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# A pseudo-spectral based method for time-fractional advection-diffusion equation

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**Abstract** In this paper, a pseudo-spectral method with the Lagrange polynomial basis is proposed to solve the time-fractional advection-diffusion equation. A semi-discrete approximation scheme is used for conversion of this equation to a system of ordinary fractional differential equations. Also, to protect the high accuracy of the spectral approximation, the Mittag-Leffler function is used for the integration along the time variable. Some examples are performed to illustrate the accuracy and efficiency of the proposed method.

**Keywords.** Time-fractional advection-diffusion equations, Mittag-Leffler functions, Fractional derivative, Pseudo-spectral methods.

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## 1. Introduction

The theory of derivatives of non-integer order goes back to the Leibniz's note to L'Hospital, dated 30 September 1695, in which the meaning of the derivate of order one half is discussed [15].

As is mentioned in [8], fractional partial differential equations (FPDEs) have been defined to depict phenomena in the fields of engineering, science and economics, such as carrier transport in amorphous semiconductors, system identification and control, anomalous diffusion in electro-chemistry, fractance circuits, electrode electrolyte interface, viscoelasticity, fractional neural modeling in bio-sciences, nuclear reactor dynamics, chaos theory, finance, sustainable environment, renewable energy and so on (for more information see [8] and references therein).

Schumer et al. [19] mentioned that an important class of problems in the Earth surface sciences which leads to the time-fractional advection-diffusion equations (TFADEs) is describing the collective behavior of particles in transport and the development of stochastic partial differential equations such as the ADE begins with assumptions

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about the random behavior of a single particle: possible velocities it may experience in a flow field and the length of time it may be immobilized. When assumptions underlying the ADE are relaxed, a FADE can arise, with a non-integer-order derivative on time or space terms [19].

It is difficult to find analytical solutions of TFADEs, or time-fractional wave-diffusion equations (TFWDEs). Although they are available for some simple cases, the solutions often refer to some special functions, which are quite sophisticated in the calculation [8].

In the last decade, a lot of works was dedicated to the numerical schemes for solving the fractional differential equations. Saadatmandi et al. [18] established a Sinc-Legendre collocation method for solving a class of fractional convention-diffusion equations with variable coefficients with Caputo fractional derivate. Gao and Sun [8] determined a three-point combined compact difference scheme with the L1 formula to solve a class of TFADEs. Their method is globally (2-$\gamma$)th-order accurate in time and at least fifth-order accurate in space for the constant coefficient TFADEs. Dehghan et al. [5] applied a high order difference scheme and Galerkin spectral technique for the numerical solution of multi-term time-fractional partial differential equations. The time-fractional derivatives are approximated by a scheme of order $O(\tau^{3-\alpha})$ and the space derivative is discretized with a fourth-order compact finite difference procedure and Galerkin spectral method. Later, Du et al. [6], Gao et al. [9], Hu and Zhang [11, 12], Mohebbi and Abbaszadeh [14] and Re et al. [17] studied the spatial fourth-order compact schemes for solving several types of time-fractional partial differential equations.

As is said in Boyd [3], spectral methods generate algebraic equations with full matrices, but in compensation, the high order of the basis functions gives high accuracy for a given $N$. When fast iterative matrix solvers are used, spectral methods can be much more efficient than finite element or finite difference methods for many classes of problems. However, they are most useful when the geometry of the problem is fairly smooth and regular. For most high-resolution numerical calculations, Boyd [3] said that the best advice is still this: use pseudo-spectral methods instead of spectral, and use Fourier series and Chebyshev polynomials in preference to more exotic functions.

The purpose of this paper is to propose a pseudo-spectral method using the Lagrange polynomial basis based on the method of lines for the numerical solution of the time-fractional advection-diffusion equations in one, two and three dimensions. At first, the pseudo-spectral method is used to reduce the original TFADE to a linear system of ordinary fractional differential equations in time and, then the matrix functions approximation is used for the integration along the fractional time derivative.

The organization of this paper is as follows: In Section 2, we briefly explain the time-fractional advection-diffusion equation. In Section 3, we describe the pseudo-spectral method, obtain the semi-discretization of the time-fractional equation and introduce the matrix functions. Convergence analysis is given in Section 4. Numerical results using different examples are reported in Section 5. Finally, a conclusion is given in Section 6.

## 2. Time fractional advection-diffusion equation

In this paper, we study the time-fractional advection-diffusion equation with a reaction term in the following form

$$\begin{aligned} D_t^\alpha u(\mathbf{x},t) &= \zeta\Delta u(\mathbf{x},t) - \eta\nabla u(\mathbf{x},t) - \lambda^2 u(\mathbf{x},t) + f(\mathbf{x},t) \\ &= \mathcal{L}u(\mathbf{x},t) + f(\mathbf{x},t), \quad 1 < \alpha \leq 2, \quad \mathbf{x} \in \Omega, \quad t \in (0,T), \end{aligned} \tag{2.1}$$

with initial conditions

$$u(\mathbf{x},0) = \phi_1(\mathbf{x}), \quad u_t(\mathbf{x},0) = \phi_2(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{2.2}$$

and boundary conditions

$$u(\mathbf{x},t) = \psi(t), \quad x \in \partial\Omega, \quad 0 < t < T, \tag{2.3}$$

where

$$\mathcal{L}u(\mathbf{x},t) = \zeta\Delta u(\mathbf{x},t) - \eta\nabla u(\mathbf{x},t) - \lambda^2 u(\mathbf{x},t), \tag{2.4}$$

is a linear differential operator. $\Omega \subset \mathbb{R}^d (d \leq 3)$ be a bounded space domain with sufficiently smooth boundary $\partial\Omega$ and $D_t^\alpha$ denotes the fractional derivative of order $\alpha$ concerning $t$. Also, $\zeta, \eta$ and $\lambda$ are arbitrary coefficients, $\phi_1(\mathbf{x}), \phi_2(\mathbf{x})$ and $\psi(t)$ are given continuous functions and, $f$ is a given function in $\Omega \times (0,T)$. For $\alpha = 1$, (2.1) is reduced to the diffusion equation. The main focus of this paper would be on the case $1 < \alpha \leq 2$.

In the particle, transport on the Earth surface model described in [19], the time-fractional parameter $\alpha$ codes retention, since it implies that the memory function $t^{-\alpha}$ is a power law, and so it can be estimated from the late-time tail of the breakthrough curve. The solution $u(\mathbf{x},t)$ to the TFADE (2.1) is the probability density of the subordinated process that models the location of a randomly selected particle [18].

Note that the Caputo fractional derivative $D_t^\alpha$ of the order $\alpha, 1 < \alpha \leq 2$ in (2.1) with starting point $t = 0$, is defined by

$$D_t^\alpha f(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t f^{(2)}(s)(t-s)^{\alpha-1} ds, \quad t > 0. \tag{2.5}$$

## 3. Pseudo-spectral methods

The pseudo-spectral methods are based on spectral methods. Spectral methods are a powerful approach for the numerical solution of PDEs, which is traced back to 1970s. Pseudo-spectral methods are a highly accurate class of techniques for the solution of partial differential equations. In these methods, the global basis functions are used for expansion of solutions. Interested readers for more information in this setting can see [3, 13].

In this paper, based on the method of lines, the pseudo-spectral method is used to reduce the original TFADE to a linear system of ordinary fractional differential equations in time and, then the matrix functions approximation is used for the integration along the fractional time derivative.

3.1. **Differentiation matrices.** To introduce the basis functions notice that any interval $[a,b]$ can be scaled to $[-1,1]$ and therefore the solution can be just considered over $[-1,1]$. Now consider the $n+1$ Chebyshev points in this interval

$$x_j = \cos(j\pi/n), \quad j = 0,1,\dots,n, \tag{3.1}$$

which are the zeros of Chebyshev polynomial $T_n(x) = \cos(n\cos^{-1}x)$. In this work, the Lagrange polynomials over Chebyshev points (3.1) are used as a polynomial basis, that are defined as follows

$$\ell_j(x) = \prod_{\substack{k=0\\k\neq j}}^{n} \left(\frac{x - x_k}{x_j - x_k}\right), \quad j = 0,1,\dots,n. \tag{3.2}$$

The Lagrange polynomials (3.2) have the Kronecker delta property, i.e. $\ell_j(x_k) = \delta_{kj}, j,k = 0,1,\dots,n$ and form a basis for all polynomials of degree not exceed n.

By using the Lagrange polynomials (3.2), the unique interpolating polynomial $I_n$ of degree $\leq n$ that interpolates the data $\{(x_j, u(x_j))\}_{j=0}^{n}$ is given by a linear combination

$$I_n(x) = \sum_{j=0}^{n} u(x_j)\ell_j(x).$$

Let $\mathbf{u} = (u_0, u_1, \dots, u_n)^T$. A discrete derivative $\mathbf{w} = (w_0, w_1, \dots, w_n)^T$ on the Chebyshev points (3.1) can be obtained by using the Chebyshev differentiation matrix $D_n$ as [1, 7, 20, 21]

$$\mathbf{w} = D_n\mathbf{u}, \tag{3.3}$$

where the elements of the differentiation matrix $D_n$ is

$$[D_n]_{ij} = \ell'_j(x_i), \quad i,j = 0,1,\dots,n.$$

The off-diagonal elements of $D_n$ can be obtained from an explicit formula

$$[D_n]_{ij} = \frac{\lambda_j}{\lambda_i(x_i - x_j)}, \quad i \neq j,$$

where $\lambda_j^{-1} = \prod_{i\neq j}^{n}(x_j - x_i)$ . Also, all the diagonal entries obtained by negative sum trick as

$$[D_n]_{ii} = -\sum_{\substack{j=0\\j\neq i}}^{n}[D_n]_{ij}, \quad i = 0,1,\dots,n.$$

Moreover, the following formula can be used to obtain the elements of $p$-order differentiation matrix $D_n^{(p)}$ analytically

$$\left[D_n^{(p)}\right]_{ij} = \ell_j^{(p)}(x_i), \quad i,j = 0,1,\dots,n.$$

More efficient techniques for accurate and stable evaluation of $p$-order differentiation matrices is proposed in [1].

3.2. **The discrete linear operator.** Now, it's time to discretize the linear operator (2.4). For this purpose, the linear operator (2.4) is applied on the Lagrange polynomials (3.2) with the Chebyshev points $\{x_r\}$ as

$$\begin{aligned}\mathcal{L}\left(\ell_i(x_r)\right) &= \zeta\partial_x^2\ell_i(x_r) - \eta\partial_x\ell_i(x_r) - \lambda^2\ell_i(x_r)\\ &= \zeta\big[D_n^2\big]_{ri} - \eta\big[D_n\big]_{ri} - \lambda^2\ell_i(x_r), \quad r, i = 0, 1, \dots, n,\end{aligned} \tag{3.4}$$

where $D_n^2$ and $D_n$ are differentiation matrices of order $(n+1)^2$.

For the square domain $\Omega = [-1,1]^2 \subset \mathbb{R}^2$, assume the points $\{\mathbf{x}_{ij}\}$ where both coordinates coincide with the one-dimensional Chebyshev points in $[-1,1]$,

$$\mathbf{x}_{ij} = (\cos(i\pi/n), \cos(j\pi/n), ), \quad i, j = 0, 1, \dots n.$$

The Lagrange polynomial regarded to these points are written as

$$\ell_{ij}(\mathbf{x}) = \ell_i(x)\ell_j(y), \quad i, j = 0, 1, \dots n. \tag{3.5}$$

Note that, in this case again the Kronecker delta property exists, i.e. $\ell_{ij}(\mathbf{x}_{ij}) = \delta_{ij},\ i, j = 0, 1, \dots n$. The second derivatives of Lagrange polynomials (3.5) concerning $x$ and $y$ are given as

$$\begin{aligned}\partial_x^2\ell_{ij}(\mathbf{x}_{rs}) &= \ell_i''(x_r)\ell_j(y_s) = [D_n^2]_{ri}\delta_{js},\\ \partial_y^2\ell_{ij}(\mathbf{x}_{rs}) &= \ell_i(x_r)\ell_j''(y_s) = \delta_{ri}[D_n^2]_{sj},\end{aligned}$$

where $D_n^2$ is the second order differentiation matrix. Thus, same as one-dimensional case, applying the linear operator (2.4) on the Lagrange polynomials (3.5) at the Chebyshev points $\{\mathbf{x}_{rs}\}$ gives

$$\begin{aligned}\mathcal{L}\left(\ell_{ij}(\mathbf{x}_{rs})\right) =& \zeta\Big(\big[D_n^2\big]_{ri}\delta_{js} + \delta_{ri}\big[D_n^2\big]_{sj}\Big) - \eta\Big(\big[D_n\big]_{ri}\delta_{js} + \delta_{ri}\big[D_n\big]_{sj}\Big)\\ &- \lambda^2\Big(\delta_{ri}\delta_{sj}\Big), \quad r, s = 0, \dots, n.\end{aligned} \tag{3.6}$$

It's possible to write the linear operator (3.6) in a matrix form as

$$\mathcal{L} = \zeta\big(I_n \otimes D_n^2 + D_n^2 \otimes I_n\big) - \eta\big(I_n \otimes D_n + D_n \otimes I_n\big) - \lambda^2\big(I_n \otimes I_n\big), \tag{3.7}$$

where the symbol $\otimes$ illustrates the Kronecker product, $D_n^2$ and $D_n$ are differentiation matrices and $I_n$ is an identity matrix, all of the order $(n+1)^2$. Notice that the same works can be done to find the discrete form of linear operator $\mathcal{L}$ in three dimensions.

3.3. **Semi-discretization of time-fractional advection-diffusion equation.** In this section, the prescribed pseudo-spectral method is used to discretize the initial-boundary value problem (2.1)-(2.3) in 2D (It's easy to write the same algorithm for 1D and 3D cases). The solution $u_n(\mathbf{x}, t)$ and the forcing term $f_n(\mathbf{x}, t)$ in (2.1) can be approximated by the Lagrange polynomials (3.5) as

$$\begin{aligned}u_n(\mathbf{x}, t) &:= \sum_{i,j=1}^{n-1} u_{ij}(t)\ell_{ij}(\mathbf{x}),\\ f_n(\mathbf{x}, t) &:= \sum_{i,j=1}^{n-1} f_{ij}(t)\ell_{ij}(\mathbf{x}),\end{aligned}$$

where $u_{ij}(t) := u(\mathbf{x}_{ij}, t)$ and $f_{ij}(t) := f(\mathbf{x}_{ij}, t)$. Values of $u(\mathbf{x}_{ij}, t)$ for $i, j = 0, n$ are known from boundary conditions (2.3). To find a system of ordinary FDEs, we introduce the residual function

$$R_n(\mathbf{x}, t) := D_t^{\alpha} u_n(\mathbf{x}, t) - \mathcal{L} u_n(\mathbf{x}, t) - f_n(\mathbf{x}, t), \tag{3.8}$$

that should vanish at the internal points. Using the discrete operator $\mathcal{L}$ in (3.6), we obtain

$$D_t^{\alpha} u_{rs}(t) = \sum_{i,j=1}^{n-1} \Big\{ \zeta \left( \left[D_n^2\right]_{ri} \delta_{js} + \delta_{ri} \left[D_n^2\right]_{sj} \right) - \eta \left( \left[D_n\right]_{ri} \delta_{js} + \delta_{ri} \left[D_n\right]_{sj} \right) \\ - \lambda^2 \left( \delta_{ri} \delta_{sj} \right) \Big\} u_{rs}(t) + f_{rs}(t), \quad r, s = 1, \ldots, n-1, \tag{3.9}$$

for all $t > 0$. To find the matrix form of this system, assume $\mathbf{U}_n(t)$ and $\mathbf{F}_n(t)$ are $(n-1)$-dimensional vectors of functions $u_n(\mathbf{x}, t)$ and $f_n(\mathbf{x}, t)$, respectively. Therefore, the following system of ordinary FDEs is obtained for interior points

$$\begin{aligned} D_t^{\alpha} \mathbf{U}_n(t) &= \tilde{\mathcal{L}}_n \mathbf{U}_n(t) + \mathbf{F}_n(t), \\ \mathbf{U}_n(0) &= [\phi_1], \\ \mathbf{U}_n'(0) &= [\phi_2], \end{aligned} \tag{3.10}$$

where $\tilde{\mathcal{L}}_n$ is a matrix of order $(n-1)^2$ which is obtained as

$$\tilde{\mathcal{L}}_n = \zeta \big( \tilde{I}_n \otimes \tilde{D}_n^2 + \tilde{D}_n^2 \otimes \tilde{I}_n \big) - \eta \big( \tilde{I}_n \otimes \tilde{D}_n + \tilde{D}_n \otimes \tilde{I}_n \big) - \lambda^2 \big( \tilde{I}_n \otimes \tilde{I}_n \big). \tag{3.11}$$

$\tilde{D}_n^2$, $\tilde{D}_n$ and $\tilde{I}_n$ are matrices of order $(n-1)^2$ extracted from $D_n^2$, $D_n$ and $I_n$ by removing rows and columns related to the boundary points (the first and the last rows and columns). A very important property of the derivative matrix $\tilde{D}_n^2$ is that all the eigenvalues of this matrix are distinct, real and also negative that makes the derivative matrix is diagonalizable. That is, it can be decomposed into a diagonal matrix including its eigenvalues and an invertible matrix including its corresponding eigenvectors. The exact solution of (3.10) can be obtained as

$$\mathbf{U}_n(t) = e_{\alpha,1}(t; \tilde{\mathcal{L}}_n) \left( \mathbf{U}_n(0) + \mathbf{U}_n'(0) \right) + \int_0^t e_{\alpha,\alpha}(t-s; \tilde{\mathcal{L}}_n) \mathbf{F}_n(s) ds, \tag{3.12}$$

where

$$e_{\alpha,\beta}(t; \lambda) = t^{\beta-1} E_{\alpha,\beta}(t^{\alpha} \lambda), \tag{3.13}$$

is the generalization of Mittag-Leffler (ML) function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \ \beta > 0, \tag{3.14}$$

which is playing an important role in the theory of fractional differential equations [7]. For computation of the integration in the right-hand side of (3.12), the following ML function integration property can be used

$$\int_0^t e_{\alpha,\beta}(t-s; z) s^r ds = \Gamma(r+1) e_{\alpha,\beta+r+1}(t; z), \quad r > -1. \tag{3.15}$$

3.4. **Matrix functions computation.** There are many equivalent ways for defining matrix functions such as $e_{\alpha,\alpha}(t-s;\tilde{\mathcal{L}}_n)$ in (3.12). In this work, for computation of these matrix functions appeared in (3.12) and (3.15), the Jordan canonical form is used which is defined in [10] as follows.

**Definition 3.1.** Any matrix $A \in \mathbb{C}^{n\times n}$ can be expressed in the Jordan canonical form

$$Z^{-1}AZ = J = diag(J_1, J_2, \ldots, J_p), \tag{3.16}$$

where $Z$ is non-singular and

$$J_k = J_k(\lambda_k) = \begin{bmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix}, \tag{3.17}$$

is called a Jordan block respect to eigenvalue $\lambda_k$ of size $m_k$ and $m_1 + m_2 + ... + m_p = n$. The Jordan matrix $J$ is unique up to the ordering of the blocks $J_i$, but the transforming matrix $Z$ is not unique.

The following definition of $f(A)$ is suitable for this work because just use the values of $f$ on the spectrum of $A$ and does not require any other information about $f$.

**Definition 3.2.** [10, p. 3][Matrix function via Jordan canonical form] Let $f$ be defined on the spectrum of $A \in \mathbb{C}^{n\times n}$ and let $A$ have the Jordan canonical form (3.16). Then

$$f(A) := Zf(J)Z^{-1} = Zdiag(f(J_k))Z^{-1}, \tag{3.18}$$

where

$$f(J_k) := \begin{bmatrix} f(\lambda_k) & f'(\lambda_k) & \cdots & \frac{f^{(m_k-1)}(\lambda_k)}{(m_k-1)!} \\ & f(\lambda_k) & \ddots & \vdots \\ & & \ddots & f'(\lambda_k) \\ & & & f(\lambda_k) \end{bmatrix}, \tag{3.19}$$

Note that if A is diagonalizable, then the Jordan canonical form reduces to an eigen-decomposition [10]. Let $A = ZDZ^{-1}$ be the eigenvalue decomposition of $A$, then

$$f(A) = Zf(D)Z^{-1} = Zdiag(f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_m))Z^{-1}, \tag{3.20}$$

where the $j$th column of $Z$ is an eigenvector of $A$ and the $j$th entry of $D$ is the corresponding eigenvalue [7].

## 4. Error analysis

Consider the interpolation operator according to Chebyshev points (3.1) and Lagrange polynomials (3.2) as follows

$$I_n : C(\Omega) \longrightarrow \mathbb{P}_n \tag{4.1}$$
$$I_n(u) = \sum_{j=0}^{n} u(x_j)\ell_j(x).$$

Note that in this section we follow a procedure which is already employed in the Börm et al. [2]. To construct an interpolation error bound, let $M_n$ be a constant satisfying the stability estimate

$$\|I_n(u)\|_\infty \leq M_n\|u\|_\infty, \quad \textit{for all } u \in C[-1,1]. \tag{4.2}$$

Furthermore,

$$I_n(u) = u \quad \textit{for all } u \in \mathbb{P}_n. \tag{4.3}$$

In the case of Chebyshev interpolation, it is possible to show that

$$M_n = \frac{2}{\pi}\ln(n+1) + 1 \leq n+1. \tag{4.4}$$

So the stability constant grows very slowly depending on $n$.

Also, for a function $u \in C^{n+1}[-1,1]$, the following approximation error bound holds [2]

$$\|u - I_n(u)\|_\infty \leq \frac{2^{-n}}{(n+1)!}\|u^{(n+1)}\|_\infty. \tag{4.5}$$

**Theorem 4.1.** [2, Theorem 3.15] *Let $k \in \{0,\dots,n\}$ and $u \in C^{n+1}[-1,1]$. If (4.2) and (4.5) hold, we have*

$$\|u^{(k)} - I_n(u)^{(k)}\|_\infty \leq \frac{2(M_n^{(k)}+1)}{(n-k+1)!}\left(\frac{1}{2}\right)^{n-k+1}\|u^{(n+1)}\|_\infty \tag{4.6}$$

*with the stability constant*

$$M_n^{(k)} = \frac{M_n}{k!}\left(\frac{n!}{(n-k)!}\right).$$

Now, using the (4.5)-(4.6) for the TFADE (2.1) in one dimension, we have the following error estimate

$$\begin{aligned} error &= \|(D_t^{\alpha} u - \mathcal{L}u) - (D_t^{\alpha} I_n(u) - \mathcal{L} I_n(u))\|_{\infty} \\ &= \|D_t^{\alpha}(u - I_n(u)) - \mathcal{L}(u - I_n(u))\|_{\infty} \\ &\leq \|D_t^{\alpha}(u - I_n(u))\|_{\infty} + \|\mathcal{L}(u - I_n(u))\|_{\infty} \\ &\leq \|D_t^{\alpha}(u - I_n(u))\|_{\infty} + |\zeta| \|u_{xx} - I_n(u)_{xx}\|_{\infty} \\ &\quad + |\eta| \|u_x - I_n(u)_x\|_{\infty} + \lambda^2 \|u - I_n(u)\|_{\infty} \\ &\leq \|D_t^{\alpha}(u - I_n(u))\|_{\infty} + |\zeta| \frac{2(M_n^{(2)} + 1)}{(n-1)!} \left(\frac{1}{2}\right)^{n-1} \|u^{(n+1)}\|_{\infty} \\ &\quad + |\eta| \frac{2(M_n^{(1)} + 1)}{n!} \left(\frac{1}{2}\right)^{n} \|u^{(n+1)}\|_{\infty} + \lambda^2 \frac{2^{-n}}{(n+1)!} \|u^{(n+1)}\|_{\infty} \end{aligned}$$

Since the time derivative is precisely calculated using the evaluation of the ML function by matrix functions, therefore the error bound of $\|D_t^{\alpha}(u - I_n(u))\|_{\infty}$ is the same order of $\|u - I_n(u)\|_{\infty}$. Finally, we have

$$error \leq C \|u^{(n+1)}\|_{\infty}.$$

where $C$ is the summation of the coefficients of the $\|u^{(n+1)}\|_{\infty}$ in the right-hand side of the inequality.

Same error bounds can be found in the two-dimensional case using tensor-product interpolation operators.

## 5. Numerical results

In this section, some numerical results are presented to illustrate the behavior of the pseudo-spectral method described in the previous sections. The main goal is to check the convergence behavior of the numerical solutions concerning the polynomial degrees $n$ for different values of $\alpha$. All the algorithms are executed by Matlab and the Matlab code mlf from [16] is used. The accuracy of the estimated solutions worked out by measuring $L_{\infty}$ error norms.

**Example 5.1.** In this example, we consider the following two cases with known exact solutions for the one-dimensional time-fractional equation (2.1) with parameters $\zeta = 1, \eta = 0$ and $\lambda = 0$ on $\Omega = (0, 1)$.

Case 1: If the right-hand-side function chosen as

$$f(x,t) = \big(\frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} - t^3\big)e^x,$$

then the exact solution would be $u(x,t) = t^3 e^x$. Therefore, initial and boundary conditions take form

$$\begin{aligned} &u(x,0) = 0, \qquad u_t(x,0) = 0, \qquad 0 < x < 1, \\ &u(0,t) = t^3, \qquad u(1,t) = t^3 e, \qquad 0 \leq t \leq 1. \end{aligned}$$

TABLE 1. The $L_\infty$ errors as functions of $n$ at $t = 1$ (Example 5.1).

| $n$ | $\alpha = 1.2$ | $\alpha = 1.5$ | $\alpha = 1.8$ |
|---|---|---|---|
| 2 | $2.8227e-03$ | $2.3751e-03$ | $1.8185e-03$ |
| 4 | $2.4699e-05$ | $2.3882e-05$ | $2.2979e-05$ |
| 6 | $2.9229e-08$ | $2.9023e-08$ | $2.8797e-08$ |
| 8 | $2.1205e-11$ | $1.9267e-11$ | $1.9122e-11$ |
| 10 | $1.7500e-12$ | $1.4860e-13$ | $4.5169e-13$ |
| 12 | $1.7375e-12$ | $4.7407e-14$ | $4.0656e-13$ |
| 14 | $1.8655e-12$ | $2.8116e-13$ | $6.3582e-13$ |

In Table 1, the $L_\infty$ errors of numerical solutions concerning the polynomial degrees $n$ for three values $\alpha = 1.2, 1.5$ and $1.8$ are shown. The numerical results demonstrate the efficiency and good convergence of the prescribed method.

Case 2: Now, if the right-hand-side function set as

$$f(x,t) = \big(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\big)\sec(x) - t^2\big(\frac{1}{\cos(x)} + \frac{2\sin^2(x)}{\cos^3(x)}\big),$$

and the initial and boundary conditions given as

$$u(x,0) = 0, \qquad u_t(x,0) = 0, \qquad 0 < x < 1,$$
$$u(0,t) = t^2, \qquad u(1,t) = t^2\sec(1), \qquad 0 \le t \le 1.$$

then the exact solution can be $u(x,t) = t^2\sec(x)$. The good convergence of the numerical solutions can be seen in Figure 1 where the $L_\infty$ errors are plotted in semilog scale concerning the polynomial degrees $n$ for three values $\alpha = 1.2, 1.5$ and $1.8$.

**Example 5.2.** For this example, consider the two-dimensional time-fractional equation (2.1) with Dirichlet boundary conditions on the bounded domain $\Omega = (0, 2\pi)^2$ with coefficients $\zeta = 1$, $\eta = -1$ and $\lambda = 1$. It's easy to see if the right-hand-side function chosen as

$$\begin{aligned} f(x,y,t) = {} & \frac{\Gamma(3)}{\Gamma(3-\alpha)}t^{2-\alpha}\sin(\pi x)\sin(\pi y) + 2\pi^2 t^2\sin(\pi x)\sin(\pi y) \\ & - \pi t^2\big(\cos(\pi x)\sin(\pi y) + \sin(\pi x)\cos(\pi y)\big) + t^2\sin(\pi x)\sin(\pi y), \end{aligned}$$

then, the exact solution is given by $u(x,y,t) = t^2\sin(\pi x)\sin(\pi y)$.

The $L_\infty$ errors of numerical solutions are plotted in semilog scale concerning the polynomial degrees $n$ for three values $\alpha = 1.2, 1.5$ and $1.8$ in Figure 2. This figure demonstrates the efficiency and good convergence of the prescribed method.

**Example 5.3.** Consider the following variable coefficient time-fractional advection-diffusion equation

$$D_t^\alpha u(x,t) = a(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} - b(x,t)\frac{\partial u(x,t)}{\partial x} + f(x,t), \quad x \in \Omega,\ t \in (0,T),$$

FIGURE 1. The $L_\infty$ errors versus polynomial degrees $n$ for $\alpha$ at $t = 1$ (Example 5.1).

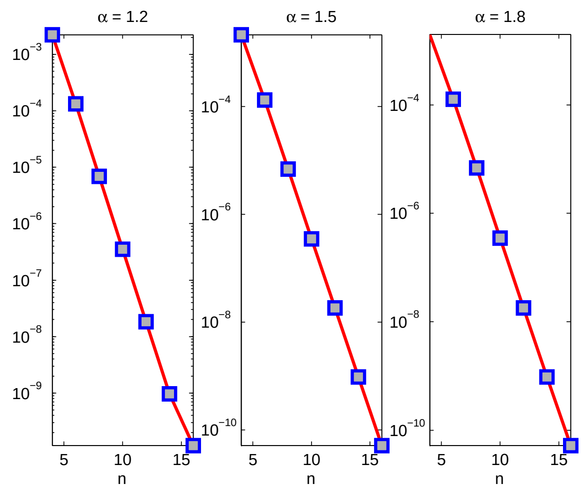
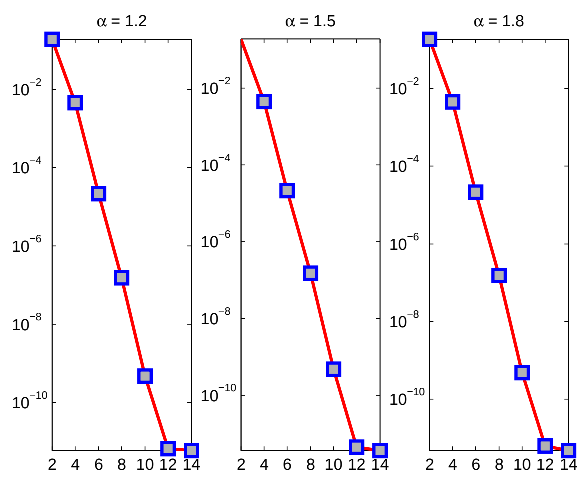



FIGURE 2. The $L_\infty$ errors versus polynomial degrees $n$ for $\alpha$ at $t = 1$ (Example 5.2).




where $\Omega = (0, 2\pi)$. We note that coefficients are not constant and considered as $a(x,t) = x+1$ and $b(x,t) = -t\sin(x)$. With a simple study can be seen if right-hand-side function set as

$$f(x,t) = t^3 \sin(x)\Big(\frac{\Gamma(4+\alpha)}{\Gamma(4)} + (x+1)t^\alpha - \cos(x)t^{1+\alpha}\Big),$$

Table 2. The $L_\infty$ errors as functions of $n$ at $t = 1$ (Example 5.3).

| $n$ | $\alpha = 1.2$ | $\alpha = 1.5$ | $\alpha = 1.8$ |
|---|---|---|---|
| 4 | $9.1738e-02$ | $6.4938e-02$ | $4.0330e-02$ |
| 6 | $4.1012e-03$ | $3.5779e-03$ | $3.3014e-03$ |
| 8 | $1.2438e-04$ | $1.0601e-04$ | $1.0298e-04$ |
| 10 | $2.3205e-06$ | $2.1578e-06$ | $2.0896e-06$ |
| 12 | $3.1182e-08$ | $3.0344e-08$ | $2.9825e-08$ |
| 14 | $3.2233e-10$ | $3.1919e-10$ | $3.1480e-10$ |

Table 3. The $L_\infty$ errors as functions of $n$ at $T = 1$ (Example 5.5).

| $n$ | $\alpha = 1.2$ | $\alpha = 1.5$ | $\alpha = 1.8$ |
|---|---|---|---|
| 4 | $1.8003e-04$ | $2.4935e-04$ | $1.6412e-03$ |
| 6 | $2.2174e-06$ | $1.1543e-05$ | $3.6865e-05$ |
| 8 | $1.0244e-08$ | $3.7311e-08$ | $5.1327e-08$ |
| 10 | $3.6764e-11$ | $1.7849e-10$ | $4.2009e-10$ |
| 12 | $9.0206e-14$ | $3.7242e-13$ | $8.3378e-13$ |
| 14 | $2.0157e-15$ | $9.2149e-15$ | $4.5131e-14$ |

then the exact solution takes form $u(x,t) = \sin(x)t^{3+\alpha}$. The initial and boundary conditions can be obtained from the exact solution.

In Table 2, the convergence behavior of the numerical solutions by a growth of the polynomial degrees $n$ for three values $\alpha = 1.2, 1.5$ and $1.8$ is shown. This table demonstrates the spectral convergence of numerical solutions.

**Example 5.4.** Now, consider the TFADE (2.1) on $\Omega = [0,1]^2$ with coefficients $\zeta = 1, \eta - 1$ and $\lambda = 1$ and constant $\gamma = 2^{12}$ with boundary conditions

$$\psi(x,y,t) = 0, \quad (x,y) \in \partial\Omega, \quad t \in (0,1]$$

and initial condition $\varphi(x,y,0) = 0$ for all $(x,y) \in \Omega$. By considering the exact solution as:

$$u(x,y,t) = \gamma t^{2+\alpha} x^3 (1-x)^3 y^3 (1-y)^3.$$

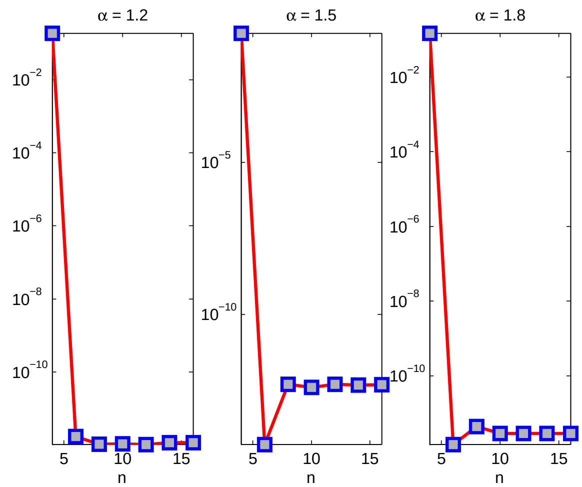
the right hand side function $f(x,y,t)$ can be found by some calculations. The Figure 3 shows the $L_\infty$ errors for different vales of $\alpha$ for $T = 1$.

**Example 5.5.** And the last example, consider the boundary value problem (2.1) with $\zeta = 1, \eta = 0$ and $\lambda = 0$ on $\Omega = [-1,1]^2$ and $t \in [0,1]$. The exact solution is chosen from [4] as

$$u(x,y,t) = E_{\alpha,1}(\frac{-1}{2}\pi^2 t^\alpha)\cos(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y),$$

where $E_{\alpha,1}$ is ML function. Table 3 shows the convergence behavior of the numerical solution concerning the polynomials of degree $n$ for different values of $\alpha$ at the final time $T = 1$.

Figure 3. The $L_\infty$ errors versus polynomial degrees $n$ for different vales of $\alpha$ at $T = 1$ (Example 5.4).




## 6. Conclusion

Pseudo-spectral methods are a highly accurate subclass of spectral methods for the numerical solution of PDEs. In this paper, an approximate scheme based on the pseudo-spectral method with the Lagrange polynomial basis on Chebyshev points is proposed to solve the time-fractional advection-diffusion equation. Also, to exchange this equation to a system of ordinary fractional differential equations, the semi-discrete approximation scheme based on the method of lines is used. The advantage of this work is using the Mittag-Leffler function for the integration along the time variable to protect the high accuracy of the spectral approximation. Some examples include 1D, 2D, and variable coefficients are performed to illustrate the spectral convergence and efficiency of the proposed method.

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