.

2. STATEMENT OF THE PROBLEM

Let (Ω, \mathcal{F}, P) be a probability space and $(W_t)_{0 \leq t \leq T}$ be a standard n -dimensional Brownian motion defined on this space, whose natural filtration is $\mathcal{F}_t = \sigma\{W_s, 0 \leq s \leq t\}$. We denote by \mathcal{P} the σ -field of \mathcal{F}_t -progressively measurable sets on $[0, T] \times \Omega$.

The aim of this paper is to study the required conditions for the existence of solutions to the following reflected forward-backward stochastic differential equation (RFBSDE):

$$\begin{aligned} X_t &= x + \int_0^t b(s, X_s, Y_s) ds + \int_0^t \sigma(s, X_s, Y_s) dW_s + \eta_t, \\ Y_t &= g(X_T) + \int_t^T h(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s + K_T - K_t, \\ Y_t &\geq L_t, \int_0^T (Y_t - L_t) dK_t = 0. \end{aligned} \tag{2.1}$$

First, we present some notations. Let

(1) \mathcal{S}^2 be the set of continuous, real-valued and adapted processes $\{X_t\}_{t \in [0, T]}$ such that:

$$\|X\|_{\mathcal{S}^2}^2 := \mathbb{E} \left[\sup_{t \in [0, T]} |X_t|^2 \right] < +\infty.$$



(2) \mathcal{H}^2 be the set of \mathcal{P} -measurable processes $\{Z_t\}_{t \in [0, T]}$ with values in \mathbb{R}^n such that:

$$\|Z\|_{\mathcal{H}^2}^2 := \mathbb{E} \left[\int_0^T |Z_t|^2 dt \right] < +\infty.$$

(3) \mathcal{S}_{ci}^2 be the subset of \mathcal{S}^2 of increasing processes.

For simplification, we suppose that $n = 1$ and only discuss the one-dimensional RFBSDE. Also, in this section, we introduce some assumptions on the coefficients of the RFBSDE (2.1).

Assumption I. $b : [0, T] \times \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $\sigma : [0, T] \times \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $h : [0, T] \times \Omega \times \mathbb{R}^3 \rightarrow \mathbb{R}$ are \mathcal{P} -measurable, continuous processes for any choice of the spatial variables and for each fixed (t, ω) , $b(t, \omega, \cdot, \cdot)$, $h(t, \omega, \cdot, \cdot, \cdot)$, $\sigma(t, \omega, \cdot, \cdot)$ are continuous functions. Moreover, we assume that for any $s \in [0, T], \omega \in \Omega, x, x', y, y', z \in \mathbb{R}$:

- (a) b is increasing in y and h is increasing in x ;
- (b) there exists a constant $M \geq 0$, such that

$$\begin{aligned} |b(s, x, y)| &\leq M(1 + |x| + |y|), \\ |b(s, x, y) - b(s, x', y')| &\leq M(|x - x'| + |y - y'|), \\ |\sigma(s, x, y)| &\leq M(1 + |x| + |y|), \\ |\sigma(s, x, y) - \sigma(s, x', y')| &\leq M(|x - x'| + |y - y'|) \\ |h(t, x, y, z)| &\leq M(1 + |y| + |z|). \end{aligned}$$

Assumption II. $g : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is a given \mathcal{F}_T -measurable continuous bounded increasing function satisfying $g(X_T) \in \mathbf{L}^2(\mathcal{F}_T)$ and L is a continuous obstacle which is \mathcal{P} -measurable, real valued, satisfying $\mathbb{E}[\sup_{0 \leq t \leq T} (L_t^+)^2] < \infty$ and $L_T \leq g(X_T)$ a.s.

Assumption III. If we denote $V([0, T], \mathbb{R}^n)$ to be the set of all \mathbb{R}^n -valued functions of bounded variation, then $\eta \in V_{\mathcal{F}}([0, T], \mathbb{R}^n)$, the set of all $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted processes η with paths in $V([0, T], \mathbb{R}^n)$.

Note that the coefficients b and σ of the RFSDE contain Y , which is the solution variable of RBSDE. Therefore the reflected FSDE and reflected BSDE in (2.1) are coupled together. The problem is that in what conditions there is at least one solution for the reflected FBSDE (2.1).

3. THE BASIC CONCLUSION

To prove the main result, we require the following lemma on the approximation of continuous functions by Lipschitz condition. This lemma was first used by Lepeltier and San Martin [10] in the study of the existence of a solution for BSDEs.

Lemma 3.1. *Let $h : \mathbb{R}^p \rightarrow \mathbb{R}$ be a continuous function with linear growth, that is, there exists a constant $M < \infty$ such that $\forall x \in \mathbb{R}^p, |h(x)| \leq M(1 + |x|)$. Then the sequence of functions*

$$h_n(x) = \inf_{y \in \mathbb{Q}^p} \{h(y) + n|x - y|\}$$



is well defined for $n \geq M$ and satisfies

- (i) linear growth: $\forall x \in \mathbb{R}^p, |h_n(x)| \leq M(1 + |x|)$;
- (ii) monotonicity in n : $\forall x \in \mathbb{R}^p, h_n(x) \leq h_{n+1}(x)$;
- (iii) Lipschitz condition: $\forall x, y \in \mathbb{R}^p, |h_n(x) - h_n(y)| \leq n|x - y|$;
- (iv) strong convergence: if $x_n \rightarrow x$, then $h_n(x_n) \rightarrow h(x)$.

Lemma 3.2. [11] Under the Assumptions (I)-(III), the following Reflected FSDE

$$X_t = x + \int_0^t b(s, X_s, Y_s)ds + \int_0^t \sigma(s, X_s, Y_s)dW_s + \eta_t,$$

has a unique strong solution.

Now, we give the main result of this paper.

Theorem 3.3. Under the Assumptions I, II and III, there exists at least one solution $(X, Y, Z, K, \eta) \in \mathcal{S}^2 \otimes \mathcal{S}^2 \otimes \mathcal{H}^2 \otimes \mathcal{S}_{ci}^2 \otimes \mathcal{S}_{ci}^2$ for equation (2.1).

Proof. To create a solution of (2.1), the main idea is the following replication:

$$\begin{aligned} X_t^n &= x + \int_0^t b(s, X_s^n, Y_s^n)ds + \int_0^t \sigma(s, X_s^n, Y_s^n)dW_s + \eta_t^n, \\ Y_t^n &= g(X_T) + \int_t^T h(s, X_s^{n-1}, Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dW_s + K_T^n - K_t^n, \\ Y_t^n &\geq L_t, \int_0^T (Y_t^n - L_t)dK_t^n = 0. \end{aligned} \tag{3.1}$$

We show that the limits of monotonic sequence (X^n, Y^n, Z^n) satisfies in equation (2.1). First, we construct the beginning point. Consider the following standard RBSDEs:

$$\begin{aligned} Y_t^0 &= g(X_T) - \int_t^T M(1 + |Y_s^0| + |Z_s^0|)ds - \int_t^T Z_s^0 dW_s + K_T^0 - K_t^0, \\ Y_t^0 &\geq L_t, \int_0^T (Y_t^0 - L_t)dK_t^0 = 0, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} U_t &= |g(X_T)| + \int_t^T M(1 + |U_s| + |V_s|)ds - \int_t^T V_s dW_s + B_T - B_t, \\ U_t &\geq L_t, \int_0^T (U_t - L_t)dB_t = 0. \end{aligned} \tag{3.3}$$

Because both of the generators are Lipschitz continuous, (according to Theorem 5.2 in El Karoui et al. [6]), each equation has a unique strong solution in $\mathcal{S}^2 \otimes \mathcal{H}^2 \otimes \mathcal{S}_{ci}^2$, denoted by (Y^0, Z^0, K^0) and (U, V, B) respectively. Furthermore, we know that $|Y_t^0| \leq U_t$. From the comparison theorem between K^0 and B (see for example [4]), we understand that $K_t^0 \geq B_t$ and then Ito's formula will result that there exists a constant C_1 (only depending on $M, T, \mathbb{E}|g(X_T)|^2$), s.t.

$$\|Y^0\|_{\mathcal{S}^2} + \|Z^0\|_{\mathcal{H}^2} + \|K^0\|_{\mathcal{S}_{ci}^2} \leq C_1, \text{ (resp. } \|U\|_{\mathcal{S}^2} + \|V\|_{\mathcal{H}^2} + \|B\|_{\mathcal{S}_{ci}^2} \leq C_1).$$



Next, we construct X^0 . Take the following reflected forward equation

$$X_t^0 = x + \int_0^t b(s, X_s^0, Y_s^0)ds + \int_0^t \sigma(s, X_s^0, Y_s^0)dW_s + \eta_t^0, \tag{3.4}$$

where Y^0 is the solution of (3.2).

Let $\{b_k(s, x, y)\}_{n \geq 0}$ be the sequence defined in Lemma (3.1), so we know (from Lemma(3.2)) that the following RFSDE has a unique solution, i.e.,

$$X_t^{0,k} = x + \int_0^t b(s, X_s^{0,k}, Y_s^0)ds + \int_0^t \sigma(s, X_s^{0,k}, Y_s^0)dW_s + \eta_t^{0,k}. \tag{3.5}$$

From the comparison theorem (for more information on the comparison theorem see Ikida and Watanabe [5], Chapter IV) we know that $X_t^{0,k} \leq X_t^{0,k+1} \leq S_t$, $\eta_t^{0,k} \leq \eta_t^{0,k+1} \leq A_t$, where $S \in \mathcal{S}^2$ and $A \in \mathcal{S}_{ci}^2$ are the unique solutions of the following RFSDE:

$$S_t = x + \int_0^t M(1 + |S_s| + |U_s|)ds + \int_0^t \sigma(s, S_s, U_s)dW_s + A_t. \tag{3.6}$$

Thus, there exists two lower semi-continuous processes X^0 and η^0 s.t. $\forall t \leq T$,

$$P - a.s. \quad \begin{aligned} X_t^{0,k} &\longrightarrow X_t^0 \leq S_t, \text{ and } \|X^{0,k} - X^0\|_{\mathcal{H}^2} \rightarrow 0, \\ \eta_t^{0,k} &\longrightarrow \eta_t^0 \leq A_t, \text{ and } \|\eta^{0,k} - \eta^0\|_{\mathcal{S}_{ci}^2} \rightarrow 0. \end{aligned}$$

Using the dominated convergence theorem and Lemma(3.1)-(iv) we get that

$$\begin{aligned} &\mathbb{E} \left[\int_0^T |b_k(s, X_s^{0,k}, Y_s^0) - b(s, X_s^0, Y_s^0)|^2 ds \right] \\ &\leq C \mathbb{E} \left[\int_0^T |X_s^{0,k} - X_s^0|^2 + \int_0^T |b_k(s, X_s^0, Y_s^0) - b(s, X_s^0, Y_s^0)|^2 ds \right] \rightarrow 0, \\ &\text{as } k \rightarrow +\infty. \end{aligned}$$

Again, since $|\sigma(s, X_s^{0,k}, Y_s^0) - \sigma(s, X_s^0, Y_s^0)| \leq M|X_s^{0,k} - X_s^0|$, then because of the Burkholder-Davis-Gundy inequality we have, when $k \rightarrow \infty$,

$$\begin{aligned} &\mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \int_0^t (\sigma(s, X_s^{0,k}, Y_s^0) - \sigma(s, X_s^0, Y_s^0))dW_s \right|^2 \right] \\ &\leq C \mathbb{E} \int_0^T |\sigma(s, X_s^{0,k}, Y_s^0) - \sigma(s, X_s^0, Y_s^0)|^2 ds \rightarrow 0. \end{aligned}$$

Now, by getting a limit in equation (3.5) and using the optional section theorem we can check that equation (3.4) is established for X^0 , i.e. X^0 is continuous. In addition, through the dominated convergence theorem and Dini's theorem, we have $\mathbb{E} [\sup_{0 \leq t \leq T} |X_t^{0,k} - X_t^0|^2] \rightarrow 0$, as $k \rightarrow \infty$.

Then, we construct (X^n, Y^n, Z^n) . First, we want to construct (X^1, Y^1, Z^1) . We can construct Y^1 using X^0 . In fact, if we denote $h^1(s, w, y, z) = h(s, X_s^0(w), y, z)$, then we can observe that $|h^1(s, w, y, z)| \leq M(1 + |y| + |z|)$. Again, if we define $h^1(s, w, y, z)$



the approximating sequence in Lemma(3.1), then we have a triple $(Y^{1,k}, Z^{1,k}, K^{1,k})$ satisfying the following equation

$$\begin{aligned} Y_t^{1,k} &= g(X_T) + \int_t^T h_k^1(s, \omega, Y_s^{1,k}, Z_s^{1,k})ds - \int_t^T Z_s^{1,k} dW_s + K_T^{1,k} - K_t^{1,k}, \\ Y_t^{1,k} &\geq L_t, \int_0^T (Y_t^{1,k} - L_t)dK_t^{1,k} = 0, \end{aligned} \tag{3.7}$$

and by using the comparison theorem we deduce

$$Y_t^0 \leq Y_t^{1,k} \leq Y_t^{1,k+1} \leq U_t, K_t^0 \geq K_t^{1,k} \geq K_t^{1,k+1} \geq B_t.$$

Since $\int_0^T (Y_t^{1,k} - L_t)dK_t^{1,k} = 0$, thus we see from Ito's formula that

$$\begin{aligned} &\mathbb{E} \left[|Y_t^{1,k}|^2 + \int_t^T |Z_s^{1,k}|^2 ds \right] \\ &= \mathbb{E} \left[|g(X_T)|^2 + 2 \int_t^T Y_s^{1,k} h_k^1(s, \omega, Y_s^{1,k}, Z_s^{1,k})ds + 2 \int_t^T L_s dK_s^{1,k} \right] \\ &\leq \mathbb{E} |g(X_T)|^2 + \alpha^2 \mathbb{E} \int_t^T |h_k^1(s, \omega, Y_s^{1,k}, Z_s^{1,k})|^2 ds + \frac{1}{\alpha^2} \mathbb{E} \int_t^T |Y_t^{1,k}|^2 ds \\ &\quad + 2 \mathbb{E} \int_t^T L_s dK_s^{1,k} \\ &\leq C_2 \left(1 + \mathbb{E} |g(X_T)|^2 + \mathbb{E} \int_t^T |Y_s^{1,k}|^2 ds \right) + \frac{1}{3} \mathbb{E} \int_t^T |Z_s^{1,k}|^2 ds \\ &\quad + \frac{1}{\beta} \mathbb{E} \left[\sup_{t \leq s \leq T} (L_s^+)^2 \right] + \beta \mathbb{E} [(K_T^{1,k} - K_t^{1,k})^2], \end{aligned}$$

the last inequality has obtained by taking $\alpha = \frac{1}{3M}$, and C_2 is a constant depending only on M and T .

However, because

$$K_T^{1,k} - K_t^{1,k} = Y_t^{1,k} - g(X_T) - \int_t^T h_k^1(s, Y_s^{1,k}, Z_s^{1,k})ds + \int_t^T Z_s^{1,k} dW_s,$$

then

$$\mathbb{E}[(K_T^{1,k} - K_t^{1,k})^2] \leq C_3 [1 + \mathbb{E}g(X_T)^2 + \mathbb{E}(Y_t^{1,k})^2 + \mathbb{E} \int_t^T (|Y_s^{1,k}|^2 + |Z_s^{1,k}|^2)ds].$$

Selecting β suitably, we can obtain from Gronwall's inequality that

$$\mathbb{E}|Y_t^{1,k}|^2 \leq C, \mathbb{E} \int_0^T |Z_s^{1,k}|^2 ds \leq C, \mathbb{E}|K_T^{1,k}|^2 \leq C.$$

Because the sequence $\{Y^{1,k}\}_{k \geq 1}$ is increasing, we can mark the limit by Y^1 . In addition, by using Ito's formula and Burkholder-Davis-Gundy inequality, we get that $\mathbb{E}\{\sup_{0 \leq t \leq T} |Y_t^{1,k}|^2\} \leq C$. Hence, Fatou's lemma indicates that $\mathbb{E}\{\sup_{0 \leq t \leq T} |Y_t^1|^2\} \leq C$ and from the dominated convergence theorem we see that

$$\mathbb{E} \int_0^T |Y_t^1 - Y_t^{1,k}|^2 dt \rightarrow 0. \tag{3.8}$$



Using Ito's formula to $|Y_t^{1,j} - Y_t^{1,k}|^2$ and taking expectation, we have

$$\begin{aligned}
& \mathbb{E}|Y_0^{1,j} - Y_0^{1,k}|^2 + \mathbb{E} \int_0^T |Z_t^{1,j} - Z_t^{1,k}|^2 dt \\
&= 2\mathbb{E} \int_0^T (Y_t^{1,j} - Y_t^{1,k}) [h_j^1(t, \omega, Y_t^{1,j}, Z_t^{1,j}) - h_k^1(t, \omega, Y_t^{1,k}, Z_t^{1,k})] dt \\
&+ 2\mathbb{E} \int_0^T (Y_t^{1,j} - Y_t^{1,k}) dK_t^{1,j} + 2\mathbb{E} \int_0^T (Y_t^{1,k} - Y_t^{1,j}) dK_t^{1,k} \\
&\leq 2\mathbb{E} \int_0^T (Y_t^{1,j} - Y_t^{1,k}) [h_j^1(t, \omega, Y_t^{1,j}, Z_t^{1,j}) - h_k^1(t, \omega, Y_t^{1,k}, Z_t^{1,k})] dt \\
&+ 2\mathbb{E} \int_0^T (Y_t^{1,j} - L_t) dK_t^{1,j} + 2\mathbb{E} \int_0^T (Y_t^{1,k} - L_t) dK_t^{1,k} \\
&\leq 2 \left(\mathbb{E} \int_0^T |Y_t^{1,j} - Y_t^{1,k}|^2 dt \right)^{1/2} \\
&\quad \cdot \left(\mathbb{E} \int_0^T |h_j^1(t, \omega, Y_t^{1,j}, Z_t^{1,j}) - h_k^1(t, \omega, Y_t^{1,k}, Z_t^{1,k})|^2 dt \right)^{1/2} \\
&\leq C \left(\mathbb{E} \int_0^T |Y_t^{1,j} - Y_t^{1,k}|^2 dt \right)^{1/2}.
\end{aligned}$$

Therefore $\{Z_t^{1,k}\}_{k \geq 1}$ is a Cauchy sequence in \mathcal{H}^2 and we know that

$$\mathbb{E} \int_0^T (|Y_t^{1,j} - Y_t^{1,k}|^2 + |Z_t^{1,j} - Z_t^{1,k}|^2) dt \rightarrow 0, \text{ as } j, k \rightarrow \infty,$$

and we can conclude from Ito's formula that

$$\begin{aligned}
& |Y_t^{1,j} - Y_t^{1,k}|^2 + \int_t^T |Z_s^{1,j} - Z_s^{1,k}|^2 ds \\
&= 2 \int_t^T (Y_s^{1,j} - Y_s^{1,k}) [h_j^1(s, \omega, Y_s^{1,j}, Z_s^{1,j}) - h_k^1(s, \omega, Y_s^{1,k}, Z_s^{1,k})] ds \\
&+ 2 \int_t^T (Y_s^{1,j} - Y_s^{1,k}) (dK_s^{1,j} - dK_s^{1,k}) \\
&- 2 \int_t^T (Y_s^{1,j} - Y_s^{1,k}) (Z_s^{1,j} - Z_s^{1,k}) dW_s.
\end{aligned}$$

Just as we have already demonstrated, $\forall j \geq k$, $\int_t^T (Y_s^{1,j} - Y_s^{1,k}) (dK_s^{1,j} - dK_s^{1,k}) \leq 0$, then

$$\begin{aligned}
|Y_t^{1,j} - Y_t^{1,k}|^2 &\leq 2 \int_t^T (Y_s^{1,j} - Y_s^{1,k}) [h_j^1(s, \omega, Y_s^{1,j}, Z_s^{1,j}) - h_k^1(s, \omega, Y_s^{1,k}, Z_s^{1,k})] ds \\
&- 2 \int_t^T (Y_s^{1,j} - Y_s^{1,k}) (Z_s^{1,j} - Z_s^{1,k}) dW_s.
\end{aligned}$$



Thus, we conclude that

$$\begin{aligned} \mathbb{E}(\sup_{0 \leq t \leq T} |Y_t^{1,j} - Y_t^{1,k}|^2) &\leq 2\left(\mathbb{E} \int_0^T |Y_t^{1,j} - Y_t^{1,k}|^2 dt\right)^{1/2} \\ &\quad \cdot \left(\mathbb{E} \int_0^T |h_j^1(s, \omega, Y_s^{1,j}, Z_s^{1,j}) - h_k^1(s, \omega, Y_s^{1,k}, Z_s^{1,k})|^2 ds\right)^{\frac{1}{2}} \\ &\quad + 2\mathbb{E}\left(\sup_{0 \leq t \leq T} \left| \int_0^t (Y_s^{1,j} - Y_s^{1,k})(Z_s^{1,j} - Z_s^{1,k}) dW_s \right|\right) \\ &\leq C\left(\mathbb{E} \int_0^T |Y_t^{1,j} - Y_t^{1,k}|^2 dt\right)^{1/2} \\ &\quad + 2C'\mathbb{E}\left[\sup_{0 \leq t \leq T} |Y_t^{1,j} - Y_t^{1,k}| \left(\int_0^T |Z_s^{1,j} - Z_s^{1,k}|^2 ds\right)^{\frac{1}{2}}\right] \\ &\leq C\left(\mathbb{E} \int_0^T |Y_t^{1,j} - Y_t^{1,k}|^2 dt\right)^{1/2} + \frac{1}{2}\mathbb{E}(\sup_{0 \leq t \leq T} |Y_t^{1,j} - Y_t^{1,k}|^2) \\ &\quad + C'\mathbb{E} \int_0^T |Z_s^{1,j} - Z_s^{1,k}|^2 ds. \end{aligned}$$

Then,

$$\mathbb{E}(\sup_{0 \leq t \leq T} |Y_t^{1,j} - Y_t^{1,k}|^2) \rightarrow 0, \quad \mathbb{E}(\sup_{0 \leq t \leq T} |K_t^{1,j} - K_t^{1,k}|^2) \rightarrow 0, \quad \text{as } j, k \rightarrow \infty. \quad (3.9)$$

Therefore, there exists a progressively measurable process K^1 such that

$$\mathbb{E}(\sup_{0 \leq t \leq T} |K_t^{1,j} - K_t^1|^2) \rightarrow 0, \quad \text{as } j \rightarrow \infty.$$

Therefore, the process $\{K_t^1, 0 \leq t \leq T\}$ is increasing and continuous with $K_0^1 = 0$. Then again, we have deduced that $K_T^{1,k} \rightarrow K_T^1$ in the sense that $E[\sup_{0 \leq t \leq T} |K_t^{1,k} - K_t^1|^2] \rightarrow 0, \text{ as } k \rightarrow \infty$. Let $K_t^{1,0} = K_t^0, 0 \leq t \leq T$, and the associated Radon measure on $[0, T]$ noted by μ^0 . Next, define $\tilde{K}^{1,k}$ by

$$\tilde{K}_t^{1,k} = \begin{cases} 0, & t < 0, \\ K_t^{1,k}, & 0 \leq t \leq T, \\ K_T^{1,k}, & t > T, \end{cases}$$

and μ_k^1 (resp. μ^1) the density function of $\tilde{K}^{1,k}$ (resp. \tilde{K}^1) such that

$$\mu_k^1([0, T]) = K_T^{1,k}, \quad (\text{resp. } \mu^1([0, T]) = K_T^1).$$

For almost all ω , we have

$$K_T^{1,k} \rightarrow K_T^1 \Leftrightarrow \mu_k^1([0, T]) \rightarrow \mu^1([0, T]).$$

From the assumption of $K^{1,k}$ we realize that μ_k^1 is bounded and $\sup_k \mu_k^1([0, T]) \leq \mu^0([0, T])$. Then there exists a subsequence, still denoted by $\{\mu_k^1\}$, such that $\mu_k^1 \rightarrow \mu^1$ in the weak sense, i.e.,

$$\forall h \in C_b(\mathbb{R}), \quad \mu_k^1(h) \rightarrow \mu^1(h).$$



Suppose that $\tilde{h} \in C_b(\mathbb{R})$ and $\tilde{h} = Y_t^1 - L_t$ on $[0, T]$, then

$$\int_0^T (Y_t^1 - L_t) dK_t^{1,k} = \int_{\mathbb{R}} \tilde{h}(t) d\mu_k^1(t) \rightarrow \int_{\mathbb{R}} \tilde{h}(t) d\mu^1(t) = \int_0^T (Y_t^1 - L_t) dK_t^1.$$

Eventually,

$$\begin{aligned} 0 &\leq \int_0^T (Y_t^1 - L_t) dK_t^1 \\ &= \int_0^T (Y_t^1 - L_t) d(K_t^1 - K_t^{1,k}) + \int_0^T (Y_t^1 - Y_t^{1,k}) dK_t^{1,k} + \int_0^T (Y_t^{1,k} - L_t) dK_t^{1,k} \\ &= \int_0^T (Y_t^1 - L_t) d(K_t^1 - K_t^{1,k}) + \int_0^T (Y_t^1 - Y_t^{1,k}) dK_t^{1,k}. \end{aligned}$$

Because $|\int_0^T (Y_t^1 - Y_t^{1,k}) dK_t^{1,k}| \leq \sup_t |Y_t^1 - Y_t^{1,k}| \cdot K_T^{1,k} \rightarrow 0$, so the right hand side of the above inequality vanishes, i.e.

$$\int_0^T (Y_t^1 - L_t) dK_t^1 = 0.$$

Thus we see that the following equation holds for the triple $\{(Y_t^1, Z_t^1, K_t^1), 0 \leq t \leq T\}$:

$$\begin{aligned} Y_t^1 &= g(X_T) + \int_t^T h^1(s, \omega, Y_s^1, Z_s^1) ds - \int_t^T Z_s^1 dW_s + K_T^1 - K_t^1, \\ Y_t^1 &\geq L_t, \int_0^T (Y_t^1 - L_t) dK_t^1 = 0. \end{aligned} \tag{3.10}$$

Next, similar to the way that we use on X^0 , we can form X^1 based on Y^1 . In addition, since b and σ are monotonic on y and $Y^0 \leq Y^1$, we conclude from the comparison on SDEs that $X^0 \leq X^1$.

Now, we can obtain the existence of a sequence $(X^n, Y^n, Z^n, K^n, \eta^n)$ which is a solution of (3.1) and P-a.s. for any $t \leq T$,

$$X_t^n \leq X_t^{n+1} \leq S_t, Y_t^n \leq Y_t^{n+1} \leq U_t, K_t^n \geq K_t^{n+1} \geq B_t, \eta_t^n \leq \eta_t^{n+1} \leq A_t,$$

Note that in the above iterations, we set $h^n(s, \omega, y, z) = h(s, X_s^{n-1}(\omega), y, z)$.

Now, we study the convergence of $(X^n, Y^n, Z^n, K^n, \eta^n)$. Clearly, there exist two lower semi-continuous processes $(X_t)_{t \leq T}$, $(Y_t)_{t \leq T}$ and two continuous increasing processes $(K_t)_{t \leq T}$, $(\eta_t)_{t \leq T}$ such that

$$X_t = \lim_{n \rightarrow \infty} X_t^n, Y_t = \lim_{n \rightarrow \infty} Y_t^n, K_t = \lim_{n \rightarrow \infty} K_t^n, \eta_t = \lim_{n \rightarrow \infty} \eta_t^n.$$

Thus, we can easily get that

$$X_t = x + \int_0^t b(s, X_s, Y_s) ds + \int_0^t \sigma(s, X_s, Y_s) dW_s + \eta_t. \tag{3.11}$$

We use the following formula to complete the proof

$$\begin{aligned} Y_t^n &= g(X_T) + \int_t^T h(s, X_s^{n-1}, Y_s^n, Z_s^n) ds - \int_t^T Z_s^n dW_s + K_T^n - K_t^n, \\ Y_t^n &\geq L_t, \int_0^T (Y_t^n - L_t) dK_t^n = 0. \end{aligned} \tag{3.12}$$



Again, we apply the Ito formula and conclude the following (for $n \geq m$):

$$\begin{aligned} (Y_t^n - Y_t^m)^2 + \int_t^T |Z_t^n - Z_t^m|^2 ds &= 2 \int_t^T (Y_s^n - Y_s^m)[h(s, X_s^{n-1}, Y_s^n, Z_s^n) \\ &\quad - h(s, X_s^{m-1}, Y_s^m, Z_s^m)] ds \\ &\quad + 2 \int_t^T (Y_s^n - Y_s^m) d(K_s^n - K_s^m) \\ &\quad - 2 \int_t^T (Y_s^n - Y_s^m)(Z_s^n - Z_s^m) dW_s, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \int_t^T (Y_s^n - Y_s^m) d(K_s^n - K_s^m) &= \int_t^T (Y_s^n - Y_s^m) dK_s^n + \int_t^T (Y_s^m - Y_s^n) dK_s^m \\ &\leq \int_t^T (Y_s^n - L_s) dK_s^n + \int_t^T (Y_s^m - L_s) dK_s^m \\ &= 0, \end{aligned}$$

so,

$$\begin{aligned} \mathbb{E}(Y_0^n - Y_0^m)^2 + \mathbb{E} \int_0^T |Z_t^n - Z_t^m|^2 ds &\leq 2 \mathbb{E} \int_0^T (Y_s^n - Y_s^m)[h(s, X_s^{n-1}, Y_s^n, Z_s^n) - h(s, X_s^{m-1}, Y_s^m, Z_s^m)] ds \\ &\leq C_1 \left[\mathbb{E} \int_0^T (Y_s^n - Y_s^m)^2 ds \right]^{\frac{1}{2}} \\ &\quad \left[\mathbb{E} \int_0^T (h(s, X_s^{n-1}(\omega), Y_s^n, Z_s^n) - h(s, X_s^{m-1}(\omega), Y_s^m, Z_s^m))^2 ds \right]^{\frac{1}{2}} \end{aligned}$$

As $\{X^n\}$, $\{Y^n\}$ are bounded by the processes S and U respectively, and $\|U\|_{\mathcal{S}^2} + \|S\|_{\mathcal{S}^2} \leq C_1$, then $\mathbb{E} \int_0^T (h(s, X_s^{n-1}(\omega), y, z) - h(s, X_s^{m-1}(\omega), y, z))^2 ds$ is bounded, too. Thus

$$\mathbb{E} \int_0^T |Z_t^n - Z_t^m|^2 ds \leq C_2 \left[\mathbb{E} \int_0^T (Y_s^n - Y_s^m)^2 ds \right]^{\frac{1}{2}} \rightarrow 0,$$

i.e. the sequence $\{Z^n\}$ is Cauchy with $Z_t = \lim_{n \rightarrow \infty} Z_t^n$. We can demonstrate the following using (3.13) and Burkholder-Davis-Gundy inequality

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |Y_t^n - Y_t^m|^2 \right) \rightarrow 0.$$

Now, using (3.12) we have

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq t \leq T} |K_t^n - K_t^m|^2 \right) &\rightarrow 0 \\ \int_0^T (Y_t^n - L_t) dK_t^n &\rightarrow \int_0^T (Y_t - L_t) dK_t, \end{aligned} \tag{3.14}$$

therefore

$$\int_0^T (Y_t - L_t) dK_t = 0.$$



Thus, we derive that (X, Y, Z, K, η) is a solution of (2.1). This result will finish the proof. ■

4. EXAMPLES

In this section, some examples are provided to show the effectiveness of the proposed Theorem (3.3).

Example 4.1. *As mentioned previously, RFBSDEs have applications in financial marketing. Now consider the following example, which is called optimal stopping problem (American option). In an American call option, the wealth process Y_t holds in the following RBSDE,*

$$\begin{aligned} X_t &= X_0 + \int_0^t \mu X_s ds + \int_0^t \sigma X_s dW_s, \quad 0 \leq t \leq 1, \\ Y_t &= (X_T - K)^+ - \int_t^T [rY_s + (\mu - r)Z_s] ds - \int_t^T \sigma Z_s dW_s, \end{aligned}$$

and $Y_t \geq (X_t - K)^+, 0 \leq t \leq \inf\{t, Y_t - (X_t - K)^+\}$. Here σ is the volatility rate, r is uniformly bounded and K is a constant.

The RBSDE apply to American put option in the following case:

$$Y_T = \xi = (K - X_T)^+.$$

For option pricing with differential interest rates, $(\mu - r)$ is related to Y_t and Z_t in this Equation.

We assume that $X = \{X_t, 0 \leq t \leq T\}$ is the risk asset and r is constant. Under some assumptions, the equation is given through a reflected BSDE, together with the forward equation of X . According to the Theorem (3.3), this equation has a solution, i.e. Y_0 , which is the option value. ■

Example 4.2. *Consider the following reflected forward-backward stochastic differential equation:*

$$\begin{cases} dX = \frac{X(1+X^2)}{(2+X^2)^3} dt + \frac{1+X^2}{2+X^2} \sqrt{\frac{1+2Y^2}{1+Y^2+\exp(-\frac{2X^2}{t+1})}} dW(t) + \exp(t), \\ dY = -g(t, X, Y)dt - f(t, X, Y)Zdt + ZdW(t) + t, \\ X(0) = x, \quad Y(T) = \exp(-\frac{X^2(T)}{T+1}). \end{cases}$$

where $g(t, x, u) = \frac{1}{1+t} \exp(-\frac{x^2}{t+1}) \left[\frac{4x^2(1+x^2)}{(2+x^2)^3} + \left(\frac{1+x^2}{2+x^2} \right)^2 \left(1 - \frac{2x^2}{t+1} - \frac{x^2}{t+1} \right) \right]$ and $f(t, x, u) = \frac{x}{(2+x^2)^2} \sqrt{\frac{1+u^2+\exp(-\frac{2x^2}{t+1})}{1+2u^2}}$. As one can easily see, all of the requirements in the Assumptions I-III are satisfied. Thus, according to Theorem (3.3), this RFBSDE has a



solution. ■

Remark 4.3. As we stated in the assumptions, h has sub-linear growth independent of x . If we assume that

$$|h(t, x, y, z)| \leq M(1 + |x| + |y| + |z|), \quad \forall s \in [0, T], \quad x, y, z \in \mathbb{R}$$

then b must have a sub-linear growth independent of y , i.e.

$$|b(s, x, y)| \leq M(1 + |x|).$$

Remark 4.4. Stochastic integrals in applications are often taken in the sense of Stratanovich calculus. This calculus is designed in such a way that its basic rules, such as the chain rule and integration by parts are the same as in the standard calculus (e.g. Rogers and Williams [15]) and integrals in the Stratanovich definition are easier to manipulate. In general, the Ito integral is the usual choice in applied mathematics while the Stratanovich integral is frequently used in physics.

Suppose that $X(t)$ satisfies the following SDE in the Stratanovich sense

$$dX(t) = \mu(X(t))dt + \sigma(X(t))\partial W(t),$$

with $\sigma(x)$ twice continuously differentiable. Then $X(t)$ satisfies the Ito SDE

$$dX(t) = \left(\mu(X(t)) + \frac{1}{2}\sigma'(X(t))\sigma(X(t)) \right) dt + \sigma(X(t))dW(t).$$

Thus the infinitesimal drift coefficient in Ito diffusion is $\mu(x) + \frac{1}{2}\sigma'(x)\sigma(x)$ and the diffusion coefficient is the same $\sigma(x)$ (see [7] for details).

5. CONCLUSION

In this paper, we used Ito's formula to prove the main theorem. But according to the above statements, if we add the twice continuously differentiability condition of σ to the assumptions, a proof using Stratanovich calculus could be also provided.

Also, the proof is true if the Brownian motion $W(t)$ is replaced with any Ito process.

FUTURE WORKS

Due to the randomness of Brownian motion, the existence of a unique solution for stochastic differential equations is not discussed. In these types of equations, the existence of a unique weak solution or a unique strong solution is studied. Future research can be dedicated to reviewing the conditions and requirements under which the existence of a unique strong solution to the equation (2.1) can be ensured. Also, the application of RFBSDEs in other disciplines such as physics and financial mathematics could be studied in the future researches.



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