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Fractional Ostrovsky equation

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Abstract In this paper, we find an integral representation for the fundamental solution of the fractional Ostrovsky equation in terms of the Airy and Bessel-Wright functions. The equation is studied in the sense of the Weyl fractional derivative and the solution is presented as the Airy transforms of Wright functions. Using the asymptotic expansion of Wright function the asymptotic behavior of solution is also discussed.

Keywords. Fractional Ostrovsky equation, Wright function, Airy function, Asymptotic behavior.

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1. Introduction

The Ostrovsky equation as the generalization of Korteweg-de Vries equation (KdV) governs the propagation of weakly nonlinear long surface and internal waves of small amplitude in a incompressible and inviscid rotational fluid. This equation which is nonintegrable by the inverse scattering transform can be presented by the following PDE [14]

$$\left(u_t - \beta u_{xxx} + (u^2)_x\right)_x = \gamma u, \quad \gamma > 0,\ x \in \mathbb{R},\ t > 0,$$

where parameter β shows the type of dispersion. In the case $\beta = -1$ (negative dispersion), the Ostrovsky equation can be considered for the surface and internal waves in the ocean and surface waves in a shallow channel with an uneven bottom. For $\beta = 1$ (positive dispersion), the Ostrovsky equation can be considered for the capillary waves on the surface of liquid or for the oblique magneto-acoustic waves in plasma. The parameter $\gamma > 0$ is also considered as the effect of rotation or Coriolis effect. For more details see [3, 4, 6, 7, 9].

Valramov in the year 2005, found the fundamental solution of Ostrovsky equation in terms of the Airy and Bessel functions and use this solution for his next works on the Riesz fractional derivatives of Airy functions and the conservation laws for KdV

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equations [16, 17, 18, 19, 20, 21]. In this paper, we study a fractionalization of the Ostrovsky equation as

$$W_+^{\alpha}\big[u_t - \beta u_{xxx} + (u^2)_x\big] = \gamma u, \quad \gamma > 0, \quad 0 < \alpha \leq 1, \tag{1.1}$$

where W_+^{α} is the Weyl fractional derivative of order α. We intend to find the fundamental solution of *fractional Ostrovsky equation* (1.1) in terms of the Airy and Bessel-Wright functions and study the asymptotic behavior of solution. First, we rewrite the relation (1.1) as

$$u_t - \beta u_{xxx} + (u^2)_x = \gamma W_-^{\alpha} u, \quad 0 < \alpha \leq 1, \tag{1.2}$$

where W_-^{α} is the Weyl fractional integral of order α.

2. Preliminaries

2.1. **The Weyl fractional derivative.** In this section, we recall some preliminaries about the Weyl fractional integrals and derivatives and then the Wright functions.

Definition 2.1. For $n - 1 < \Re(\alpha) \leq n, n \in \mathbb{N}$, the Weyl fractional integral and derivative of order α are defined as [5, 3]

$$W_+^{-\alpha} f(x) := \frac{d^{-\alpha}}{dx^{-\alpha}} f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-\xi)^{\alpha-1} f(\xi) d\xi,$$

$$W_+^{\alpha} f(x) := \frac{d^{\alpha}}{dx^{\alpha}} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^{x} (x-\xi)^{n-\alpha-1} f(\xi) d\xi.$$

Lemma 2.2. *In view of the Fourier transform of the function* $f(x)$

$$F(\xi) = \mathcal{F}\{f(x);\xi\} = \int_{-\infty}^{\infty} e^{-i\xi x} f(x) dx,$$

and its inversion formula

$$f(x) = \mathcal{F}^{-1}\{F(\xi);x\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} F(\xi) d\xi,$$

the Fourier transform of the Weyl fractional integral is given by [5]

$$\mathcal{F}\{W_+^{-\alpha} f(x);\xi\} = \frac{e^{-\frac{i\pi\alpha}{2}}}{\xi^{\alpha}} \mathcal{F}\{f(x);\xi\}. \tag{2.1}$$

2.2. **The Wright function.** The Wright function was introduced by the British mathematician Edward Maitland Wright in 1930's and was developed by himself [22, 23, 24, 25, 26, 27]. This function is considered as a contour integral representation on the Hankel path in complex plane with a cut along the negative real semi-axis $\arg \tau = \pi$

$$W(c, d; z) = \frac{1}{2\pi i} \int_{\mathrm{Ha}} \tau^{-d} e^{\tau + z\tau^{-c}} d\tau, \quad c > -1, d \in \mathbb{C}, \ z \in \mathbb{C}. \tag{2.2}$$

This function can be also presented by the following series using the integral representation of reciprocal gamma function [15]

$$W(c,d;z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(cn+d)},\quad c>-1, d\in\mathbb{C},\ z\in\mathbb{C}. \tag{2.3}$$

The Wright function has a important role in the theory and applications of fractional calculus particularly in the initial and boundary value problems for the fractional diffusion-wave equations [1, 8, 12]. For simplicity of the relation (2.2) and its asymptotic behaviors in the various values of parameters c and d, see [11]. As a special case of the Wright function (2.3), we consider the Bessel-Wright function $\mathcal{J}_{\nu}^{\mu}(z)$ [10]

$$\mathcal{J}_{\nu}^{\mu}(z)=W(\mu,\nu+1;-z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma(\mu n+\nu+1)},\quad \mu>-1. \tag{2.4}$$

The Airy function of first kind $\mathrm{Ai}(z)$ can be also derived from the Wright function [2, 13]

$$W(-1/3,2/3;-z)=3^{\frac{2}{3}}\mathrm{Ai}(3^{-\frac{1}{3}}z),$$

where

$$\mathrm{Ai}(z)=\frac{1}{\pi}\int_0^{\infty}\cos(zt+\frac{t^3}{3})dt,$$

Moreover, for $z\to+\infty$ the asymptotic behavior of the Wright function is given by

$$W(c,d;-z)\sim z^{p(\frac{1}{2}-d)}e^{\sigma z^p\cos(p\pi)}\cos\Big(\pi p(\frac{1}{2}-d)+\sigma z^p\sin(p\pi)\Big)\{c_1+O(z^{-p})\}, \tag{2.5}$$

where $p=\frac{1}{1+c}$ and $\sigma=(1+c)c^{-\frac{c}{1+c}}$, and c_1 can be evaluated exactly. We consider following asymptotic formula that we needs in the next sections

$$\mathrm{Ai}(-z)\sim\frac{1}{\sqrt{\pi}z^{\frac{1}{4}}}\Big[\cos(\frac{2}{3}z^{\frac{3}{2}}-\frac{\pi}{4})+O(\frac{1}{z^{\frac{3}{2}}})\Big], \tag{2.6}$$

$$\int_0^z\mathrm{Ai}(-\varsigma)d\varsigma\sim\frac{2}{3}-\frac{1}{\sqrt{\pi}z^{\frac{3}{4}}}\cos(\frac{2}{3}z^{\frac{3}{2}}+\frac{\pi}{4})+O(\frac{1}{z^{\frac{9}{4}}}), \tag{2.7}$$

$$\int_0^z\mathrm{Ai}(\varsigma)d\varsigma\sim\frac{1}{3}-\frac{1}{2\sqrt{\pi}z^{\frac{3}{4}}}\exp(-\frac{2}{3}z^{\frac{3}{2}}). \tag{2.8}$$

3. Main Theorem

In this section, we consider the linear form of equation (1.2) and study the fundamental solution of the Cauchy problem

$$u_t-\beta u_{xxx}-\gamma W_{-}^{\alpha}u=0,\quad 0<\alpha\le 1,\ \gamma>0,\quad u(x,0)=\delta(x). \tag{3.1}$$

We first apply the Fourier transform on the equation (3.1) and use the relation (2.1) to get an integral representation for the fundamental solution of the Cauchy problem as follows

$$E^{(\alpha)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\Big(i\xi x - i\beta\xi^3 t + \frac{\gamma t e^{-\frac{i\pi\alpha}{2}}}{\xi^\alpha}\Big)d\xi.$$

In order to obtain a different form for the above solution we state the following lemma for the inverse Fourier transform of the function $\exp\big(\frac{\gamma t e^{-\frac{i\pi\alpha}{2}}}{\xi^\alpha}\big)$.

Lemma 3.1. *For* $\alpha = \frac{2m-1}{2n-1}, m < n,\ m, n \in \mathbb{N}$, *the inverse Fourier transform of* $\exp\big(\frac{\gamma t e^{-\frac{i\pi\alpha}{2}}}{\xi^\alpha}\big)$ *is given by*

$$\mathcal{F}^{-1}\{\exp\big(\frac{\gamma t e^{-\frac{i\pi\alpha}{2}}}{\xi^\alpha}\big); x\} = -\alpha|x|^{\alpha-1}\gamma t W\Big(\alpha, \alpha+1; -|x|^\alpha \gamma t\Big)H(-x) + \delta(x).$$

where H *is the Heaviside unite step function.*

Proof. First, we intend to find the inverse Fourier transform of $\frac{1}{\xi}\exp\big(\frac{\gamma t e^{-\frac{i\pi\alpha}{2}}}{\xi^\alpha}\big)$. We consider the following integral

$$\Lambda^{(\alpha)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\xi}\exp\Big(i\xi x + \frac{\gamma t e^{-\frac{i\pi\alpha}{2}}}{\xi^\alpha}\Big)d\xi.$$

and apply the change of variables $i\xi = \tau$ to get

$$\Lambda^{(\alpha)}(x,t) = \frac{1}{2\pi}\int_{-i\infty}^{i\infty} \frac{1}{\tau}\exp\Big(\tau x + \frac{\gamma t}{\tau^\alpha}\Big)d\tau.$$

The above integral is a particular case of the integral (2.2) on the Hankel path and is written in terms of the Wright function for $x < 0$ as [11, Theorem 2.1]

$$\Lambda^{(\alpha)}(x,t) = -iW(\alpha, 1; -|x|^\alpha\gamma t)H(-x),$$

or equivalently

$$\Lambda^{(\alpha)}(x,t) = -i\mathcal{J}_0^\alpha(|x|^\alpha\gamma t)H(-x).$$

At this point, using the fact that

$$\frac{d}{d\xi}W(c,d;\xi) = W\Big(c, c+d; \xi\Big),$$

for $f(x) \in C^1(\mathbb{R})$ we have

$$\begin{aligned}
\mathcal{F}^{-1}\{\exp\big(\frac{\gamma t e^{-\frac{i\pi\alpha}{2}}}{\xi^\alpha}\big); x\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\Big(i\xi x + \frac{\gamma t e^{-\frac{i\pi\alpha}{2}}}{\xi^\alpha}\Big)d\xi = \frac{1}{i}\frac{\partial}{\partial x}\Lambda^{(\alpha)}(x,t)\\
&= -\frac{\partial}{\partial x}\Big[W(\alpha, 1; -|x|^\alpha\gamma t)H(-x)\Big]\\
&= -\alpha|x|^{\alpha-1}\gamma t W(\alpha, \alpha+1; -|x|^\alpha\gamma t)H(-x) + \delta(x),
\end{aligned}$$

where we used the following formula of differentiation

$$\Big(f(x)H(x)\Big)' = f'(x)H(x) + \delta(x)f(0).$$

This completes the proof. □

Theorem 3.2. *The fundamental solutions of positive and negative dispersions for the Cauchy problem* (3.1) *are given by*

$$\begin{aligned} {}^{+}E^{(\alpha)}(x,t) &= \frac{1}{\sqrt[3]{3\beta t}}\mathrm{Ai}\Big(-\frac{x}{\sqrt[3]{3\beta t}}\Big) - \frac{\alpha\gamma t}{\sqrt[3]{3\beta t}} \\ &\quad \int_0^\infty \mathrm{Ai}\Big(-\frac{x+u}{\sqrt[3]{3\beta t}}\Big)u^{\alpha-1}W(\alpha,\alpha+1;-u^\alpha\gamma t)du, \end{aligned} \tag{3.2}$$

$$\begin{aligned} {}^{-}E^{(\alpha)}(x,t) &= \frac{1}{\sqrt[3]{3|\beta|t}}\mathrm{Ai}\Big(\frac{x}{\sqrt[3]{3|\beta|t}}\Big) - \frac{\alpha\gamma t}{\sqrt[3]{3|\beta|t}} \\ &\quad \int_0^\infty \mathrm{Ai}\Big(\frac{x+u}{\sqrt[3]{3|\beta|t}}\Big)u^{\alpha-1}W(\alpha,\alpha+1;-u^\alpha\gamma t)du, \end{aligned} \tag{3.3}$$

where the superindex $+$ *corresponds to the positive dispersion case and the superindex* $-$ *is used for negative dispersion.*

Proof. We first consider Lemma 3.1 and the following integral with respect to the Airy function

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\Big(i\xi x - i\beta\xi^3 t\Big)d\xi =$$
$$\begin{cases} \frac{1}{\pi}\int_0^\infty \cos\Big(\xi x - \beta\xi^3 t\Big)d\xi = \frac{1}{\sqrt[3]{3\beta t}}\mathrm{Ai}\Big(-\frac{x}{\sqrt[3]{3\beta t}}\Big), & \beta > 0, \\ \frac{1}{\pi}\int_0^\infty \cos\Big(\xi x + |\beta|\xi^3 t\Big)d\xi = \frac{1}{\sqrt[3]{3|\beta| t}}\mathrm{Ai}\Big(\frac{x}{\sqrt[3]{3|\beta| t}}\Big), & \beta < 0, \end{cases} \tag{3.4}$$

and then we apply the convolution theorem for the Fourier transform to rewrite the solution (3.4) as

$$\begin{aligned} E^{(\alpha)}(x,t) &= \frac{1}{\sqrt[3]{3\beta t}}\mathrm{Ai}\Big(-\frac{x}{\sqrt[3]{3\beta t}}\Big) - \frac{\alpha\gamma t}{\sqrt[3]{3\beta t}} \\ &\quad \int_{-\infty}^0 \mathrm{Ai}\Big(-\frac{x-y}{\sqrt[3]{3\beta t}}\Big)|y|^{\alpha-1}W(\alpha,\alpha+1;-|y|^\alpha\gamma t)dy. \end{aligned}$$

We change the variable of integration $y = -u$ and get the result for two cases of parameter β (the positive dispersion and the negative dispersion). □

We know that the functions $\mathrm{Ai}(-\frac{x}{\sqrt[3]{3\beta t}})$ and $\mathrm{Ai}(\frac{x}{\sqrt[3]{3\beta t}})$ are fundamental solutions of following KdV equation for $\beta > 0$ and $\beta < 0$ respectively

$$\begin{cases} u_t - \beta u_{xxx} = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \delta(x). \end{cases}$$

For this reason, we consider the following representation

$$\begin{cases} {}^{+}E^{(\alpha)}(x,t) = E_0^{+}(x,t) + {}^{+}E_\gamma^{(\alpha)}(x,t), \\ {}^{-}E^{(\alpha)}(x,t) = E_0^{-}(x,t) + {}^{-}E_\gamma^{(\alpha)}(x,t), \end{cases}$$

where

$$E_0^{+}(x,t) = \frac{1}{\sqrt[3]{3\beta t}}\mathrm{Ai}(-\frac{x}{\sqrt[3]{3\beta t}}), \tag{3.5}$$

$${}^{+}E_\gamma^{(\alpha)}(x,t) = -\frac{\alpha\gamma t}{\sqrt[3]{3\beta t}}\int_0^\infty \mathrm{Ai}\Big(-\frac{x+u}{\sqrt[3]{3\beta t}}\Big)u^{\alpha-1}W(\alpha,\alpha+1,-u^\alpha\gamma t)du \tag{3.6}$$

and

$$E_0^{-}(x,t) = \frac{1}{\sqrt[3]{3\beta t}}\mathrm{Ai}(\frac{x}{\sqrt[3]{3\beta t}}), \tag{3.7}$$

$${}^{-}E_\gamma^{(\alpha)}(x,t) = -\frac{\alpha\gamma t}{\sqrt[3]{3|\beta|t}}\int_0^\infty \mathrm{Ai}\Big(\frac{x+u}{\sqrt[3]{3|\beta|t}}\Big)u^{\alpha-1}W(\alpha,\alpha+1,-u^\alpha\gamma t)du. \tag{3.8}$$

It is important to mention that the integral of ${}^{+}E^{(\alpha)}$ converges conditionally and the integral of ${}^{-}E^{(\alpha)}$ is absolutely convergent because of the exponential decay of the Airy function for the positive argument.

Corollary 3.3. *In special case $\alpha = 1$, the relations (3.2) and (3.3) are reduces to the following representations which confirms the results of paper [21]*

$$E^{+}(x,t) = \frac{1}{\sqrt[3]{3\beta t}}\mathrm{Ai}\Big(-\frac{x}{\sqrt[3]{3\beta t}}\Big) - \frac{\sqrt{\gamma t}}{\sqrt[3]{3\beta t}}\int_0^\infty \mathrm{Ai}\Big(-\frac{x+u}{\sqrt[3]{3\beta t}}\Big)\frac{J_1(2\sqrt{\gamma t u})}{\sqrt{u}}du,$$

$$E^{-}(x,t) = \frac{1}{\sqrt[3]{3|\beta|t}}\mathrm{Ai}\Big(\frac{x}{\sqrt[3]{3|\beta|t}}\Big) - \frac{\sqrt{\gamma t}}{\sqrt[3]{3|\beta|t}}\int_0^\infty \mathrm{Ai}\Big(\frac{x+u}{\sqrt[3]{3|\beta|t}}\Big)\frac{J_1(2\sqrt{\gamma t u})}{\sqrt{u}}du,$$

where J_1 is the Bessel function of order one.

Corollary 3.4. *Using the integrating by parts and applying the asymptotic expansion (2.5) for the Wright function, we get other integral representations for the solutions ${}^{+}E_\gamma^{(\alpha)}(x,t)$ and ${}^{-}E_\gamma^{(\alpha)}(x,t)$ in terms of the Wright functions as follows*

$${}^{+}E^{(\alpha)}(x,t) = \frac{1}{(3\beta t)^{\frac{2}{3}}}\int_0^\infty \mathrm{Ai}'\Big(-\frac{x+u}{\sqrt[3]{3\beta t}}\Big)W(\alpha,1;-u^\alpha\gamma t)du, \tag{3.9}$$

$${}^{-}E^{(\alpha)}(x,t) = -\frac{1}{(3|\beta|t)^{\frac{2}{3}}}\int_0^\infty \mathrm{Ai}'\Big(\frac{x+u}{\sqrt[3]{3|\beta|t}}\Big)W(\alpha,1;-u^\alpha\gamma t)du. \tag{3.10}$$

Also, from the above relations we can deduce that

$$\int_{-\infty}^{\infty} {}^{\pm}E^{(\alpha)}(x,t)dx = 0.$$

4. Some Properties of The Fundamental Solution

In this section, we intend to obtain other integral representations for ${}^{\pm}E_{\gamma}^{(\alpha)}(x,t)$ that are useful to get the associated asymptotic expansions.

4.1. **Some Integral Representations.** We consider the change of variables

$$y=\frac{u}{\sqrt[3]{3|\beta|t}},\quad \chi=\chi(x,t)=\frac{x}{\sqrt[3]{3|\beta|t}},\quad a=a_{\gamma}(t)=2\sqrt{\gamma t}(3|\beta|t)^{\frac{\alpha}{6}},$$

and use the definition (2.4) for the Bessel-Wright function $\mathcal{J}_{\nu}^{\mu}(z)$ to rewrite the relations (3.5)- (3.8), respectively, in the following forms

$$E_0^{+}(x,t)=\frac{1}{\sqrt[3]{3\beta t}}\mathrm{Ai}(-\chi), \tag{4.1}$$

$${}^{+}E_{\gamma}^{(\alpha)}(x,t)=-\frac{\alpha a^2}{4\sqrt[3]{3\beta t}}\int_0^{\infty}\mathrm{Ai}\big(-(\chi+y)\big)y^{\alpha-1}\mathcal{J}_{\alpha}^{\alpha}(\frac{a^2}{4}y^{\alpha})dy, \tag{4.2}$$

$$E_0^{-}(x,t)=\frac{1}{\sqrt[3]{3|\beta|t}}\mathrm{Ai}(\chi), \tag{4.3}$$

$${}^{-}E_{\gamma}^{(\alpha)}(x,t)=-\frac{\alpha a^2}{4\sqrt[3]{3|\beta|t}}\int_0^{\infty}\mathrm{Ai}(\chi+y)y^{\alpha-1}\mathcal{J}_{\alpha}^{\alpha}(\frac{a^2}{4}y^{\alpha})dy. \tag{4.4}$$

Using the relations (3.9) and (3.10), we also get the following relations in terms of the Bessel-Wright functions

$${}^{+}E^{(\alpha)}(x,t)=\frac{1}{(3\beta t)^{\frac{1}{3}}}\int_0^{\infty}\mathrm{Ai}^{'}\big(-(\chi+y)\big)\mathcal{J}_{0}^{\alpha}(\frac{a^2}{4}y^{\alpha})dy,$$

$${}^{-}E^{(\alpha)}(x,t)=-\frac{1}{(3|\beta|t)^{\frac{1}{3}}}\int_0^{\infty}\mathrm{Ai}^{'}\big(-(\chi+y)\big)\mathcal{J}_{0}^{\alpha}(\frac{a^2}{4}y^{\alpha})dy.$$

Lemma 4.1. *For $p=\frac{1}{1+\alpha}$, the asymptotic expansion of ${}^{+}E_{\gamma}^{(\alpha)}(x,t)$ is given by*

$${}^{+}E_{\gamma}^{(\alpha)}(x,t)\sim\frac{\alpha a^{p(1-2\alpha)+2}\cos\big(\pi p(\frac{1}{2}-\alpha)\big)}{2^{p(1-2\alpha)+1}\sqrt{\pi}\sqrt[3]{3\beta t}}\int_0^{\infty}U(a,\chi,\xi)\Big[S^{-}(a,\chi,\varsigma)-S^{+}(a,\chi,\varsigma)\Big]d\varsigma,$$

where

$$U(a,\chi,\varsigma)=\frac{\varsigma^{2\alpha+p-2\alpha p-1}\chi^{\alpha+\frac{p\alpha}{2}-p\alpha^2-\frac{1}{4}}e^{\sigma(\frac{a^2p}{4}\varsigma^{2\alpha p})}\chi^{\alpha p}\cos(\pi p)}{(1+\varsigma^2)^{\frac{1}{4}}},$$

and

$$\begin{aligned} S^{-}(a,\chi,\varsigma) &= \sin\big(\pi p(\frac{1}{2}-\alpha)\big)\sin\Big(\sigma(\frac{a^2}{4}\varsigma^{2\alpha}\chi^{\alpha})^p\sin(p\pi)\Big) \\ &\quad \cos\big(\frac{2}{3}\chi^{\frac{3}{2}}(1+\varsigma^2)^{\frac{3}{2}}+\frac{\pi}{4}\big), \\ S^{+}(a,\chi,\varsigma) &= \cos\big(\pi p(\frac{1}{2}-\alpha)\big)\cos\Big(\sigma(\frac{a^2}{4}\varsigma^{2\alpha}\chi^{\alpha})^p\sin(p\pi)\Big) \\ &\quad \cos\big(\frac{2}{3}\chi^{\frac{3}{2}}(1+\varsigma^2)^{\frac{3}{2}}-\frac{\pi}{4}\big). \end{aligned}$$

Proof. We employ the relation (4.2) and apply the change of variables $\xi=\chi+y$ and $\varsigma=\sqrt{\frac{\xi-\chi}{\chi}}$ to obtain

$$^{+}E_{\gamma}^{(\alpha)}(x,t) = -\frac{\alpha a^2\chi^{\alpha}}{2\sqrt[3]{3\beta t}}\int_0^{\infty}\mathrm{Ai}\big(-\chi(1+\varsigma^2)\big)\varsigma^{2\alpha-1}\mathcal{J}_{\alpha}^{\alpha}(\frac{\mathrm{a}^2}{4}\chi^{\alpha}\varsigma^{2\alpha})\mathrm{d}\varsigma.$$

We now use the relations (2.5) and (2.6) for the asymptotic expansions of the Wright and Airy functions (as $\chi\to\infty$) to get the result. □

Lemma 4.2. *There exists $C\in\mathbb{R}$ such that*

$$\mathcal{J}_{\alpha}^{\alpha}(\frac{a^2}{4}y^{\alpha}) \le C y^{p(\frac{\alpha}{2}-\alpha^2)}e^{\sigma(\frac{a^2}{4}y^{\alpha})^p\cos(p\pi)}, \quad y>0, \quad p=\frac{1}{1+\alpha}. \tag{4.5}$$

Also

$$\mathcal{J}_{\alpha}^{\alpha}(\frac{a^2}{4}y^{\alpha}) \sim \frac{1}{\Gamma(\alpha+1)}, \qquad y\to 0, \tag{4.6}$$

Proof. Applying the asymptotic expansion (2.5), we can find $C\in\mathbb{R}$ to establish the relation (4.5). We also use the series representation (2.3) for the Wright function and obtain the approximation (4.6).

□

4.2. **The Airy Transforms.** In view of the relations (4.1)-(4.4), we define the Airy transform as

$$(\mathrm{Ai}\psi)(x) = \int_{-\infty}^{\infty}\mathrm{Ai}(x+y)\psi(y)dy, \tag{4.7}$$

where $\psi(y)$ is given by

$$\psi(y) = \begin{cases} \frac{a}{2}y^{\alpha-1}\mathcal{J}_{\alpha}^{\alpha}(\frac{a^2}{4}y^{\alpha}), & y\ge 0, \\ 0, & y<0. \end{cases}$$

It is clear that by putting $\bar{\psi}(y)=\psi(-y)$, we can derive an

$$\int_{-\infty}^{\infty} \mathrm{Ai}\big(-(x+y)\big)\psi(y)dy = \int_{-\infty}^{\infty} \mathrm{Ai}(\bar{x}+y)\bar{\psi}(y)dy,$$

where

$$\bar{\psi}(y) = \begin{cases} 0, & y > 0, \\ \frac{a}{2} y^{\alpha-1}\mathcal{J}_{\alpha}^{\alpha}(\frac{a^2}{4}y^{\alpha}), & y \leq 0. \end{cases}$$

Now, we state the following theorem for the norms of fundamental solutions and the Airy transforms of Bessel-Wright functions.

Theorem 4.3. *The L_2-norms of ${}^{-}E_{\gamma}^{(\alpha)}(x,t)$ and ${}^{+}E_{\gamma}^{(\alpha)}(x,t)$ are given by*

$$|{}^{-}E_{\gamma}^{(\alpha)}(\chi,t)|_2^2 = \frac{\alpha^2 a^2}{4(3|\beta|t)^{\frac{1}{3}}}|\psi|_2^2,$$

and

$$|{}^{+}E_{\gamma}^{(\alpha)}(\chi,t)|_2^2 = \frac{\alpha^2 a^2}{4(3\beta t)^{\frac{1}{3}}}|\bar{\psi}|_2^2,$$

where

$$\begin{aligned} |\psi|_2^2 &= \frac{a^2}{4}\int_0^{\infty} y^{2\alpha-2}\left(\mathcal{J}_{\alpha}^{\alpha}(\frac{a^2}{4}y^{\alpha})\right)^2 dy, \\ |\bar{\psi}|_2^2 &= \frac{a^2}{4}\int_{-\infty}^{0} y^{2\alpha-2}\left(\mathcal{J}_{\alpha}^{\alpha}(\frac{a^2}{4}y^{\alpha})\right)^2 dy. \end{aligned}$$

Proof. Using the definition of Airy transform (4.7) and applying the relation (4.4) we have

$${}^{-}E_{\gamma}^{(\alpha)}(x,t) = -\frac{\alpha a}{2\sqrt[3]{3\beta t}}(\mathrm{Ai}\psi)(x),$$

and

$${}^{+}E_{\gamma}^{(\alpha)}(x,t) = -\frac{\alpha a}{2\sqrt[3]{3\beta t}}(\mathrm{Ai}\bar{\psi})(\bar{x}).$$

We now consider the L_2-norm of ${}^{-}E_{\gamma}^{(\alpha)}(\chi,t)$

$$|{}^{-}E_{\gamma}^{(\alpha)}(\chi,t)|_2^2 = \frac{\alpha^2 a^2}{4(3\beta t)^{\frac{1}{3}}}\int_{-\infty}^{\infty}|(\mathrm{Ai}\psi)(\chi)|^2 \mathrm{d}\chi,$$

and use the Parseval theorem for Airy transform

$$|\mathrm{Ai}\psi|_2 = |\psi|_2,$$

to get

$$|{}^{-}E_{\gamma}^{(\alpha)}(\chi,t)|_2^2=\frac{\alpha^2a^2}{4(3\beta t)^{\frac{1}{3}}}|\psi|_2^2,$$

where

$$|\psi|_2^2=\frac{a^2}{4}\int_0^{\infty}y^{2\alpha-2}\Big(\mathcal{J}_{\alpha}^{\alpha}(\frac{a^2}{4}y^{\alpha})\Big)^2dy.$$

Analogous result holds for ${}^{+}E_{\gamma}^{(\alpha)}(x,t)$.

□

Theorem 4.4. *The following asymptotic expansions hold for* ${}^{+}E_{\gamma}^{(\alpha)}(x,t)$ *and* ${}^{+}E_{\gamma}^{(\alpha)}(x,t)$ *as* $\chi\to\infty$

$$ {}^{+}E_{\gamma}^{(\alpha)}(x,t)\sim\frac{\alpha a^2}{\sqrt[3]{3\beta t}}\int_0^{\infty}F(z)\Big\{\frac{\cos(\frac{2}{3}\chi^{\frac{3}{2}}+\frac{\pi}{4})}{\chi^{\frac{3}{4}}}-\frac{\cos\big((\frac{2}{3}(\chi+\frac{z^2}{a^2})^{\frac{3}{2}}+\frac{\pi}{4})}{\chi^{\frac{3}{4}}}\Big\}dz, \tag{4.8}$$

and

$$ {}^{-}E_{\gamma}^{(\alpha)}(x,t)\sim\frac{\alpha a^2}{\sqrt[3]{3|\beta|t}}\int_0^{\infty}F(z)\Big\{\frac{\exp(-\frac{2}{3}\chi^{\frac{3}{2}})}{\chi^{\frac{3}{4}}}-\frac{\exp\big((-\frac{2}{3}(\chi+\frac{z^2}{a^2})^{\frac{3}{2}})}{\chi^{\frac{3}{4}}}\Big\}dz, \tag{4.9}$$

where $z=a\sqrt{y}$ *and*

$$F(z)=\frac{a^2}{2z}\frac{d}{dz}\Big[(\frac{z}{a})^{2\alpha-2}\mathcal{J}_{\alpha}^{\alpha}\Big(\frac{a^2}{4}(\frac{z}{a})^{2\alpha}\Big)\Big].$$

Proof. First for $\alpha\in[0,1]$, we use the relation (4.5) and we deduce that

$$\lim_{y\to\infty}y^{\alpha-1}\mathcal{J}_{\alpha}^{\alpha}(\frac{a^2}{4}y^{\alpha})=0.$$

Hence, by integration by parts we can write

$$\begin{aligned}
&{}^{+}E_{\gamma}^{(\alpha)}(x,t)=\\
&=-\frac{\alpha a^2}{4\sqrt[3]{3\beta t}}\int_0^{\infty}\Big(\Big[y^{\alpha-1}\mathcal{J}_{\alpha}^{\alpha}(\frac{a^2}{4}y^{\alpha})\Big]\frac{d}{dy}\Big[\int_0^{y}\mathrm{Ai}\big(-(\chi+\eta)\big)d\eta\Big]\Big)dy,\\
&=-\frac{\alpha a^2}{4\sqrt[3]{3\beta t}}\int_0^{\infty}\Big(\frac{d}{dy}\Big[y^{\alpha-1}\mathcal{J}_{\alpha}^{\alpha}(\frac{a^2}{4}y^{\alpha})\Big]\Big[\int_0^{y}\mathrm{Ai}\big(-(\chi+\eta)\big)d\eta\Big]\Big)dy.
\end{aligned}\tag{4.10}$$

Here, we apply the relation $\mathcal{J}_{\alpha}^{\alpha}(\frac{a^2}{4}y^{\alpha})=W(\alpha,\alpha+1;-(3\beta t)^{\frac{\alpha}{3}}y^{\alpha}\gamma t)$ to get

$$\frac{d}{dy}\Big[y^{\alpha-1}\mathcal{J}^{\alpha}_{\alpha}(\frac{a^2}{4}y^{\alpha})\Big] = (\alpha-1)y^{\alpha-2}\mathcal{J}^{\alpha}_{\alpha}(\frac{a^2}{4}y^{\alpha}) - \frac{a^2}{4}y^{2\alpha-2}\mathcal{J}^{\alpha}_{2\alpha-1}(\frac{a^2}{4}y^{\alpha}) + \frac{a^2}{4}\alpha y^{2\alpha-2}\mathcal{J}^{\alpha}_{2\alpha}(\frac{a^2}{4}y^{\alpha}), \tag{4.11}$$

and rewrite the relation (4.10) as

$$\begin{aligned} {}^{+}E^{(\alpha)}_{\gamma}(x,t) &= \frac{\alpha(\alpha-1)a^2}{4\sqrt[3]{3\beta t}}\int_0^{\infty} y^{\alpha-2}\mathcal{J}^{\alpha}_{\alpha}(\frac{a^2}{4}y^{\alpha})\int_0^{y}\mathrm{Ai}\big(-(\chi+\eta)\big)d\eta dy, \\ &- \frac{\alpha a^4}{16\sqrt[3]{3\beta t}}\int_0^{\infty} y^{2\alpha-2}\mathcal{J}^{\alpha}_{2\alpha-1}(\frac{a^2}{4}y^{\alpha})\int_0^{y}\mathrm{Ai}\big(-(\chi+\eta)\big)d\eta dy, \\ &+ \frac{\alpha^2 a^4}{16\sqrt[3]{3\beta t}}\int_0^{\infty} y^{2\alpha-2}\mathcal{J}^{\alpha}_{2\alpha}(\frac{a^2}{4}y^{\alpha})\int_0^{y}\mathrm{Ai}\big(-(\chi+\eta)\big)d\eta dy. \end{aligned}$$

We make the change of variables $z = a\sqrt{y}$ and $\chi+\eta=\xi$ and obtain

$$\begin{aligned} {}^{+}E^{(\alpha)}_{\gamma}(x,t) &= \frac{\alpha(\alpha-1)a^{4-2\alpha}}{2\sqrt[3]{3\beta t}}\int_0^{\infty} z^{2\alpha-3}\mathcal{J}^{\alpha}_{\alpha}\Big(\frac{a^2}{4}(\frac{z}{a})^{2\alpha}\Big)\int_{\chi}^{\chi+\frac{z^2}{a^2}}\mathrm{Ai}(-\xi)d\xi dz, \\ &- \frac{\alpha a^{6-4\alpha}}{8\sqrt[3]{3\beta t}}\int_0^{\infty} z^{4\alpha-3}\mathcal{J}^{\alpha}_{2\alpha-1}\Big(\frac{a^2}{4}(\frac{z}{a})^{2\alpha}\Big)\int_{\chi}^{\chi+\frac{z^2}{a^2}}\mathrm{Ai}(-\xi)d\xi dz, \\ &+ \frac{\alpha^2 a^{6-4\alpha}}{8\sqrt[3]{3\beta t}}\int_0^{\infty} z^{4\alpha-3}\mathcal{J}^{\alpha}_{2\alpha}\Big(\frac{a^2}{4}(\frac{z}{a})^{2\alpha}\Big)\int_{\chi}^{\chi+\frac{z^2}{a^2}}\mathrm{Ai}(-\xi)d\xi dz. \end{aligned}$$

Using the relation (4.11), we set the function $F(z)$ in the following form

$$\begin{aligned} F(z) &= \frac{a^2}{2z}\frac{d}{dz}\Big[(\frac{z}{a})^{2\alpha-2}\mathcal{J}^{\alpha}_{\alpha}\Big(\frac{a^2}{4}(\frac{z}{a})^{2\alpha}\Big)\Big] \\ &= \frac{(\alpha-1)a^{2-2\alpha}}{2}z^{2\alpha-3}\mathcal{J}^{\alpha}_{\alpha}\Big(\frac{a^2}{4}(\frac{z}{a})^{2\alpha}\Big) \\ &- \frac{a^{4-4\alpha}}{8}z^{4\alpha-3}\Big[\mathcal{J}^{\alpha}_{2\alpha-1}\Big(\frac{a^2}{4}(\frac{z}{a})^{2\alpha}\Big)+\mathcal{J}^{\alpha}_{2\alpha}\Big(\frac{a^2}{4}(\frac{z}{a})^{2\alpha}\Big)\Big], \end{aligned}$$

and reconstruct ${}^{+}E^{(\alpha)}_{\gamma}(x,t)$ as

$${}^{+}E^{(\alpha)}_{\gamma}(x,t) = \frac{\alpha a^2}{\sqrt[3]{3\beta t}}\int_0^{\infty}F(z)\int_{\chi}^{\chi+\frac{z^2}{a^2}}\mathrm{Ai}(-\xi)d\xi.$$

Finally, using the asymptotic expansion (2.7) for $\chi\to\infty$, we obtain (4.8). For ${}^{-}E^{(\alpha)}_{\gamma}(x,t)$ we similarly get

$${}^{-}E^{(\alpha)}_{\gamma}(x,t) = \frac{\alpha a^2}{\sqrt[3]{3|\beta|t}}\int_0^{\infty}F(z)\int_{\chi}^{\chi+\frac{z^2}{a^2}}\mathrm{Ai}(\xi)d\xi,$$

and apply the asymptotic expansion (2.8) for deriving the relation (4.9). □

5. Conclusion

This paper provided a study on the Ostrovsky equation in the fractional derivative case for the space variable. This fractionalization was shown in sense of the Weyl fractional derivative for the integral term of the equation. We saw that how the fundamental solution of the fractional Ostrovsky equation could be changed from the Bessel function to Bessel-Wright function and also the associated asymptotic analysis. The role of Airy function as the Green's function of the KdV equation is still determinative for the introduced fractional Ostrovsky equation. As a future work, we can develop our results for a new fractionalization of the term u_{xxx} and extend the associated fundamental solution.

References

[1] A. Ansari, *On the Volterra μ-functions and the M-Wright functions as kernels and eigenfunctions of Volterra type integral operators*, Fract. Calc. Appl. Anal, *19*(2) (2016), 567-572.

[2] A. Ansari, *The fractional Airy transform*, Hacet. J. Math. Stat, *44*(4) (2015), 761766.

[3] H. Askari and A. Ansari, *Airy functions and Riesz fractional trigonometric operators*, Integral Transforms Spec. Funct, *29*(8) (2018), 1-20.

[4] E. S. Benilov, *On the surface waves in a shallow channel with an uneven bottom*, Stud. Appl. Math, *87* (1992), 1–14.

[5] L. Debnath and D. Bhatta, *Integral Transforms and Their Applications*, second ed., Chapman and Hall, New York, 2006.

[6] V. N. Galkin and Yu.A. Stepanyants, *On the existence of stationary solitary waves in a rotating fluid*, J. Appl. Math. Mech, *55*(6) (1991), 939–943.

[7] O. A. Gilman, R. Grimshaw, and Yu. A. Stepanyants, *Appoximate Analytical and Numerical Solutions of the Stationary Ostrovsky Equation*, Stud. Appl. Math, *95* (1995), 115–126.

[8] R. Gorenflo, Yu. Luchko, and F. Mainardi, *Analytical properties and applications of the Wright function*, Fract. Calc. Appl. Anal., *2*(4) (1999), 383–415.

[9] R. Grimshaw, *Evolution equations for weakly nonlinear long internal waves in a rotating fluid*, Stud. Appl. Math., *73* (1985), 1–33.

[10] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman & J. Wiley, Harlow, New York, 1994.

[11] Yu. Luchko, *Algorithms for Evaluation of the Wright Function for the Real Arguments' Values*, Fract. Calc. Appl. Anal., *11*(1) (2008), 57–75.

[12] L. Moslehi and A. Ansari, *On M-Wright transforms and time-fractional diffusion equations*, Integral Transforms Spec. Funct., *28*(2) (2017), 113-129.

[13] L. Moslehi and A. Ansari, *Integral representations of products of Airy functions related to fractional calculus,* J. Class. Anal., *7*(2) (2015), 99112.

[14] L. A. Ostrovsky, *Nonlinear internal waves in a rotating ocean*, Oceanology, *18* (1978), 181–191.

[15] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[16] V. Varlamov, *Semi-integer derivatives of the Airy functions and related properties of the Korteweg-de Vries-type equations*, Z. Angew. Math. Phys., *59* (2008), 381–399.

[17] V. Varlamov, *Fractional derivatives of products of Airy functions*, J. Math. Anal. Appl., *337* (2008), 667–685.

[18] V. Varlamov, *Differential and integral relations involving fractional derivatives of Airy functions and applications*, J. Math. Anal. Appl., *348* (2008), 101–115.

[19] V. Varlamov, *Integrals involving products of Airy functions, their derivatives and Bessel functions*, J. Math. Anal. Appl., *370* (2010), 687–702.

[20] V. Varlamov, *Riesz fractional derivatives of the product of Airy transforms*, Phys. Scr., T*136* (2009), 014004 (5pp).
[21] V. Varlamov, *Oscillatory integrals generated by the Ostrovsky equation*, Z. Angew. Math. Phys., *56* (2005), 957–985.
[22] E. M. Wright, *On the coefficients of power series having essential singularities*, J. London Math. Soc., *8* (1933), 71–80.
[23] E. M. Wright, *The asymptotic expansion of the generalized Bessel function*, Proc. London Math. Soc., *38* (2) (1934), 257–270.
[24] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, J. London Math. Soc., *10* (1935), 286–293.
[25] E. M. Wright, *The asymptotic expansion of integral functions defined by Taylor Series*, Philos. Trans. Roy. Soc. London A., *238* (1940), 423–451.
[26] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function II*, Proc. London Math. Soc., *46*(2) (1940), 389–408.
[27] E. M. Wright, *The generalized Bessel function of order greater than one*, Quart. J. Math. Oxford Ser., *11* (1940), 36–48.