Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 8, No. 3, 2020, pp. 468-479
DOI:10.22034/cmde.2020.32139.1502

Preserving asymptotic mean-square stability of stochastic theta scheme for systems of stochastic delay differential equations

Omid Farkhondeh Rouz*
Faculty of Mathematical Sciences,
University of Tabriz, Tabriz, Iran.
E-mail: omid_farkhonde_7088@yahoo.com

Abstract This article examines asymptotic mean-square stability analysis of stochastic linear theta (SLT) scheme for n-dimensional stochastic delay differential equations (SDDEs). We impose some conditions on drift and diffusion terms, which admit that the diffusion coefficient can be highly nonlinear and does not necessarily satisfy a linear growth or global Lipschitz condition. We prove that the proposed scheme is asymptotically mean square stable if the employed stepsize is smaller than a given and easily computable upper bound. In particular, based on our investigation in the case $\theta \in \left[\frac{1}{2}, 1\right]$, the stepsize is arbitrary. Eventually, numerical examples are given to demonstrate the effectiveness of our work.

Keywords. Stochastic delay differential equations, Stochastic linear theta scheme, Asymptotic mean-square stability,

2010 Mathematics Subject Classification. 65C20, 60H35, 65C30, 65L20.

1. Introduction

Stochastic differential systems, including stochastic differential equations (SDEs), stochastic delay differential equations (SDDEs) and neutral stochastic delay differential equations (NSDDEs), have been greatly developed and played an important role in many ways such as economics, finance, physics and financial markets (see [6, 9, 30]). In some applications (see [2, 10, 11, 13, 18, 20, 23]), SDEs are generalized to SDDEs, i.e. equations whose coefficients are allowed to depend also on the solution at previous time values. SDDEs serve as models of physical processes whose time evolution depends on their past history with noise disturbance. In many fields of science, there has been an increasing interest in the investigation of SDDEs, in particular, in the combined effects of noise and delay in dynamical systems (see, for example, [4, 5]). The fundamental theory of existence and uniqueness of the solution of SDDEs has been studied in [21, 24].

SDDEs which are arising in many applications can not be solved analytically. Hence, one needs to develop effective numerical schemes for such systems. The stability theory of numerical solutions is one of the fundamental research topics in numerical analysis. In particular, two numerical schemes that are frequently applied to SDDEs

Received: 24 February 2019 ; Accepted: 14 May 2019.
∗ Corresponding author.

are the Euler's one-step schemes (either the explicit or backward Euler-Maruyama scheme) and the Milstein scheme (see [5, 16, 21]).

Recently, stability theorems of stochastic differential systems, for example, moment stability (M-stability), (see [12, 20]) and almost sure asymptotic stability (T-stability), (see [26]), have attracted much attention. There are many results on stability theorems for stochastic differential systems (see [20, 21]). In fact, several authors investigate the stability for various kinds of SDDEs (see [3, 7, 15, 19, 22, 25, 28, 29, 32]). Moreover, other scholars analyze the stability of theta scheme [31] for SDDEs.

In order to develop the classical schemes used in SDEs, the Euler's one-step schemes and the Milstein scheme can be combined with either or both of the θ-scheme and split-step techniques. In particular, in [8], it proposes and analyzes a split-step θ-Euler-Maruyama (SSTEM) scheme for solving SDEs, whereas in [33], it develops the split-step θ-Milstein (SSTM) scheme for SDEs and establishes its convergence. The SSTEM scheme is also used by [14, 27] for approximating SDDEs. In particular, these authors investigate the stability of the SSTEM approach, which turns out to be conditionally stable when $\theta \in \left[0, \frac{1}{2}\right]$ and unconditionally stable when $\theta \in \left(\frac{1}{2}, 1\right]$.

Based on the proposed papers, and to the best of our knowledge, the asymptotic mean-square stability of the SLT scheme for n-dimensional SDDEs has not been considered yet. In particular, as the main contribution of the present paper, following the theoretical analysis, is shown that the SLT scheme is asymptotically mean-square stable if the employed stepsizes are smaller than a given and easily computable upper bound. Finally, we present illustrate experiments that confirm the validity of our theoretical findings.

The remainder of the paper is organized as follows. In Section 2 we recall some preliminary results and introduces the SLT scheme for n-dimensional SDDEs. Section 3 studies asymptotic mean-square stability of the proposed numerical scheme. Finally, some figures are reported to illustrate the stability results in Section 4.

2. Preliminary Results and Stochastic Linear Theta scheme

All over this article, we will use $|\cdot|$ to denotes the Euclidean norm in $\mathbb{R}^n$ and $\langle x, y\rangle$ be the Euclidean inner product of vectors $x, y \in \mathbb{R}^n$ and is defined $\langle x, y\rangle = x^T y$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual assumptions (i.e. it is right continuous, increasing and such that $\mathcal{F}_0$ contains all the $\mathbf{P}$-null sets), $W(t)$ be a one-dimensional Brownian motion defined on this probability space, $f : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ be Borel measurable functions. In this article, we consider n-dimensional SDDE as follows:

$$dx(t) = f\big(t, x(t), x(t-\tau)\big)dt + g\big(t, x(t), x(t-\tau)\big)dW(t), \quad t > 0, \tag{2.1}$$

with initial data:

$$x(t) = \psi(t), \quad -\tau \leq t \leq 0, \tag{2.2}$$

where $\tau > 0$ is a constant. To avoid the use of non-anticipative calculus, we only consider diffusion functions such that (see [16] and [17]):

$$g\big(t, x(t), x(t-\tau)\big) = g_1\big(t, x(t)\big) + g_2\big(t, x(t-\tau)\big). \tag{2.3}$$

Algorithm 2.1. We introduce the stochastic linear theta (SLT) scheme by choosing a discretization stepsize $\Delta = \dfrac{\tau}{(m-\delta)}$ for $\delta \in [0,1)$, and m denotes the integer part of $\dfrac{\tau}{\Delta}$. Let us consider the time discretization levels $t_k = k\Delta$, $k = -m, -m+1, \dots$, and $x_k = \psi(k\Delta)$, $k = -m, \dots, -1$. Then, compute the approximation $\{x_k\}_{k\geq 0}$ according to the following scheme:

$$\begin{aligned} x_{k+1} = x_k &+ (1-\theta) f\big(t_k, x_k, \overline{x}_k\big)\Delta + \theta f\big(t_{k+1}, x_{k+1}, \overline{x}_{k+1}\big)\Delta \\ &+ g\big(t_k, x_k, \overline{x}_k\big)\Delta W_k, \qquad k = 0, 1, 2, \dots, \end{aligned} \tag{2.4}$$

where $\Delta W_k := W(t_{k+1}) - W(t_k)$, $\theta \in [0,1]$ and $\overline{x}_k$ denotes an approximation to the delay argument $x(t_k - \tau)$. In this paper, noting that $x(t_k - \tau) = x(t_{k-m} + \delta\Delta)$ and following [1], we compute $\overline{x}_k$ by linear interpolation of x at two successive delay points:

$$\overline{x}_k = \delta x_{k-m+1} + (1-\delta) x_{k-m}, \tag{2.5}$$

where $\overline{x}_k = \psi(t_k)$ for $k \leq 0$. It is well-known that the SLT scheme includes the Euler-Maruyama (EM) scheme ($\theta = 0$), the trapezoidal scheme ($\theta = \frac{1}{2}$) and the backward Euler-Maruyama (BEM) scheme ($\theta = 1$).

Definition 2.1. The numerical scheme (2.4) is said to be asymptotically stable in mean-square if, for any stepsize $\Delta > 0$, the following relation holds:

$$\lim_{k\to\infty} \mathbb{E}\big(x_k^T x_k\big) = 0. \tag{2.6}$$

Theorem 2.2. *(see [21]) Assume that there exists a symmetric, positive definite $n \times n$ matrix Q and two constants α and β such that for all $(x, \overline{x}) \in \mathbb{R}^n \times \mathbb{R}^n$ and $t > 0$,*

$$x_k^T Q f(t_k, x_k, \overline{x}_k) + \frac{1}{2} trace\big(g^T(t_k, x_k, \overline{x}_k) Q g(t_k, x_k, \overline{x}_k)\big) \leq \alpha x^T Q x + \beta \overline{x}^T Q \overline{x}, \tag{2.7}$$

with $\alpha + \beta < 0$. Then, the trivial solution of equation (2.1) is mean-square stable.

3. Asymptotically Mean-Square Stability Analysis

In this section, we discuss the asymptotically mean-square stability of SLT approximation $\{x_k\}_{k\geq 0}$ for $\theta \in [0,1]$. For the purpose of stability, assume that $f(t,0,0) = g(t,0,0) = 0$. This shows that (2.1) admits a trivial solution.

Theorem 3.1. *Let all the conditions in Theorem 2.2 hold, and suppose that there exist two positive constants σ_1 and σ_2 such that, the function f satisfies the following linear growth condition:*

$$f^T(t, x, \overline{x}) Q f(t, x, \overline{x}) \leq \sigma_1 x^T Q x + \sigma_2 \overline{x}^T Q \overline{x}, \tag{3.1}$$

for all $(x, \overline{x}) \in \mathbb{R}^n \times \mathbb{R}^n$, and $t > 0$.

(i) *Then the SLT approximation (2.4) with $\theta \in [\frac{1}{2}, 1]$ is asymptotically mean-square stable for any stepsize $\Delta > 0$.*

(ii) *Let $\theta \in [0, \frac{1}{2})$. Then, there exists a stepsize bound Δ^* depending on θ, such that for any $\Delta \in (0, \Delta^*)$ the SLT approximation (2.4) is asymptotically mean-square stable.*

Proof. *Case (i)*: We can compute from (2.4) that:

$$
\begin{aligned}
&\big(x_{k+1}-\theta f(t_{k+1},x_{k+1},\overline{x}_{k+1})\Delta\big)^T Q\big(x_{k+1}-\theta f(t_{k+1},x_{k+1},\overline{x}_{k+1})\Delta\big)\\
&\qquad=\big(x_k+(1-\theta)f\big(t_k,x_k,\overline{x}_k\big)\Delta+g\big(t_k,x_k,\overline{x}_k\big)\Delta W_k\big)^T\\
&\qquad\quad\times Q\big(x_k+(1-\theta)f\big(t_k,x_k,\overline{x}_k\big)\Delta+g\big(t_k,x_k,\overline{x}_k\big)\Delta W_k\big)\\
&\qquad=\big(x_k-\theta f(t_k,x_k,\overline{x}_k)\Delta\big)^T Q\big(x_k-\theta f(t_k,x_k,\overline{x}_k)\Delta\big)\\
&\qquad\quad+(1-2\theta)f^T(t_k,x_k,\overline{x}_k)Qf(t_k,x_k,\overline{x}_k)\Delta^2\\
&\qquad\quad+2x_k^TQf(t_k,x_k,\overline{x}_k)\Delta+\Delta W_k^Tg^T(t_k,x_k,\overline{x}_k)Qg(t_k,x_k,\overline{x}_k)\Delta W_k\\
&\qquad\quad+2\Delta W_k^Tg^T(t_k,x_k,\overline{x}_k)Q\big(x_k+(1-\theta)f\big(t_k,x_k,\overline{x}_k\big)\Delta\big). \qquad (3.2)
\end{aligned}
$$

Since $W(t)$ is a Brownian motion, we have $\Delta W_k \sim N(0,\Delta I_d)$, where I_d is the identity matrix. Noting that $g^T(t_k,x_k,\overline{x}_k)Qg(t_k,x_k,\overline{x}_k)$ is independent of ΔW_k, we can obtain:

$$
\mathbb{E}\Big(\Delta W_k^Tg^T(t_k,x_k,\overline{x}_k)Qg(t_k,x_k,\overline{x}_k)\Delta W_k\Big)=\Delta\mathbb{E}\Big(trace\big(g^T(t_k,x_k,\overline{x}_k)Qg(t_k,x_k,\overline{x}_k)\big)\Big). \qquad (3.3)
$$

By taking expectation at both sides of (3.2), by using (3.3) and the fact that ΔW_k is independent on x_k and $\overline{x}_k$, we can obtain:

$$
\begin{aligned}
Z_{k+1}&:=\mathbb{E}\Big(\big(x_{k+1}-\theta f(t_{k+1},x_{k+1},\overline{x}_{k+1})\Delta\big)^TQ\big(x_{k+1}-\theta f(t_{k+1},x_{k+1},\overline{x}_{k+1})\Delta\big)\Big)\\
&=Z_k+(1-2\theta)\mathbb{E}\Big(f^T(t_k,x_k,\overline{x}_k)Qf(t_k,x_k,\overline{x}_k)\Big)\Delta^2\\
&\quad+2\Delta\mathbb{E}\Big(x_k^TQf(t_k,x_k,\overline{x}_k)+\frac{1}{2}trace\big(g^T(t_k,x_k,\overline{x}_k)Qg(t_k,x_k,\overline{x}_k)\big)\Big). \qquad (3.4)
\end{aligned}
$$

Now by using the monotone condition (2.7), we may compute:

$$
\begin{aligned}
Z_{k+1}&\le Z_k+(1-2\theta)\mathbb{E}\Big(f^T(t_k,x_k,\overline{x}_k)Qf(t_k,x_k,\overline{x}_k)\Big)\Delta^2\\
&\quad+2\Delta\mathbb{E}\Big(\alpha x^TQx+\beta\overline{x}^TQ\overline{x}\Big). \qquad (3.5)
\end{aligned}
$$

Summing up relation (3.5) from $j=0$ to $j=k$ yields:

$$
\begin{aligned}
Z_{k+1}&\le Z_0+(1-2\theta)\Delta^2\sum_{j=0}^{k}\mathbb{E}\Big(f^T(t_j,x_j,\overline{x}_j)Qf(t_j,x_j,\overline{x}_j)\Big)\\
&\quad+2\Delta\alpha\sum_{j=0}^{k}\mathbb{E}\Big(x_j^TQx_j\Big)+2\Delta\beta\sum_{j=0}^{k}\mathbb{E}\Big(\overline{x}_j^TQ\overline{x}_j\Big). \qquad (3.6)
\end{aligned}
$$

Moreover, it follows from relation (2.5) that:

$$
\mathbb{E}\Big(\overline{x}_j^TQ\overline{x}_j\Big)=\delta\mathbb{E}\Big(x_{j-m+1}^TQx_{j-m+1}\Big)+(1-\delta)\mathbb{E}\Big(x_{j-m}^TQx_{j-m}\Big),
$$

which gives:

$$\sum_{j=0}^{k}\mathbb{E}\Big(\overline{x}_j^TQ\overline{x}_j\Big)\leq\sum_{j=-m+1}^{k-m+1}\mathbb{E}\Big(x_j^TQx_j\Big)+(1-\delta)\mathbb{E}\Big(x_{-m}^TQx_{-m}\Big). \tag{3.7}$$

In addition, due to the fact that $k-m+1\leq k$ and $\delta\in[0,1)$, we have:

$$\begin{aligned}\sum_{j=-m+1}^{k-m+1}&\mathbb{E}\Big(x_j^TQx_j\Big)+(1-\delta)\mathbb{E}\Big(x_{-m}^TQx_{-m}\Big)\\&=\sum_{j=-m+1}^{-1}\mathbb{E}\Big(x_j^TQx_j\Big)+\sum_{j=0}^{k-m+1}\mathbb{E}\Big(x_j^TQx_j\Big)+(1-\delta)\mathbb{E}\Big(x_{-m}^TQx_{-m}\Big)\\&\leq\sum_{j=-m+1}^{-1}\mathbb{E}\Big(x_j^TQx_j\Big)+\sum_{j=0}^{k}\mathbb{E}\Big(x_j^TQx_j\Big)+(1-\delta)\max_{-m\leq j\leq -1}\mathbb{E}\Big(x_j^TQx_j\Big)\\&\leq\sum_{j=0}^{k}\mathbb{E}\Big(x_j^TQx_j\Big)+(m-\delta)\max_{-m\leq j\leq -1}\mathbb{E}\Big(x_j^TQx_j\Big).\end{aligned} \tag{3.8}$$

We know that $(m-\delta)\Delta=\tau$. Now, substitution of relation (3.8) into (3.6) yields:

$$\begin{aligned}Z_{k+1}&\leq Z_0+(1-2\theta)\Delta^2\sum_{j=0}^{k}\mathbb{E}\Big(f^T(t_j,x_j,\overline{x}_j)Qf(t_j,x_j,\overline{x}_j)\Big)\\&+2\Delta\alpha\sum_{j=0}^{k}\mathbb{E}\Big(x_j^TQx_j\Big)+2\Delta\beta\Big(\sum_{j=0}^{k}\mathbb{E}\Big(x_j^TQx_j\Big)+(m-\delta)\max_{-m\leq j\leq -1}\mathbb{E}\Big(x_j^TQx_j\Big)\Big)\\&=Z_0+(1-2\theta)\Delta^2\sum_{j=0}^{k}\mathbb{E}\Big(f^T(t_j,x_j,\overline{x}_j)Qf(t_j,x_j,\overline{x}_j)\Big)\\&+2\Delta(\alpha+\beta)\sum_{j=0}^{k}\mathbb{E}\Big(x_j^TQx_j\Big)+2\beta\tau\max_{-m\leq j\leq -1}\mathbb{E}\Big(x_j^TQx_j\Big).\end{aligned} \tag{3.9}$$

Noting that $\theta\in\big[\frac{1}{2},1\big]$ and $\alpha+\beta<0$, it follows from inequality (3.9) that:

$$\sum_{j=0}^{k}\mathbb{E}\Big(x_j^TQx_j\Big)\leq-\frac{Z_0}{2\Delta(\alpha+\beta)}-\frac{\beta\tau\max_{-m\leq j\leq -1}\mathbb{E}\Big(x_j^TQx_j\Big)}{\Delta(\alpha+\beta)}. \tag{3.10}$$

By using inequality (3.10), we see that $\sum_{j=0}^{k}\mathbb{E}\big(x_j^TQx_j\big)<\infty$.

So we have $\lim_{k\to\infty}\mathbb{E}\big(x_k^TQx_k\big)=0$, which implies that the scheme is asymptotically mean-square stable for all stepsize $\Delta>0$.

Case (ii): By applying the linear growth condition (3.1) for $\theta \in \big[0, \frac{1}{2}\big)$, we can obtain:

$$\begin{aligned}
&\sum_{j=0}^{k} \mathbb{E}\Big(f^T(t_j, x_j, \overline{x}_j) Q f(t_j, x_j, \overline{x}_j)\Big) \\
&\quad \le \sigma_1 \sum_{j=0}^{k} \mathbb{E}\Big(x_j^T Q x_j\Big) + \sigma_2 \sum_{j=0}^{k} \mathbb{E}\Big(\overline{x}_j^T Q \overline{x}_j\Big) \\
&\quad \le \sigma_1 \sum_{j=0}^{k} \mathbb{E}\Big(x_j^T Q x_j\Big) + \sigma_2 \Big(\sum_{j=-m+1}^{k-m+1} \mathbb{E}\Big(x_j^T Q x_j\Big) + (1-\delta) \mathbb{E}\Big(x_{-m}^T Q x_{-m}\Big)\Big) \\
&\quad \le \big(\sigma_1 + \sigma_2\big) \sum_{j=0}^{k} \mathbb{E}\Big(x_j^T Q x_j\Big) + \sigma_2 (m-\delta) \max_{-m \le j \le -1} \mathbb{E}\Big(x_j^T Q x_j\Big),
\end{aligned} \tag{3.11}$$

which together with (3.9) gives:

$$\begin{aligned}
Z_{k+1} &\le Z_0 + (1-2\theta)\Delta^2 \Big(\big(\sigma_1 + \sigma_2\big) \sum_{j=0}^{k} \mathbb{E}\Big(x_j^T Q x_j\Big) \\
&\quad + \sigma_2 (m-\delta) \max_{-m \le j \le -1} \mathbb{E}\Big(x_j^T Q x_j\Big)\Big) \\
&\quad + 2\Delta\big(\alpha + \beta\big) \sum_{j=0}^{k} \mathbb{E}\Big(x_j^T Q x_j\Big) + 2\beta\tau \max_{-m \le j \le -1} \mathbb{E}\Big(x_j^T Q x_j\Big) \\
&\le Z_0 + \Delta\Big((1-2\theta)\big(\sigma_1 + \sigma_2\big)\Delta + 2\big(\alpha+\beta\big)\Big) \sum_{j=0}^{k} \mathbb{E}\Big(x_j^T Q x_j\Big) \\
&\quad + \Big(2\beta\tau + \sigma_2(1-2\theta)(m-\delta)\Delta^2\Big) \max_{-m \le j \le -1} \mathbb{E}\Big(x_j^T Q x_j\Big).
\end{aligned} \tag{3.12}$$

We shall choose stepsize $\Delta \in (0, \Delta^*)$, where Δ^* defined as follows:

$$\Delta^* = -\frac{2(\alpha+\beta)}{(1-2\theta)\big(\sigma_1+\sigma_2\big)}. \tag{3.13}$$

In fact, if $\Delta \in (0, \Delta^*)$, we have that $\Delta\Big((1-2\theta)\big(\sigma_1+\sigma_2\big)\Delta + 2\big(\alpha+\beta\big)\Big) < 0$. Similar to the proof of (3.10) in *Case (i)*, we can obtain:

$$\begin{aligned}
\sum_{j=0}^{k} \mathbb{E}\Big(x_j^T Q x_j\Big) &\le \frac{Z_0}{-\Delta\Big((1-2\theta)\big(\sigma_1+\sigma_2\big)\Delta + 2\big(\alpha+\beta\big)\Big)} \\
&\quad + \frac{\Big(2\beta\tau + \sigma_2(1-2\theta)(m-\delta)\Delta^2\Big) \max_{-m \le j \le -1} \mathbb{E}\Big(x_j^T Q x_j\Big)}{-\Delta\Big((1-2\theta)\big(\sigma_1+\sigma_2\big)\Delta + 2\big(\alpha+\beta\big)\Big)}.
\end{aligned} \tag{3.14}$$

By inequality (3.14), it is easy to see that $\sum_{j=0}^{k} \mathbb{E}\big(x_j^T Q x_j\big) < \infty$.

Thus we have $\lim_{k\to\infty} \mathbb{E}\big(x_k^T Q x_k\big) = 0$, which implies that the proposed scheme is asymptotically mean-square stable for any stepsize $\Delta \in (0, \Delta^*)$. This completes the proof of Theorem 3.1. □

4. Numerical Illustrations

In this section, we give two numerical examples to illustrate the obtained theoretical results in the previous section. Note that in all of the test cases, we choose $\psi(t)$ such that $\mathbb{E}\big(\sup_{-\tau\le t\le 0} |\psi(t)|^2\big) < +\infty$.

Example 4.1. Let us consider one-dimensional nonlinear SDDE as follows:

$$dx(t) = \Big(-8x(t)+2\sin(x(t-1))\Big)dt+\Big(\frac{x(t)}{1+x^2(t)}+\sin(x(t-1))\Big)dW(t), \quad t>0, \tag{4.1}$$

with initial data $x(t) = 1$ for $t \in [-1, 0]$. For any $x, \overline{x} \in \mathbb{R}$, $t > 0$, and for any positive number Q we have:

$$\begin{aligned} x^T Q f(t, x, \overline{x}) &+ \frac{1}{2} trace\Big(g^T(t, x, \overline{x}) Q g(t, x, \overline{x})\Big) \\ &= -8x^T Q x + 2x^T Q \sin(\overline{x}) + \frac{1}{2} trace\Big(\big(\frac{x}{1+x^2}\big)^T Q \big(\frac{x}{1+x^2}\big)\Big) \\ &+ trace\Big(\big(\frac{x}{1+x^2}\big)^T Q \sin(\overline{x})\Big) + \frac{1}{2} trace\Big(\big(\sin(\overline{x})\big)^T Q \sin(\overline{x})\Big) \\ &\le -6x^T Q x + 2\overline{x}^T Q \overline{x}, \end{aligned} \tag{4.2}$$

where we used the fact that $\left|\frac{x}{1+x^2}\right| \le x$, $|\sin(\overline{x})| \le |\overline{x}|$ and the Cauchy-Schwartz inequality. Moreover, similar arguments yield:

$$f^T(t, x, \overline{x}) Q f(t, x, \overline{x}) \le 80x^T Q x + 20\overline{x}^T Q \overline{x}. \tag{4.3}$$

Therefore, Theorem 2.2 and the linear growth condition (3.1) hold with $Q = 1$, $\alpha = -6$, $\beta = 2$, $\sigma_1 = 80$ and $\sigma_2 = 20$. Substituting these value in (3.13) yields $\Delta^* = \frac{8}{100(1-2\theta)}$. To empirically check these theoretical findings, the SLT approximation of the SDDE (4.1) is performed using both $\theta = 0.25$ and $\theta = 0.75$. Therefore, according to Theorem 3.1, if $\theta = 0.25$, the SLT scheme is stable for stepsizes $\Delta < 0.16$, where as, if $\theta = 0.75$, the SLT scheme is stable for any stepsizes $\Delta > 0$.

The results obtained are reported in Figures 1-3. As we may see in Figure 1, for the case $\theta = 0.25$, if $\Delta \le 2^{-3}$, the SLT scheme turns out to be asymptotically mean-square stable. However, asymptotic mean-square stability is lost if $\Delta \ge 2^{-2}$, which is shown in Figure 2. Also, Figure 3 shows that the SLT scheme with $\theta = 0.75$ can share the asymptotic mean-square stability even for any large stepsize $\Delta > 0$.

FIGURE 1. Stable numerical solutions of SLT scheme for different stepsizes with $\theta = 0.25$.

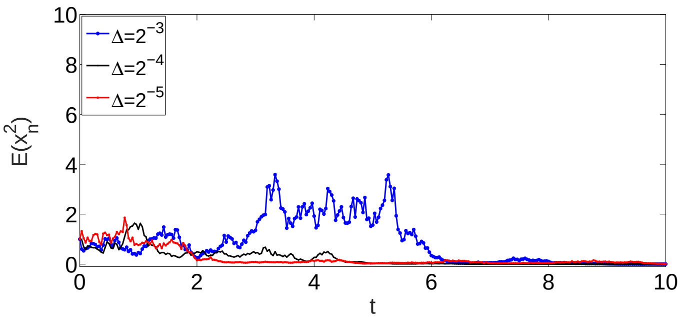


FIGURE 2. Unstable numerical solutions of SLT scheme for different stepsizes with $\theta = 0.25$.

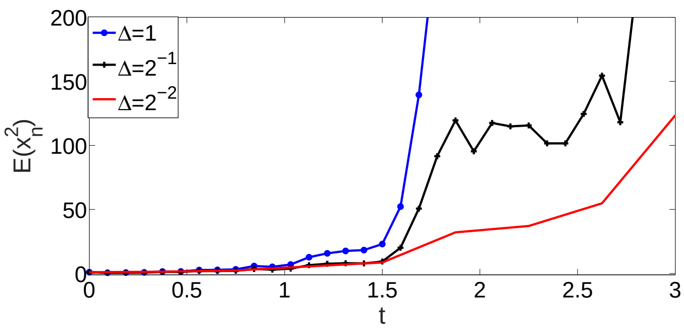


FIGURE 3. Stable numerical solutions of SLT scheme for different stepsizes with $\theta = 0.75$.

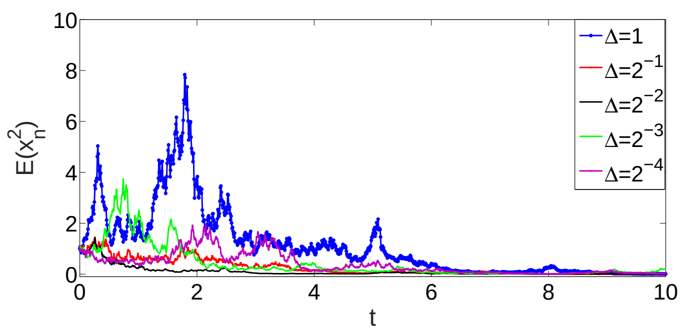


FIGURE 4. Unstable and stable numerical solutions of SLT scheme for different stepsizes with $\theta = 0.25$.

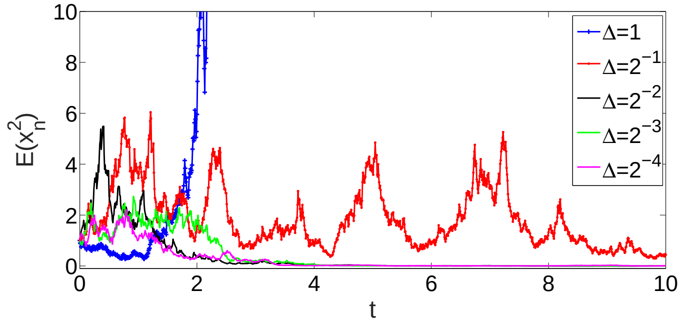
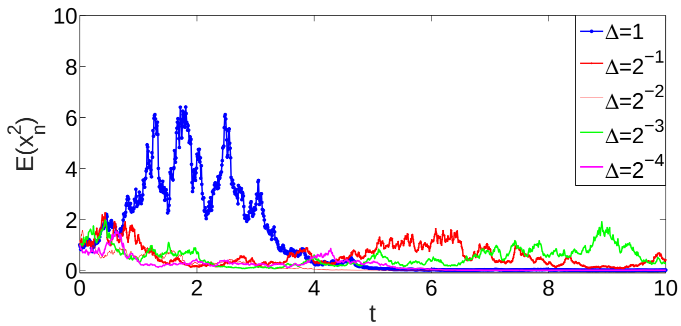


FIGURE 5. Stable numerical solutions of SLT scheme for different stepsizes with $\theta = 0.75$.



Example 4.2. Let us consider two-dimensional nonlinear SDDE as follows:

$$dx(t) = Vx(t)dt + g(t, x(t), x(t-\tau))dW(t), \quad t > 0, \tag{4.4}$$

with initial data:

$$\psi = \big(\psi_1(t), \psi_2(t)\big)^T \in C\big([-1, 0]; \mathbb{R}^2\big), \tag{4.5}$$

where $\psi_1(t) = \sin(2\pi t)$, $\psi_2(t) = \cos(2\pi t)$ and $W(t)$ is a scalar Brownian motion. We set the following matrix V as follows:

$$V = \begin{bmatrix} -2 & 1 \\ -1 & -1 \end{bmatrix}, \tag{4.6}$$

and $x = \big(x_1(t), x_2(t)\big)^T$, $\overline{x} = \big(x_1(t-1), x_2(t-1)\big)^T$, $g(t, x, \overline{x}) = \big(\frac{1}{4}\ln\big(1 + \overline{x}_1^2(t)\big), \frac{1}{4}\sin^2(\overline{x}_2(t))\big)^T$. We are going to show that Theorem 2.2 and the linear growth condition (3.1) hold if we choose, for example, the following symmetric positive definite matrix Q:

$$Q = \begin{bmatrix} 8 & 1 \\ 1 & 8 \end{bmatrix}. \tag{4.7}$$

In fact, we immediately obtain:

$$x^T Q x = 8x_1^2 + 2x_1x_2 + 8x_2^2 \leq 9x_1^2 + 9x_2^2, \tag{4.8}$$

$$\overline{x}^T Q \overline{x} = 8\overline{x}_1^2 + 2\overline{x}_1\overline{x}_2 + 8\overline{x}_2^2 \geq \overline{x}_1^2 + \overline{x}_2^2, \tag{4.9}$$

$$\begin{aligned} x^T Q f(t, x, \overline{x}) &= -17x_1^2 - 3x_1x_2 - 7x_2^2 \\ &\leq -\frac{31}{2}x_1^2 - \frac{11}{2}x_2^2 \\ &\leq -\frac{11}{2}|x|^2. \end{aligned} \tag{4.10}$$

Moreover, owing to the fact that $\sin^2(\overline{x}_2) \leq |\overline{x}_2|$, $\left|\ln\left(1+\overline{x}_1^2\right)\right| \leq |\overline{x}_1|$ and to the Cauchy-Schwartz inequality, we have:

$$\begin{aligned} \frac{1}{2} trace\Big(g^T(t, x, \overline{x}) Q g(t, x, \overline{x})\Big) &\leq \frac{9}{16}\big(\ln\left(1+\overline{x}_1^2\right)\big)^2 + \frac{9}{16}\sin^4(\overline{x}_2) \\ &\leq \frac{9}{16}\big(\overline{x}_1^2 + \overline{x}_2^2\big) \leq \frac{9}{16}|\overline{x}|^2. \end{aligned} \tag{4.11}$$

Putting (4.10) and (4.11) together yields:

$$x^T Q f(t, x, \overline{x}) + \frac{1}{2} trace\Big(g^T(t, x, \overline{x}) Q g(t, x, \overline{x})\Big) \leq -\frac{11}{2}|x|^2 + \frac{9}{16}|\overline{x}|^2. \tag{4.12}$$

In addition, it can be immediately checked that:

$$\begin{aligned} f^T(t, x, \overline{x}) Q f(t, x, \overline{x}) &= 24x_1^2 + 6x_1x_2 \\ &\leq 27x_1^2 + 3\big(x_1^2 + x_2^2\big) \leq 27|x|^2. \end{aligned} \tag{4.13}$$

It follows that Theorem 2.2 and the linear growth condition (3.1) hold with Q as in (4.7), $\alpha = -\frac{11}{2}$, $\beta = \frac{9}{16}$, $\sigma_1 = 27$ and $\sigma_2 = 0$. Substituting these values in (3.13) yields $\Delta^* = \frac{79}{216(1-2\theta)}$. To empirically check these theoretical findings, the SLT approximation of the SDDE (4.4) is performed using both $\theta = 0.25$ and $\theta = 0.75$.

Therefore, according to Theorem 3.1, if stepsize $\theta = 0.25$, the SLT scheme is stable for stepsizes $\Delta < 0.73$, whereas, if $\theta = 0.75$, the SLT scheme is stable for any stepsize $\Delta > 0$. The results obtained are reported in Figures 4 and 5. As we may see in Figure 4, for the case $\theta = 0.25$, if $\Delta \leq 2^{-1}$, the SLT scheme turns out to be asymptotically mean square-stable. However, asymptotic mean-square stability is lost if $\Delta \geq 1$. Also, Figure 5 shows that the SLT scheme with $\theta = 0.75$ can share the asymptotic mean-square stability even for any large stepsize $\Delta > 0$.

5. Conclusion

In this paper, we investigated stochastic linear theta scheme for n-dimensional SDDEs under a coupled monotone condition on drift and diffusion coefficients. In this regard we examined the asymptotic mean-square stability for these kinds of equations. The parameter θ can extend the values of stepsize Δ in the asymptotic mean-square stability for SLT scheme. We obtained the stability results of the SLT scheme numerically, which is shown in Figures 1-5.

References

[1] D. Ahmadian, O. Farkhondeh Rouz, and L. V. Ballestra, *Stability analysis of split-step θ-Milstein method for a class of n-dimensional stochastic differential equations*, Appl. Math. Comput, *348* (2019), 413–424.

[2] C. T. H. Baker and E. Buckwar, *Numerical analysis of explicit one-step methods for stochastic delay differential equations*, LMS J. Comput. Math, *3* (2000), 315–335.

[3] C. T. H. Baker and E. Buckwar, *Exponential stability in p-th mean of solutions and of convergent Euler-type solutions of stochastic delay differential equations*, J. Comput. Appl. Math, *184* (2005), 404–427.

[4] E. Buckwar, R. Kuske, S. E. A. Mohammed, and T. Shardow, *The weak Euler scheme for stochastic differential delay equations*, Available on: *http* : *//eprints.ma.man.ac.uk*/149/01/*weak_mal_web*.pdf.

[5] E. Buckwar, *Introduction to the numerical analysis of stochastic delay differential equations*, J. Comput. Appl. Math, *125* (2000), 297–307.

[6] C. D. Charalambous, R. J. C. Bultitude, X. Li, and J. Zhan, *The estimation of parameters for stochastic differential equations using neural networks*, IEEE Trans. Wireless Commun, *7* (2008), 434–439.

[7] L. Chen, and F. Wu, *Almost sure exponential stability of the backward Euler-Maruyama method for stochastic delay differential equations with monotone-type condition*, J. Comput. Appl. Math, *282* (2015), 44–53.

[8] X. Ding, Q. Ma, and L. Zhang, *Convergence and stability of the split-step θ-method for stochastic differential equations*, Comput. Math. Appl, *60* (2010), 1310–1321.

[9] N. T. Dung, *Fractional stochastic differential equations with applications to finance*, J. Math. Anal. Appl, *397* (2013), 334–348.

[10] O. Farkhondeh Rouz, and D. Ahmadian, *Stability analysis of two classes of improved backward Euler methods for stochastic delay differential equations of neutral type*, Comput. Methods Differ. Equ, *3* (2017), 201–213.

[11] O. Farkhondeh Rouz, D. Ahmadian, and M. Milev, *Exponential mean-square stability of two classes of theta Milstein methods for stochastic delay differential equations*, AIP Conference Proceedings, *1910* (2017), 060015-1–060015-14.

[12] D. J. Higham, X. Mao, and C. Yuan, *Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations*, SIAM J. Numer. Anal, *45* (2007), 592–609.

[13] Y. Hu, F. Wu, and C. Huang, *Robustness of exponential stability of a class of stochastic functional differential equations with infinite delay*, Automatica, *45* (2009), 2577–2584.

[14] C. Huang, *Mean square stability and dissipativity of two classes of theta methods for systems of stochastic delay differential equations*, J. Comput. Appl. Math, *259* (2014), 77–86.

[15] C. Huang, S. Gan, and D. Wang, *Delay-dependent stability analysis of numerical methods for stochastic delay differential equations*, J. Comput. Appl. Math, *236* (2012), 3514–3527.

[16] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Appl. Math. (New York), Springer-Verlag, Berlin, *23* (1992).

[17] P. E. Kloeden and T. Shardlow, *The Milstein Scheme for Stochastic Delay Differential Equations Without Using Anticipative Calculus*, Stoch. Anal. Appl, *30* (2012), 181–202.

[18] X. Li and X. Fub, *Stability analysis of stochastic functional differential equations with infinite delay and its application to recurrent neural networks*, Random Oper. Stoch. Equ, *234* (2015), 407–417.

[19] X. Li and W. Cao, *On mean-square stability of two-step Maruyama methods for nonlinear neutral stochastic delay differential equations*, Appl. Math. Comput, *261* (2015), 373–381.

[20] X. R. Mao, *Exponential Stability of Stochastic Differential Equations*, Marcel Dekker, New York, (1994).

[21] X. Mao, *Stochastic Differential Equations and their Applications*, Horwood, Chichester, (1997).

[22] X. Mao, *Exponential stability of equidistant Euler-maruyama approximations of stochastic differential delay equations*, J. Comput. Appl. Math, *200* (2007), 297–316.

[23] X. Mao, C. Yuan, and J. Zouc, *Stochastic differential delay equations of population dynamics*, J. Math. Anal. Appl, *304* (2005), 296–320.

[24] S. E. A. Mohammed, *Stochastic Functional Differential Equations*, Pitman, London, (1984).

[25] X. Qu and C. Huang, *Delay-dependent exponential stability of the backward Euler method for nonlinear stochastic delay differential equations*, Int. J. Comput. Math, *89* (2012), 1039–1050.

[26] Y. Saito and T. Mitsui, *T-stability of numerical scheme for stochastic differential equations, in:Contributions in Numerical Mathematics*, World Sci. Ser. Appl. Anal, *2* (1993), 333–344.

[27] X. Wang, S. Gan, and D. Wang, *θ-Maruyama methods for nonlinear stochastic differential delay equations*, Appl. Numer. Math, *98* (2015), 38–58.

[28] F. Wu, X. Mao, and P. E. Kloeden, *Discrete Razumikhin-type technique and stability of the Euler maruyama method to stochastic functional differential equations*, Discrete Contin. Dyn. Syst, *33* (2013), 885–903.

[29] F. Wu, X. Mao, and P. E. Kloeden, *Almost sure exponential stability of the Euler-Maeuyama approximations for stochastic functional differential equations*, Random Oper. Stoch. Equ, *19* (2011), 165–186.

[30] P. Wilmott, *Derivatives: The Theory and Practice of Financial Engineering*, John Wiley and Sons, New York, (1998).

[31] X. Zong, F. Wu, and C. Huang, *Theta schemes for SDDEs with non-globally lipschitz continuous coefficients*, J. Comput. Appl. Math, *278* (2015), 258–277.

[32] X. Zong and F. Wu, *Exponential stability of the exact and numerical solutions for neutral stochastic delay differential equations*, Appl. Math. Modelling, *40* (2016), 19–30.

[33] X. Zong, F. Wu, and G. Xu, *Convergence and stability of two classes of theta-Milstein schemes for stochastic differential equations*, J. Comput. Appl. Math, *336* (2018), 8–29.

[34] Y. Zhu, M. Zhu, and J. Li, *Almost sure exponential stability for time-changed stochastic differential equations*, J. Nonlinear Sci. Appl, *10* (2017), 5985–5998.