

A numerical scheme for diffusion-convection equation with piecewise constant arguments

Mojgan Esmailzadeh

Department of Applied Mathematics, Lahijan Branch,
Islamic Azad University, Lahijan, Iran.
E-mail: m.esmailzadeh11@gmail.com

Hashem Saberi Najafi*

Department of Mathematics, Lahijan Branch,
Islamic Azad University, Lahijan, Iran.
E-mail: hnajafi@guilan.ac.ir

Hossein Aminikhah

Faculty of Mathematical Sciences, Department of Applied Mathematics
and Computer Science, University of Guilan, Rasht, Iran.
E-mail: aminikhah@guilan.ac.ir

Abstract This article is concerned with using a finite difference method, namely the θ -methods, to solve the diffusion-convection equation with piecewise constant arguments. The stability of this scheme is also obtained. Since there are not many published results on the numerical solution of this sort of differential equation and because of the importance of the above equation in the physics and engineering sciences, we have decided to study and present a stable numerical solution for the above mentioned problem. At the end of article some experiments are done to demonstrate the stability of the scheme. We also draw the figures for the numerical and analytical solutions which confirm our results. The numerical solutions have also been compared with analytical solutions.

Keywords. Diffusion-convection equation, Piecewise constant arguments, θ -methods, Asymptotically stability.

2010 Mathematics Subject Classification. 65L05, 37M05, 97N40, 30C30.

1. INTRODUCTION

The diffusion-convection equation or advection-diffusion equation is one of the most important partial differential equations in modeling of the real world phenomena, like biology, mechanics, and electronics.

In general, the applications of this equation lie in the fluid dynamics, heat transfer and mass transfer. There are many numerical models for solving these sort of equations in the literature, for example Dehghan [2, 3], Mohebbi and Dehghan [6]

Received: 27 December 2018 ; Accepted: 27 April 2019.

* Corresponding author.

have developed numerical methods for solving one and three-dimensional advection-diffusion equations. In this article, we study the diffusion-convection equation with piecewise constant argument.

The studies of differential equations with piecewise continuous arguments (EPCA) were initiated by Wiener [14], Cook and Wiener [1], and Shah and Wiener [9]. In the book of Wiener [14], the general theory for EPCA has been investigated. Wiener in [14], showed that partial differential equations (PDE) with piecewise constant time naturally arise in the process of approximating PDE by using piecewise constant arguments. In chapter 3 of the Wiener's book [14], we see that,

“if in the equation

$$u_t = a^2 u_{xx} - bu,$$

which describes heat flow in a rod with both diffusion $a^2 u_{xx}$ along the rod and heat loss (or gain) across the lateral sides of the rod, the lateral heat change is measured at discrete times, then we get an equation with piecewise constant argument (EPCA)

$$u_t(x, t) = a^2 u_{xx}(x, t) - bu(x, nh), \quad t \in [nh, (n+1)h], \quad n = 0, 1, \dots, \quad (1.1)$$

where $h > 0$ is some constant. This equation can be written in the form

$$u_t(x, t) = a^2 u_{xx}(x, t) - bu(x, [t/h]h),$$

where $[.]$ designates the greatest-integer function. The diffusion-convection equation $u_t = a^2 u_{xx} - ru_x$ describes, for instance, the concentration $u(x, t)_x$ of a pollutant carried along in a stream moving with velocity r . The term $a^2 u_{xx}$ is the diffusion contribution and $-ru_x$ is the convection component. If the convection part is measured at discrete time nh , the process results in the equation

$$u_t(x, t) = a^2 u_{xx}(x, t) - ru_x(x, [t/h]h).$$

These examples indicate at the considerable potential of EPCA as an analytical and computational tool in solving some complicated problems of mathematical physics. Therefore, it is important to investigate boundary-value problems (BVP) and initial-value problems (IVP) for EPCA in partial derivatives, and explore the influence of certain discontinuous delays on the behavior of solutions to some typical problems of mathematical physics.”

In the literature, considerable works have been done on the study of the (EPCA). Some of the important works in the theoretical and numerical methods are listed below.

Theoretical works:

1. Generalized solutions of functional differential equations, a book published in 1993 [14].
2. Boundary value problems for partial differential equations with piecewise constant delay [15].
3. Partial differential equations with piecewise constant delay [16].
4. A survey of partial differential equations with piecewise continuous arguments [18].
5. A wave equation with discontinuous time delay [17].



6. Bounded solutions of nonlinear parabolic equations with time delay [7].
7. Almost periodic solutions of nonlinear equations with time delay [8].
8. Existence, computability, and stability for solutions of the diffusion equation with general piecewise constant argument [11].

Numerical works:

9. Analytical and numerical stability of partial differential equations with piecewise constant arguments [12].
10. Stability of θ -methods [4, 5, 10].
11. Stability analysis [13].
12. Retarded differential equations with piecewise constant delay [1].
13. Advanced differential equations with piecewise constant argument deviations [9].

In this work, we consider the following problem

$$\begin{cases} u_t(x, t) = a^2 u_{xx}(x, t) - bu_x(x, [t]), & t > 0, \\ u(0, t) = u(1, t) = 0, \\ u(x, 0) = v(x), \end{cases} \quad (1.2)$$

where $a, b \in \mathbb{R}$, $u : \Omega \rightarrow \mathbb{R}$, $v : [0, 1] \rightarrow \mathbb{R}$, and $[.]$ designates the greatest integer function. The main goal of the work is to investigate the stability of numerical solution for 1.2 and obtaining an approximate analytical solution for comparison with numerical results.

This paper is organized as follows.

In Section 2, the existence of the solution and the conditions under which the analytical solution of 1.2 is asymptotically stable are presented and also the formula of the analytic solution of 1.2, which has been found in [14], is provided.

In Section 3, we provide the numerical stability analysis for the θ -methods to 1.2, using the similar methods in [12, 18].

In Section 4, some numerical experiments are presented, that support our analysis and confirm the results. Also, in this section, we managed to draw our numerical solution for comparison with the analytical solutions from [14].

2. ANALYTIC SOLUTION AND STABILITY CONDITIONS

Definition 2.1. ([14]) A solution of 1.2 is a function $u(x, t)$ satisfies the following conditions :

- (i) $u(x, t)$ is continuous in $\Omega = [0, 1] \times [0, \infty)$,
- (ii) The partial derivative u_t, u_x, u_{xx} exist and are continuous in Ω with the possible exception of the points (x, nh) , where one- sided derivatives exist ($n = 0, 1, 2, \dots$),
- (iii) $u(x, t)$ satisfies $u_t(x, t) = a^2 u_{xx}(x, t) - bu_x(x, [t])$ in Ω , with the possible exception of the points (x, nh) , and conditions

$$u(0, t) = u(1, t) = 0, \quad u(x, 0) = v(x).$$

Definition 2.2. ([14]) If any solution $u(x, t)$ of 1.2 satisfies $\lim_{t \rightarrow \infty} u(x, t) = 0$ then the zero solution of 1.2 is called asymptotically stable.



2.1. **Analytic solution.** ([14]) Let $u_n(x, n)$ be the analytic solution of (1.2) for $t \in [n, n + 1)$, then the analytic solution under the conditions is

$$u_n(x, t) = u_n(x) + \sum_{j=1}^{\infty} \frac{\sqrt{2} T_{nj}(nh) \left(1 - e^{-a^2 \pi^2 j^2 (t-nh)}\right) \sin(\pi j x)}{a^2 \pi^2 j^2}, \tag{2.1}$$

where

$$T_{nj}(nh) = -a^2 \pi^2 j^2 \sqrt{2} \int_0^1 u_n(x) \sin(\pi j x) dx + b \pi j \sqrt{2} \int_0^1 u_n(x) \cos(\pi j x) dx,$$

and $u_n(x) = u_n(x, nh)$.

Given the initial function $u(x, 0) = u_0(x)$, we can find the coefficients $T_{0j}(0)$ and the solution $u_0(x, t)$ on $0 \leq t \leq h$.

Furthermore, continuity of the solution $u(x, t)$ implies

$$u_n(x, (n + 1)h) = u_{n+1}(x, (n + 1)h) = u_{n+1}(x).$$

Since $u_0(x, h) = u_1(x)$, we can calculate the coefficient $T_{1j}(h)$ and the solution $u_1(x, t)$ on $h \leq t \leq 2h$. By the method of steps the solution can be extended to any interval $[nh, (n + 1)h]$.

Remark 2.3. ([14]) According to Definition 2.2 and formula 2.1, the zero solution of 1.2 is asymptotically stable if and only if

$$\lim_{n \rightarrow \infty} u_n(x, t) = 0, \tag{2.2}$$

and 2.2 holds if

$$-a^2 \pi^2 < b < a^2 \pi^2 \frac{e^{a^2 \pi^2 h} + 1}{e^{a^2 \pi^2 h} - 1}. \tag{2.3}$$

3. THE STABILITY OF NUMERICAL SOLUTION

In this section, the numerical asymptotic stability of θ -methods for 1.2 is discussed.

3.1. **θ -methods.** Let Δx and Δt are step-sizes of spatial and time directions which satisfy $\Delta x = \frac{1}{p}$ and $\Delta t = \frac{1}{m}$, respectively, where $p, m \geq 1$ are positive integers. Denote the spatial and time nodes as $x_i = i\Delta x$ and $t_n = n\Delta t$, respectively, and u_i^n as an approximation to $u(x_i, t_n)$. Applying the θ -methods to 1.1, we have

$$\begin{cases} \frac{u_i^{n+1} - u_i^n}{\Delta t} = \theta \left(a^2 \left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} \right) - b \left(\frac{u^h(x_{i+1}, [t_{n+1}]) - u^h(x_i, [t_{n+1}])}{\Delta x} \right) \right) \\ + (1 - \theta) \left(a^2 \left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \right) - b \left(\frac{u^h(x_{i+1}, [t_n]) - u^h(x_i, [t_n])}{\Delta x} \right) \right), \\ u_0^n = u_p^n = 0, \quad n = 0, 1, 2, \dots, \\ u_i^0 = v(x_i), \quad i = 0, 1, 2, \dots, p, \end{cases} \tag{3.1}$$

where $u^h(x_i, [t_n])$ is an approximation to $u(x_i, [t_n])$. If we denote $n = km + l$ ($l = 0, 1, \dots, m - 1, k = 0, 1, 2, \dots$), then both $u^h(x_i, [t_n])$ and $u^h(x_i, [t_{n+1}])$ can be defined as u_i^{km} according to Definition 2.1.



Let $r_1 = \frac{\Delta t}{\Delta x^2}$ and $r_2 = \frac{\Delta t}{\Delta x}$, then 3.1 can be written as

$$\begin{aligned}
 & -a^2\theta r_1 u_{i-1}^{km+l+1} + (1 + 2a^2\theta r_1) u_i^{km+l+1} \\
 & -a^2\theta r_1 u_{i+1}^{km+l+1} = a^2(1 - \theta) r_1 u_{i-1}^{km+l} \\
 & \quad + (1 - 2a^2(1 - \theta) r_1) u_i^{km+l} \\
 & \quad + a^2(1 - \theta) r_1 u_{i+1}^{km+l} - br_2 (u_{i+1}^{km} - u_i^{km}), \tag{3.2}
 \end{aligned}$$

and let $i = 1, 2, \dots, p - 1$, then 3.2 yields as

$$\begin{aligned}
 & \begin{bmatrix} 1 + 2a^2\theta r_1 & -a^2\theta r_1 & \cdots & 0 & 0 \\ -a^2\theta r_1 & 1 + 2a^2\theta r_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 + 2a^2\theta r_1 & -a^2\theta r_1 \\ 0 & 0 & \cdots & -a^2\theta r_1 & 1 + 2a^2\theta r_1 \end{bmatrix} \begin{pmatrix} u_1^{km+l+1} \\ u_2^{km+l+1} \\ \vdots \\ \vdots \\ u_{p-1}^{km+l+1} \end{pmatrix} \\
 & = \begin{bmatrix} \sigma & a^2(1 - \theta) r_1 & \cdots & 0 & 0 \\ a^2(1 - \theta) r_1 & \sigma & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma & a^2(1 - \theta) r_1 \\ 0 & 0 & \cdots & a^2(1 - \theta) r_1 & \sigma \end{bmatrix} \begin{pmatrix} u_1^{km+l} \\ u_2^{km+l} \\ \vdots \\ \vdots \\ u_{p-1}^{km+l} \end{pmatrix} \\
 & + br_2 \begin{bmatrix} 1 & -1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{pmatrix} u_1^{km} \\ u_2^{km} \\ \vdots \\ \vdots \\ u_{p-1}^{km} \end{pmatrix}, \tag{3.3}
 \end{aligned}$$

where $\sigma = 1 - 2a^2(1 - \theta) r_1$.

Denote $u^n = (u_1^n, u_2^n, \dots, u_{p-1}^n)^T$, $n = 0, 1, 2, \dots$ and $v(x) = (v(x_1), v(x_2), \dots, v(x_{p-1}))^T$ and $(p - 1) \times (p - 1)$ triple-diagonal matrix $F_1 = \text{diag}(-1, 2, -1)$ and

$$F_2 = \begin{bmatrix} 1 & -1 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Then 3.3 can be written as

$$\begin{aligned}
 & (I + a^2\theta r_1 F_1) u^{km+l+1} = [I - a^2(1 - \theta) r_1 F_1] u^{km+l} + br_2 F_2 u^{km}, \\
 & u^0 = v(x). \tag{3.4}
 \end{aligned}$$



3.2. Stability analysis of θ -methods.

Definition 3.1. ([12]) If any solution of 3.2 satisfies

$$\lim_{n \rightarrow \infty} u_i^n = 0, \quad i = 1, 2, \dots, p,$$

then the zero solution of 3.1 is called asymptotically stable.

Lemma 3.2. *The matrix F_1 is a positive definite matrix, with the eigenvalues*
 $\lambda_{F_1} = 4 \cos^2 \frac{k\pi}{2p}, \quad k = 1, 2, \dots, p-1.$

From 3.4, we can get

$$u^{km+l+1} = Ru^{km+l} + Su^{km}, \quad l = 0, 1, \dots, m-1, \quad (3.5)$$

where

$$\begin{cases} R = (I + a^2\theta r_1 F_1)^{-1} [I - a^2(1-\theta) r_1 F_1], \\ S = br_2 (I + a^2\theta r_1 F_1)^{-1} F_2. \end{cases}$$

Iteration of 3.5 gives

$$\begin{aligned} u^{km+l+1} &= (R^{l+1} + (R^l + R^{l-1} + \dots + R + I) S) u^{km} \\ &= (R^{l+1} + (R^{l+1} - I)(R - I)^{-1} S) u^{km}. \end{aligned}$$

After simplifying

$$(R - I)^{-1} S = -\frac{b}{a^2} \Delta x F_3,$$

where

$$F_3 = \begin{bmatrix} \frac{1}{2} & 1 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & \frac{1}{2} \end{bmatrix},$$

we have

$$u^{(k+1)m} = Gu^{km}, \quad (3.6)$$

where

$$G = \left[R^m + (R^m - I) \left(-\frac{b}{a^2} \Delta x F_3 \right) \right].$$

From 3.6, we know that the zero solution of 3.4 is asymptotically stable if and only if the eigenvalue of matrix G satisfies

$$|\lambda_G| < 1. \quad (3.7)$$

So, it is sufficient to have,

$$\max |\lambda_G| < 1. \quad (3.8)$$



For convenience, let

$$Q = \frac{1 - a^2(1 - \theta)r_1\lambda_{F_1}}{1 + a^2\theta r_1\lambda_{F_1}},$$

in which Q is an eigenvalue of matrix R . We know that, $\lambda_{F_3} = \frac{1}{2}$. So inequality 3.8 gives,

$$\begin{aligned} \max |\lambda_G| &= \max \left\{ \left| Q^m + (Q^m - 1) \left(-\frac{b\Delta x}{2a^2} \right) \right| \right\} \\ &\leq \max \left\{ |Q^m| + |Q^m - 1| \left| -\frac{b\Delta x}{2a^2} \right| \right\} \\ &< 1. \end{aligned} \tag{3.9}$$

Assume $\left| -\frac{b\Delta x}{2a^2} \right| < 1$ then we have two cases. m is even or odd. The inequality 3.9 is equivalent to $-1 < Q < 1$ for m is even and $0 < Q < 1$ when m is odd. Since $Q < 1$, then, it is sufficient we investigate the conditions $Q > -1$ if m is even and $Q > 0$ when m is odd.

Therefore, we have the following theorem.

Theorem 3.3. Under the conditions 2.3, $\frac{-2a^2}{\Delta x} < b < \frac{2a^2}{\Delta x}$ and

$$m \text{ is even, } r_1 \neq \frac{1}{a^2\lambda_{F_1}(1-\theta)} \text{ and } \begin{cases} r_1 < \min \frac{2}{a^2\lambda_{F_1}(1-2\theta)}, & 0 \leq \theta < \frac{1}{2}, \\ r_1 > 0 & \frac{1}{2} \leq \theta \leq 1, \end{cases}$$

or

$$m \text{ is odd, and } r_1 < \min \frac{1}{a^2\lambda_{F_1}(1-\theta)},$$

the zero solution of 3.3 is asymptotically stable.

Remark 3.4. If $Q^m = 0$, then the corresponding fully implicit finite difference scheme is asymptotically stable unconditionally.

4. NUMERICAL EXPERIMENTS

We divide this section into two parts.

Part 1. In this part, we will give two examples for investigating the results in the article. It is easy to verify that the coefficients in examples 4.1 and 4.2 satisfy the conditions of the theorem 3.3. In Figs.1-2, we plot the numerical solutions of example 4.1 using $(\theta = 0.7, m = 100, p = 20)$ and $(\theta = 0.4, m = 400, p = 15)$, respectively. The analytical solution of this example is also shown in Fig.3 for $(m = 100, p = 20)$. Fig.4 shows the error of this computation. Figs.5-6 show the numerical and analytical solutions of example 4.2 respectively, using $(\theta = 0.8, m = 100, p = 20)$. To show the accuracy of the results, in this case, we also plot the errors in Fig.7.

In all cases the error formula is

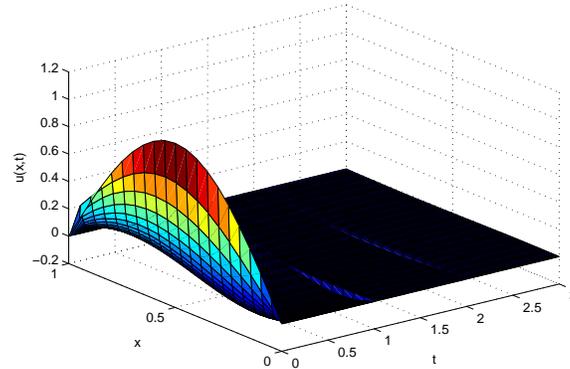
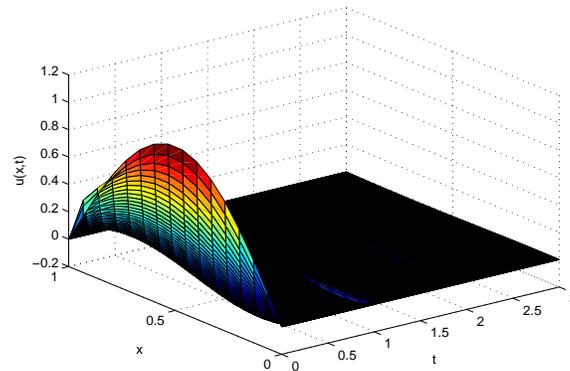
$$Error = |U(x, t) - u(x, t)|, \tag{4.1}$$

where $U(x, t)$ is an approximation for the exact solution using formula (2.1) and $u(x, t)$ is numerical solution.

Example 4.1. Consider the following problem

$$\begin{aligned} u_t(x, t) &= 4u_{xx}(x, t) - 5u_x(x, [t]), \\ u(0, t) &= u(1, t) = 0, \\ u(x, 0) &= \sin \pi x. \end{aligned}$$



FIGURE 1. The numerical solution with $m = 100$, $p = 20$ and $\theta = 0.7$.FIGURE 2. The numerical solution with $m = 400$, $p = 15$ and $\theta = 0.4$.

Example 4.2. Consider the following problem with larger "a" and "b", compared with the previous example.

$$\begin{aligned} u_t(x, t) &= 100u_{xx}(x, t) - 80u_x(x, t), \\ u(0, t) &= u(1, t) = 0, \\ u(x, 0) &= \sin \pi x. \end{aligned}$$

Part 2. In this part, we will give one example. In this example, the coefficients don't hold the condition 2.3. From Figs.8-9, we can see the numerical and the analytical solutions are not stable. So, this example confirms our theoretical results in the present article.



FIGURE 3. The analytical solution with $m = 100$ and $p = 20$.

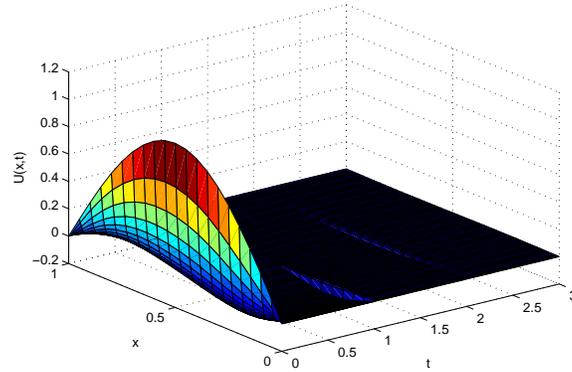
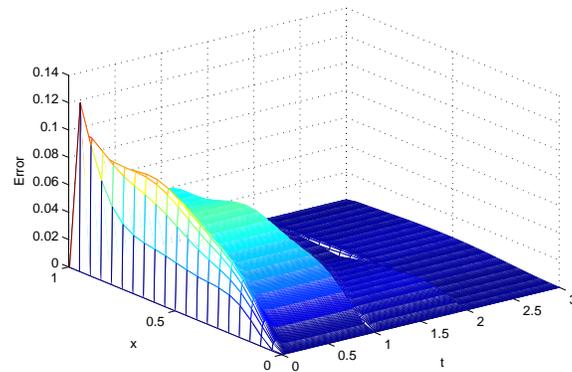


FIGURE 4. The graph of error with $m = 100$ and $p = 20$.



Example 4.3. In this example, we choose $b = 50$, while, based on the condition 2.3, b should reside in the interval $(-39.4784, 39.4784)$.

$$\begin{aligned}
 u_t(x, t) &= 4u_{xx}(x, t) - 50u_x(x, [t]), \\
 u(0, t) &= u(1, t) = 0, \\
 u(x, 0) &= \exp(\pi x).
 \end{aligned}$$



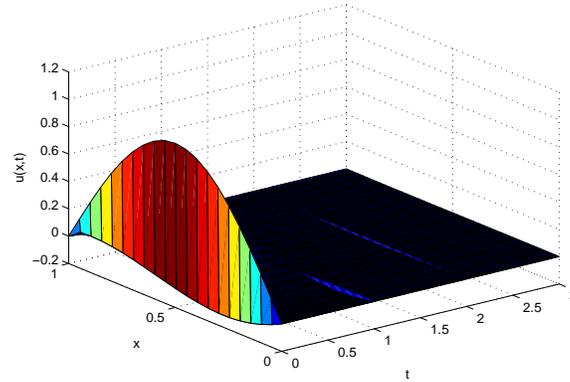
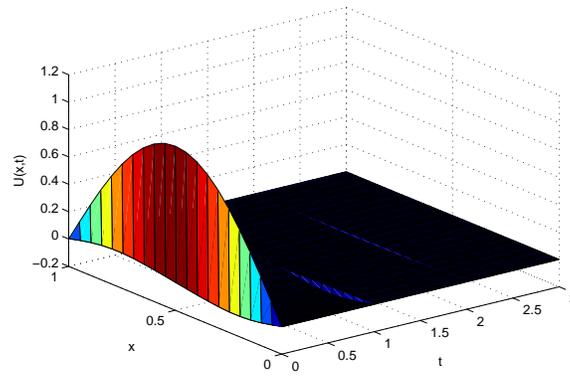
FIGURE 5. The numerical solution with $m = 100$, $p = 20$ and $\theta = 0.8$.FIGURE 6. The analytical solution with $m = 100$ and $p = 20$.

FIGURE 7. The graph of error with $m = 100$ and $p = 20$.

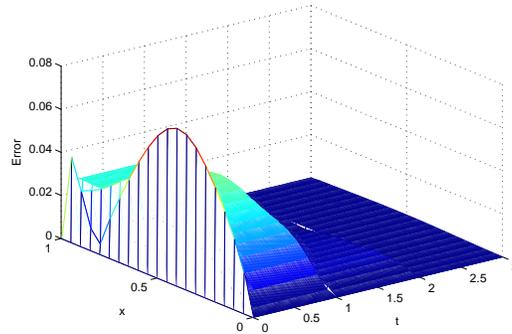


FIGURE 8. The numerical solution with $m = 150$, $p = 20$ and $\theta = 0.6$.

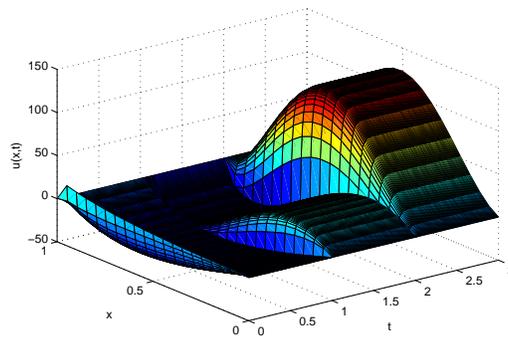
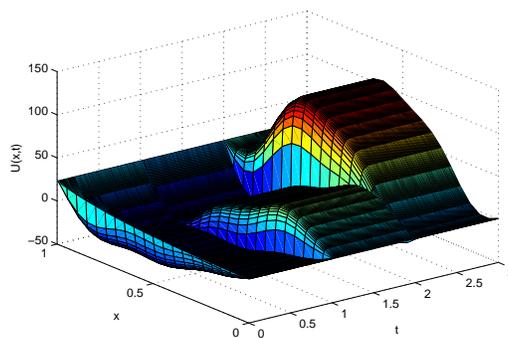


FIGURE 9. The analytical solution with $m = 100$ and $p = 20$.



5. CONCLUSIONS

In the present paper, the diffusion-convection equation with piecewise constant arguments is solved by the θ -methods. An important part of this paper is to investigate the stability that has been described in details in the Theorem 3.3. To show the accuracy of the results in all cases, we have plotted figures for errors. As figures show, the errors tend to zero.

Finally, to verify the condition of stability, we have used examples with different parameters. All the forms claim that the equations which hold in the condition of stability, have stable analytical and numerical solutions.

REFERENCES

- [1] K. L. Cooke and J. Wiener, *Retarded differential equations with piecewise constant delays*, J. Math. Anal., 99 (1984), 265-297.
- [2] M. Dehghan, *Weighted finite difference techniques for the one-dimensional advection-diffusion equation*, Appl. Math. Comput., 147 (2004), 307-319.
- [3] M. Dehghan, *Numerical solution the one-dimensional advection-diffusion equation*, Appl. Math. Comput., 150 (2004), 5-19.
- [4] H. Liang, M. Z. Liu, and W. Lv, *Stability of θ -schemes in the numerical solution of a partial differential equation with piecewise continuous arguments*, Appl. Math. Lett., 33 (2010), 198-206.
- [5] M. Z. Liu and M. N. Spiker, *The stability of θ -methods in the numerical solution of delay differential equations*, IMA J. Num. Anal., 10 (1990), 31-48.
- [6] A. Mohebbi and M. Dehghan, *High-order compact solution of the one-dimensional heat and advection-diffusion equations*, Appl. Math. Model., 147 (2010), 3071-3084.
- [7] H. Poorkarimian and J. Wiener, *Bounded solutions of nonlinear parabolic equations with time delay*, Elec. J. Dif. Eq. conf 02 (1999), 87-91.
- [8] H. Poorkarimian and J. Wiener, *Almost periodic solutions of nonlinear hyperbolic equations with time delay*, Elec. J. Dif. Eq. conf 07 (2001), 99-102.
- [9] S. M. Shah and J. Wiener, *Advanced differential equations with piecewise constant argument deviations*, Internat. J. Math. Sci., 6 (1983), 671-703.
- [10] M. H. Song, Z. W. Yang and M. Z. Liu, *Stability of θ methods for advanced differential equations with piecewise continuous arguments*, Comput. Math. Appl., 49 (2005), 1295-1301.
- [11] T. Veloz and M. Pinto, *Existence, Computability and stability for solutions of the diffusion equation with general piecewise constant argument*, J. Math. Anal. Appl., 426 (2015), 330-339.
- [12] Q. Wang and J. Wen, *Analytical and numerical stability of partial differential equations with piecewise constant arguments*, Num. Meth. Partial Diff. Eq., 30 (2014), 1-16.
- [13] Q. Wang, *Stability analysis of parabolic partial differential equations with piecewise continuous arguments*, Inc. Numer Meth. Partial Differential Eq., 33 (2017), 531-545.
- [14] J. Wiener, *Generalized solutions of functional differential equations*, World Scientific, Singapore, 1993.
- [15] J. Wiener, *Boundary value problems for partial differential equations with piecewise constant delay*, Int. J. Math. Math. Sci., 14 (1991), 301-321.
- [16] J. Wiener and L. Debnath, *Partial differential equations with piecewise constant delay*, Int. J. Math. Math. Sci., 14 (1991), 485-496.
- [17] J. Wiener and L. Debnath, *A wave equation with discontinuous time delay*, Int. J. Math. Math. Sci., 15 (1992), 781-788.
- [18] J. Wiener and L. Debnath, *A survey of partial differential equations with piecewise continuous arguments*, Int. J. Math. Math. Sci., 18 (1995), 209-228 .

