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# Solving the Fokker-Planck equation via the compact finite difference method

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**Abstract** In this study, we solve the Fokker-Planck equation by a compact finite difference method, By the finite difference method the computation of Fokker-Planck equation is reduced to a system of ordinary differential equations. Two different methods, boundary value method and cubic $C^1$-spline collocation method, for solving the resulting system are proposed. Both methods have fourth-order accuracy in time variable. By the boundary value method, some pointwise approximate solutions are only obtained. But, $C^1$-spline method gives a closed-form approximation in each space step, too. Illustrative examples are included to demonstrate the validity and efficiency of the methods. A comparison is made with existing results.

**Keywords.** Boundary value method, Collocation method, Compact method, Cubic $C^1$-spline, Fokker-Planck equation.

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## 1. Introduction

Fokker-Planck equation arises in a number of different fields in natural science, including solid-state physics, chemical physics, quantum optics, theoretical biology, and circuit theory. A Fokker-Planck equation describes the change of probability of a random function in space and time; hence it is naturally used to describe solute transport. The Fokker-Planck equation was first used by Fokker and Planck (for instance, see [14]) to describe the Brownian motion of particles.

In one variable case, the general Fokker-Planck equation is written in the following form

$$\frac{\partial u(x,t)}{\partial t} = \left[-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right]u(x,t), \tag{1.1}$$

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$$(x,t) \in [a,b] \times [0,T],$$

with initial condition

$$u(x,0) = \varphi(x),$$

and the boundary conditions

$$u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), \quad t \geq 0, \tag{1.2}$$

where $B(x) > 0$ for all $x \in [a,b]$, and $A(x)$ is the drift coefficient and $B(x)$ is the diffusion coefficient. We assume that $\psi_1$ and $\psi_2$ are smooth functions.

It is worth noting that some semi-analytic techniques are employed to solve the Fokker-Planck equation. For example, this equation is investigated in [20] using the Adomian decomposition method. Also, the variational iteration method is developed in [8] to solve this equation. For some other investigations on this model or some other similar models, the interested readers can see references [6, 8, 12, 13, 21]. Authors of [22] developed a finite difference technique to solve the type of Fokker-Planck equations describing the stochastic dynamics of a particle in a storage ring.

In [23] a finite difference procedure is given for solving the Fokker-Planck equation in two dimensions. Lakestani in [9] proposed a numerical scheme for Fokker-Planck equation using the cubic B-spline scaling functions. For more applications of the model studied in this work the interested reader can see [3, 7, 11].

The basic approach for high-order compact difference methods is to introduce the standard compact difference approximations to the differential equations and then by repeated differentiation and associated compact differencing, a new high-order compact scheme will be developed that incorporates the effect of the leading truncation error terms in the standard method [18]. Recently due to the high-order, compactness and high resolution, we have seen many publications for high-order compact difference methods in computational fluid dynamics, computational acoustics and electromagnetic [4, 17, 18].

In this paper, a forth-order compact finite difference scheme for solving the equation (1.1) is introduced. By the scheme computations, the Fokker-Planck equation is reduced to a system of ordinary differential equations. The system is solved by both boundary value method and cubic spline technique separately and pointwise approximations are obtained. Furthermore, by the cubic spline technique in each space step a closed-form approximation can be obtained.

The organization of this paper is as follows: In section 2, we introduce a forth-order compact finite difference scheme and we obtain a system of ordinary differential equations from equation (1.1). In sections 3 and 4, cubic $C^1$-spline collocation technique and boundary value methods for the time integration are presented respectively. Validation of the proposed methods is shown in section 5 through some examples.

## 2. A fourth-order compact finite difference scheme

In this section, we state the fourth-order compact finite difference scheme for the spatial derivatives of (1.1). Consider the following partial differential equation

$$\frac{\partial u}{\partial x}\left(2\frac{dB}{dx}-A\right)+\frac{\partial^2 u}{\partial x^2}B=Q(x,t),\quad a<x<b. \tag{2.1}$$

For positive integer $n$ let $\Delta x=\frac{b-a}{n}$ denotes the step size of spatial derivatives and define

$$x_r=a+rh,\ \ r=0,1,\cdots,n.$$

If we denote the central difference schemes of order two for second and first derivatives of $u$ as $\delta_x^2 u=\frac{u_{r+1}-2u_r+u_{r-1}}{h^2}$ and $\delta_x u=\frac{u_{r+1}-u_{r-1}}{2h}$, respectively, then at each point $x_r$ we have the following relation for equation (2.1)

$$\delta_x u_r\left(2\frac{dB_r}{dx}-A_r\right)+\delta_x^2 u_r B_r-\tau_r=Q_r, \tag{2.2}$$

in which

$$\tau_r=\frac{h^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_r\left(2\frac{dB_r}{dx}-A_r\right)+B_r\frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_r+O(h^4). \tag{2.3}$$

In order to obtain a fourth-order scheme, the third and fourth derivatives of $u$ in (2.3) should be approximated. By differention of (2.2) we get

$$\left.\frac{\partial^3 u}{\partial x^3}\right|_r=\frac{1}{B_r}\left[\frac{\partial Q}{\partial x}-\frac{\partial^2 u}{\partial x^2}\left(3\frac{dB}{dx}-A\right)+\frac{\partial u}{\partial x}\left(2\frac{d^2B}{dx^2}-\frac{dA}{dx}\right)\right]_r. \tag{2.4}$$

Similarly from (2.4) we can write

$$\begin{aligned}\left.\frac{\partial^4 u}{\partial x^4}\right|_r=\frac{1}{B_r}\Bigg[&\frac{\partial^2 Q}{\partial x^2}+\frac{\partial^3 u}{\partial x^3}\left(A-4\frac{dB}{dx}\right)\\&-\frac{\partial^2 u}{\partial x^2}\left(5\frac{d^2B}{\partial x^2}-2\frac{dA}{dx}\right)\\&-\frac{\partial u}{\partial x}\left(2\frac{d^3B}{dx^3}-\frac{d^2A}{dx^2}\Bigg]\right)_r.\end{aligned} \tag{2.5}$$

By Eqs. (2.2), (2.3), (2.4) and (2.5) we get

$$\begin{aligned}
&\left(2\frac{dB}{dx}-A\right)_r \delta_x u_r + B_r\delta_x^2 u_r \\
&-\frac{h^2}{12}\left[\delta_x^2 u\left(-\frac{A}{B}\left(3\frac{dB}{dx}-A\ +\left(5\frac{d^2B}{dx^2}-2\frac{dA}{dx}\right)\right)\right)\right. \\
&\left.-\delta_x u\left(-\frac{A}{B}\left(2\frac{d^2B}{dx^2}-\frac{dA}{dx}\right)+\left(2\frac{d^3B}{dx^3}-\frac{d^2A}{dx^2}\right)\right)\right]_r \\
&=\left(1+\frac{h^2}{12}(\delta_x^2-\frac{A}{B}\delta_x)\right)Q_r+O(h^4).
\end{aligned} \tag{2.6}$$

The above equation is a fourth-order compact finite difference scheme for equation (1.1). Now we rewrite the equation (1.1) as follows

$$\frac{\partial u}{\partial x}\left(2\frac{dB}{dx}-A\right)+\frac{\partial^2 u}{\partial x^2}B=\frac{\partial u}{\partial t}+\frac{dA}{dx}u-\frac{d^2B}{dx^2}u. \tag{2.7}$$

By discretization of the above equation in space and by using the scheme (2.6) we have

$$\begin{aligned}
&\left(2\frac{dB}{dx}-A\right)_r \delta_x u_r + B_r\delta_x^2 u_r - \frac{h^2}{12}\left[\delta_x^2 u\left(-\frac{A}{B}\left(3\frac{dB}{dx}-A\right)\right.\right. \\
&\left.\left.+\left(5\frac{d^2B}{dx^2}-2\frac{dA}{dx}\right)\right)-\delta_x u\left(-\frac{A}{B}\left(2\frac{d^2B}{dx^2}-\frac{dA}{dx}\right)+\left(2\frac{d^3B}{\partial x^3}-\frac{d^2A}{dx^2}\right)\right)\right]_r \\
&=\left(1+\frac{h^2}{12}(\delta_x^2-\frac{A_r}{B_r}\delta_x)\right)\left(u_r{}'+\left.\frac{dA}{dx}\right|_r u_r-\left.\frac{d^2B}{dx^2}\right|_r u_r\right),
\end{aligned} \tag{2.8}$$

in which

$$u_r(t)=u(x_r,t), \quad u'_r(t)=\frac{\partial u}{\partial t}(x_r,t).$$

We put

$$\begin{aligned}
&\delta_x A = K^1, \quad \delta_x^2 A = K^2, \quad \delta_x^3 A = K^3, \quad \delta_x^4 A = K^4, \\
&\delta_x B = H^1, \quad \delta_x^2 B = H^2, \quad \delta_x^3 B = H^3, \quad \delta_x^4 B = H^4.
\end{aligned} \tag{2.9}$$

Then we can rewrite the equation (2.8) as follows:

$$\begin{aligned}
&\frac{h^2}{12}\delta_x^2 u'_r - \frac{h^2}{12}\frac{A_r}{B_r}\delta_x^2 u_r{}' + u_r{}' = \\
&\delta_x u_r\left(2H_r^1 - A_r + \frac{h^2}{12}\frac{A_r}{B_r}\left(-3H_r^2+2K_r^1\right)+\frac{h^2}{12}\left(4H_r^3-3K_r^2\right)\right) \\
&+\delta_x u_r^2\left(B_r+\frac{h^2}{12}\frac{A_r}{B_r}\left(-3H_r^1+A_r\right)+\frac{h^2}{12}\left(6H_r^2-3K_r^1\right)\right) \\
&+u_r\left(\frac{h^2}{12}\frac{A_r}{B_r}\left(K_r^2-H_r^3\right)-\left(K_r^1-H_r^2\right)-\frac{h^2}{12}\left(K_r^3-H_r^4\right)\right).
\end{aligned} \tag{2.10}$$

The above equation can be written as

$$u'_{r-1}\left(\frac{1}{12}+\frac{h}{24}\frac{A_r}{B_r}\right)+u'_r\left(\frac{5}{6}\right)+u'_{r+1}\left(\frac{1}{12}-\frac{h}{24}\frac{A_r}{B_r}\right)=$$
$$u_{r-1}\left(\frac{P_r^{(2)}}{h^2}-\frac{P_r^{(1)}}{2h}\right)+u_r\left(P_r^{(3)}-\frac{2P_r^{(2)}}{h^2}\right)+u_{r+1}\left(\frac{P_r^{(2)}}{h^2}+\frac{P_r^{(1)}}{2h}\right), \tag{2.11}$$

where

$$\begin{aligned} P_r^{(1)} &= 2H_r^1 - A_r + \tfrac{h^2}{12}\tfrac{A_r}{B_r}\left(-3H_r^2+2K_r^1\right)+\tfrac{h^2}{12}\left(4H_r^3-3K_r^2\right),\\ P_r^{(2)} &= B_r + \tfrac{h^2}{12}\tfrac{A_r}{B_r}\left(-3H_r^1+A_r\right)+\tfrac{h^2}{12}\left(6H_r^2-3K_r^1\right),\\ P_r^{(3)} &= \tfrac{h^2}{12}\tfrac{A_r}{B_r}\left(K_r^2-H_r^3\right)-\left(K_r^1-H_r^2\right)-\tfrac{h^2}{12}\left(K_r^3-H_r^4\right). \end{aligned} \tag{2.12}$$

Thus, by Eq. (2.11) a system of ordinary differential equations is obtained as

$$Vu'(t) = Wu(t) + G(t), \tag{2.13}$$

where

$$u(t) = [u_1(t), \dots, u_{n-1}(t)]^T, \tag{2.14}$$

and the tridiagonal matrices $V$ and $W$ and the vector $G(t)$ are given by

$$V=\begin{pmatrix} \frac{5}{6} & \frac{1}{12}-\frac{h}{24}\frac{A_1}{B_1} & & & \\ \frac{1}{12}+\frac{h}{24}\frac{A_2}{B_2} & \frac{5}{6} & \frac{1}{12}-\frac{h}{24}\frac{A_2}{B_2} & & \\ & \ddots & \ddots & \ddots & \\ & \frac{1}{12}+\frac{h}{24}\frac{A_{n-2}}{B_{n-2}} & \frac{5}{6} & \frac{1}{12}-\frac{h}{24}\frac{A_{n-2}}{B_{n-2}} \\ & & \frac{1}{12}+\frac{h}{24}\frac{A_{n-1}}{B_{n-1}} & \frac{5}{6} \end{pmatrix},$$

$$W=\begin{pmatrix} P_1^{(3)}-\frac{2P_1^{(2)}}{h^2} & \frac{P_1^{(2)}}{h^2}+\frac{P_1^{(1)}}{2h} & & \\ \frac{P_2^{(2)}}{h^2}-\frac{P_2^{(1)}}{2h} & P_2^{(3)}-\frac{2P_2^{(2)}}{h^2} & \frac{P_2^{(2)}}{h^2}+\frac{P_2^{(1)}}{2h} & \\ & \ddots & \ddots & \ddots \\ & \frac{P_{n-2}^{(2)}}{h^2}-\frac{P_{n-2}^{(1)}}{2h} & P_{n-2}^{(3)}-\frac{P_{n-2}^{(2)}}{h^2} & \frac{P_{n-2}^{(2)}}{h^2}+\frac{P_{n-2}^{(1)}}{2h} \\ & & \frac{P_{n-1}^{(2)}}{h^2}-\frac{P_{n-1}^{(1)}}{2h} & P_{n-1}^{(3)}-\frac{P_{n-1}^{(2)}}{h^2} \end{pmatrix},$$

$$G(t)=\begin{pmatrix} \left(\frac{P_1^{(2)}}{h^2}-\frac{P_1^{(1)}}{2h}\right)\psi_1(t)-\left(\frac{1}{12}+\frac{h}{24}\frac{A_1}{B_1}\right)\psi_1{}'(t) \\ 0 \\ \vdots \\ 0 \\ \left(\frac{P_{n-1}^{(2)}}{h^2}+\frac{P_{n-1}^{(1)}}{2h}\right)\psi_2(t)-\left(\frac{1}{12}-\frac{h}{24}\frac{A_{n-1}}{B_{n-1}}\right)\psi_2{}'(t) \end{pmatrix},$$

in which $P_r^{(1)}$, $P_r^{(2)}$ and $P_r^{(3)}$ are obtained from (2.12) and (2.9) and $\psi_1(t)$ and $\psi_2(t)$ are initial conditions (1.2).

By defining $M = V^{-1}W$ and $Q = V^{-1}$ the equation (2.13) can be written as

$$u'(t) = Mu(t) + QG(t) = F(t, u(t)). \tag{2.15}$$

## 3. Cubic $C^1$-spline collocation method

Now we apply the cubic $C^1$ spline collocation approach [15] to the system of ordinary differential equations (2.15). The cubic $C^1$ spline collocation method is an $A$-stable method for solving the first-order ordinary differential equations and has fourth-order accuracy [10, 16].

Let $\Delta t$ denote the step size of the time variable and define

$$t_j = j\Delta t, \quad j = 0, 1, \cdots .$$

Let $U(t)$ be a vector that approximates $u(t)$ such that each of its component is a cubic spline function. Now, suppose that $U(t)$ satisfies [17] at collocation points $t_{j-1}$, $t_{j-\frac{1}{2}}$ and $t_j$ in the time interval $[t_{j-1}, t_j]$ i.e. $U'(t_l) = F(t_l, U(t_l))$, $l = j-1, j-\frac{1}{2}, j$.

From [15, 16] we have the following relation

$$U(t) = U^{j-1} + \Delta t T_1(m){U'}^{j-1} + \Delta t T_2(m){U'}^{j-\frac{1}{2}} + \Delta t T_3(m){U'}^{j}, \\ t \in [t_{j-1}, t_j] \tag{3.1}$$

where

$$\begin{array}{lll} T_1(m) = m - \frac{3}{2}m^2 + \frac{2}{3}m^3, & T_2(m) = 2m^2 - \frac{4}{3}m^3, & \\ T_3(m) = -\frac{1}{2}m^2 + \frac{2}{3}m^3, & t = t_{j-1} + m\Delta t, & m \in [0,1]. \end{array}$$

By (2.15) and (3.1) we get

$$U^j = U^{j-1} + \frac{\Delta t}{6}[MU^{j-1} + QG^{j-1} + \\ 4MU^{j-\frac{1}{2}} + 4QG^{j-\frac{1}{2}} + MU^j + QG^j], \tag{3.2}$$

and

$$U^{j-\frac{1}{2}} = U^{j-1} + \frac{\Delta t}{24}[5MU^{j-1} + 5QG^{j-1} + \\ 8MU^{j-\frac{1}{2}} + 8QG^{j-\frac{1}{2}} - MU^j - QG^j], \tag{3.3}$$

in which $U^j = U(t_j)$, $C^j = C(t_j)$, ${U'}^j = U'(t_j)$ and so on. After some manipulation (3.2) and (3.3) can be written as

$$\left(I - \frac{\Delta t}{6}M\right)U^j = \left(I - \frac{\Delta t}{6}M\right)U^{j-1} + \frac{2\Delta t}{3}MU^{j-\frac{1}{2}} + \\ \frac{\Delta t}{6}Q\left(G^{j-1} + 4G^{j-\frac{1}{2}} + G^j\right), \tag{3.4}$$

and

$$\left(I-\frac{\Delta t}{3}M\right)U^{j-\frac{1}{2}}=\left(I-\frac{5\Delta t}{24}M\right)U^{j-1}-\frac{\Delta t}{24}MU^{j}+ \frac{\Delta t}{24}Q\left(5G^{j-1}+8G^{j-\frac{1}{2}}-G^{j}\right), \tag{3.5}$$

respectively, where $I$ is the $(n-1)\times(n-1)$ identity matrix. Multiplying both sides of (3.4) and (3.5) by $\left(I-\frac{\Delta t}{3}M\right)$ and $\frac{2\Delta t}{3}M$ respectively, and summing resulted equations give us

$$\left(I-\frac{\Delta t}{2}M+\frac{\Delta t^2}{12}M^2\right)U^{j}= \left(I+\frac{\Delta t}{2}M+\frac{\Delta t^2}{12}M^2\right)U^{j-1}+\left(\frac{\Delta t}{6}Q+\frac{\Delta t^2}{12}QM\right)G^{j-1}+\frac{2\Delta t}{3}QG^{j-\frac{1}{2}} +\left(\frac{\Delta t}{6}Q-\frac{\Delta t^2}{12}QM\right)G^{j}. \tag{3.6}$$

The above relation is a linear system of $(n-1)$ equations. By solving it for $U^j$, $j=1,2,\dots,N$, the discrete and then by using (2.15), (3.18) and (3.5) a cubic spline approximation of $u(x_r,t)$ in $[t_{j-1},t_j]$ can be obtained. Note that by multiplying Eq. (3.6) in $V^2$ we can avoid of any matrix inverting.

To investigate the stability of the difference scheme (3.6), we consider the homogeneous boundary conditions case of it i.e.

$$U^{j}=\Phi U^{j-1}\ ,\quad j=1,2,\dots,N,$$

where the amplification matrix is given by

$$\Phi=\left(I-\frac{\Delta t}{2}M+\frac{\Delta t^2}{12}M^2\right)^{-1}\left(I+\frac{\Delta t}{2}M+\frac{\Delta t^2}{12}M^2\right). \tag{3.7}$$

Let $\lambda\in C$ be an eigenvalue of $M$ and $z=\Delta t\lambda$. For unconditionally stability of the method the absolute values of the eigenvalues of amplification matrix $\Phi$ must be less than one, i.e.

$$\left|\frac{1+\frac{z}{2}+\frac{z^2}{12}}{1-\frac{z}{2}+\frac{z^2}{12}}\right|<1. \tag{3.8}$$

Let $z=a+bi$, $g_1(z)=1+\frac{z}{2}+\frac{z^2}{12}$ and $g_2(z)=1-\frac{z}{2}+\frac{z^2}{12}$. We have the following relations

$$|g_1(z)|^2=\left(1+\frac{a}{2}+\frac{a^2-b^2}{12}\right)^2+b^2\left(\frac{a}{6}+\frac{1}{2}\right)^2,$$

and

$$|g_2(z)|^2 = \left(1 - \frac{a}{2} + \frac{a^2 - b^2}{12}\right)^2 + b^2\left(\frac{a}{6} - \frac{1}{2}\right)^2.$$

As we see $|g_1(z)| < |g_2(z)|$ iff $a < 0$ and $|g_1(z)| \geq |g_2(z)|$ if $a \geq 0$. Therefore the relation (3.8) holds iff $Re\lambda < 0$ and $z$ be in the left-half complex plane. So, in this case the method is A-stable.

Also, since the amplification matrix, $\Phi$, in (3.7) is the (2,2) Pade approximation of $e^{\Delta t M}$, so the method has fourth-order accuracy in time component.

## 4. Boundary value methods (BVMs)

Compared to the other initial value solvers, BVMs have the advantage of both unconditional stability and high-order accuracy. Consider the following initial value problem

$$y'(t) = f(t, y(t)), \quad t \in [t_0, T], \quad y(t_0) = y_0. \tag{4.1}$$

A k-step formula of BVMs for approximating the problem (4.1) can be written as [**?**, **?**]

$$\sum_{i=0}^{k} \alpha_i y_{i+j} = \Delta t \sum_{i=0}^{k} \beta_i f_{i+j}, \quad j = 0, 1, \ldots, N-k, \tag{4.2}$$

where $y_r \approx y(r\Delta t)$, $t_r = r\Delta t$ and $f_r = f(t_r, y_r)$. The BVM (4.2) requires $\gamma$ initial conditions and $k - \gamma$ final conditions, i. e. we need the values of $y_0, y_1, \ldots, y_{\gamma-1}$ and $y_{N-k+\gamma+1}, y_{N-k+\gamma+2}, \ldots, y_N$. The initial condition in (4.1) provides the value $y_0$. The extra $\gamma - 1$ initial and $k - \gamma$ final conditions are of the form

$$\sum_{i=0}^{k} \alpha_i^{(j)} y_i = \Delta t \sum_{i=0}^{k} \beta_i^{(j)} f_i \ , \quad j = 1, \ldots, \gamma - k, \tag{4.3}$$

and

$$\sum_{i=0}^{k} \alpha_i^{(j)} y_{N-k+i} = \Delta t \sum_{i=0}^{k} \beta_i^{(j)} f_{N-k+i} \ , \quad j = N-k+\gamma+1, \ldots, N, \tag{4.4}$$

where the coefficients ${\alpha_i}^{(j)}$ and ${\beta_i}^{(j)}$ are chosen such that truncation errors for the initial and final conditions are of the same order as for the basic formula (4.2).

A fourth-order BVM approximation of (4.2) by $k = 3$ and $\gamma = 2$ is as follows [1, 19]

$$\frac{1}{12}(y_{j+3} + 9y_{j+2} - 9y_{j+1} - y_j) = \frac{\Delta t}{2}(f_{j+2} + f_{j+1}) \ , \quad j = 0, \ldots, N-3. \tag{4.5}$$

The additional equations associated with the initial and final conditions are respectively

$$\frac{1}{24}(-17y_0 + 9y_1 + 9y_2 - y_3) = \frac{\Delta t}{4}(3f_1 + f_0), \tag{4.6}$$

and

$$\frac{1}{24}(y_{N-3} - 9y_{N-2} - 9y_{N-1} + 17y_N) = \frac{\Delta t}{4}(3f_{N-1} + f_N). \tag{4.7}$$

For Eq. (2.15) the BVM with $k = 3$ and $\gamma = 2$ leads to a linear system of equations as

$$\begin{aligned} &(A_M \otimes V - \Delta t B_M \otimes W)U = \\ &\Delta t(B_M \otimes I_{n-1})G + \Delta t(b_0 \otimes (W u_0 + G_0)) - a_0 \otimes (V u_0), \end{aligned} \tag{4.8}$$

in which

$$\begin{aligned} &u_0 = [u_1(t_0), u_2(t_0), \ldots, u_{n-1}(t_0)]^T, \\ &G_0 = [G_1(t_0), 0, \ldots, 0, G_{n-1}(t_0)]^T, \\ &U = [u_1(t_1), u_2(t_1), \ldots, u_{n-1}(t_1), u_1(t_2), u_2(t_2), \ldots, u_{n-1}(t_2), \ldots, u_1(t_N), \\ &u_2(t_N), \ldots, u_{n-1}(t_N)]^T, \\ &G = [G_1(t_1), 0, \ldots, 0, G_{n-1}(t_1), G_1(t_2), 0, \ldots, 0, G_{n-1}(t_2), \ldots, G_1(t_N), \\ &0, \ldots, 0, G_{n-1}(t_N)]^T, \end{aligned}$$

where $G_1(t)$ and $G_{n-1}(t)$ are respectively the first and the last component of the vector $G(t)$ in (2.13) and are obtained by (2.12), (2.11), (2.9) and (1.2). Also in (4.8)

$$A_M = \begin{bmatrix} \frac{9}{24} & \frac{9}{24} & \frac{-1}{24} & & & \\ \frac{-9}{12} & \frac{9}{12} & \frac{1}{12} & & & \\ \frac{-1}{12} & \frac{-9}{12} & \frac{9}{12} & \frac{1}{12} & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \frac{-1}{12} & \frac{-9}{12} & \frac{9}{12} & \frac{1}{12} \\ & & \frac{1}{24} & \frac{-9}{24} & \frac{-9}{24} & \frac{17}{24} \end{bmatrix},$$

$$B_M = \begin{bmatrix} \frac{3}{4} & 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ & \ddots & \ddots & & \\ & & \frac{1}{2} & \frac{1}{2} & \\ & & & \frac{3}{4} & \frac{1}{2} \end{bmatrix},$$

$$a_0 = \begin{bmatrix} \frac{-17}{24}, & \frac{-1}{12}, & 0, & \ldots & 0 \end{bmatrix}^T,$$

and

$$b_0 = \begin{bmatrix} \frac{1}{4}, & 0, & \ldots & 0 \end{bmatrix}^T.$$

By solving the linear system of equations (4.8), we can obtain the unknown solutions $u_i(t_j)$, $i = 1, 2, \dots, n-1$, $j = 1, 2, \dots, N$.

## 5. Numerical experiments

We applied the methods presented in this article and solved several examples. We performed our computations using **Maple 13** software.

5.1. **Test problem 1.** Consider equation (1.1) with $A(x) = 1$ and $B(x) = 1$ and following initial condition $\varphi(x) = \exp\left(-\frac{(x-1)^2}{4}\right)$. The exact solution is given with

$$u(x,t) = \frac{1}{\sqrt{1+t}} \exp\left(-\frac{(x-(t+1))^2}{4(1+t)}\right).$$

The boundary conditions can be obtained easily from the exact solution. By applying the techniques described in sections 3 and 4, the equation is solved with different values of $h$ and $\Delta t$. The maximum errors of approximate solutions for $T = 1$ , are shown in Table 1.

Table 1. Maximum errors obtained for Problem 1 at $T = 1$.

| $h = \Delta t$ | Spline method | BVM |
|---|---|---|
| 1/10 | $2.545 \times 10^{-7}$ | $1.574 \times 10^{-7}$ |
| 1/20 | $2.600 \times 10^{-8}$ | $3.300 \times 10^{-9}$ |
| 1/40 | $3.190 \times 10^{-9}$ | $2.277 \times 10^{-10}$ |

5.2. **Test problem 2.** Consider equation (1.1) with $A(x) = -1$ and $B(x) = 1$ [9]. The exact solution is given with

$$u(x,t) = x + t.$$

The boundary conditions can be obtained easily from the exact solution. By applying BVM and spline method, the equation is solved. We compared the numerical results, with the results of the method presented in [9]. The numerical results for $T = 1$ with $h = \Delta t = 0.1$ are shown in Table 2.

Table 2. Errors obtained for Problem 2 at $T = 1$.

| Grid point | Method in [9] | Spline method | BVM |
|---|---|---|---|
| 0.2 | $1.3 \times 10^{-6}$ | $3.00 \times 10^{-7}$ | $4.5 \times 10^{-12}$ |
| 0.4 | $2.0 \times 10^{-6}$ | $2.99 \times 10^{-8}$ | $1.2 \times 10^{-11}$ |
| 0.6 | $3.0 \times 10^{-6}$ | $3.00 \times 10^{-7}$ | $1.8 \times 10^{-11}$ |
| 0.8 | $4.4 \times 10^{-6}$ | $5.21 \times 10^{-7}$ | $1.7 \times 10^{-11}$ |

5.3. **Test problem 3.** Consider equation (1.1) with $A(x) = \frac{-1}{x+1}$ and $B(x) = 1$. The exact solution is given with

$$u(x,t) = (x+1)^3 + 8(x+1)t, \quad x \in [0, 0.1].$$

The boundary conditions and initial condition can be obtained easily from the exact solution. By using the introduced methods this equation is solved. The obtained errors of approximations for time $T = 0.1$ are given in Table 3. The computations have been performed by equal sizes of $h$ and $\Delta t$. The computational order $(C-order)$ of convergence is obtained by

$$C - order = \frac{log(\frac{E(2h,2\Delta t)}{E(h,\Delta t)})}{log(2)},$$

in which $E(h, \Delta t)$ is the absolute error by space step size $h$ and time step size $\Delta t$ at the time $T = 0.1$. By Table 3, corresponding to the points 0.02, 0.04, 0.06 and 0.08, the $C - order$s for the spline method are 4, 4.031, 4.033 and 3.998 and for BVM are 3.027, 3.835, 3.889 and 3.703, respectively. These confirm our theoretical findings. Our computations for $h = \Delta t$= 0.005 and 0.0025, are performed for spline method in 3.6 and 17.2 seconds and for BVM in 8.7 and 81.5 seconds, respectively.

Table 3. Errors obtained at $T = 0.1$ for Problem 3.

| | Spline method | | BVM | |
|---|---|---|---|---|
| Grid point | $h = 0.005$ | $h = 0.0025$ | $h = 0.005$ | $h = 0.0025$ |
| 0.02 | $8.96 \times 10^{-5}$ | $5.60 \times 10^{-6}$ | $1.06 \times 10^{-5}$ | $1.03 \times 10^{-6}$ |
| 0.04 | $9.19 \times 10^{-5}$ | $5.62 \times 10^{-6}$ | $3.11 \times 10^{-5}$ | $2.18 \times 10^{-6}$ |
| 0.06 | $9.23 \times 10^{-5}$ | $5.64 \times 10^{-6}$ | $4.89 \times 10^{-5}$ | $3.30 \times 10^{-6}$ |
| 0.08 | $9.01 \times 10^{-5}$ | $5.64 \times 10^{-6}$ | $5.94 \times 10^{-5}$ | $4.56 \times 10^{-6}$ |

## 6. Conclusion

In this paper, we proposed a class of finite difference schemes, for solving Fokker-Planck equation. First, we combined a high-order compact finite difference scheme of fourth-order to approximate the spatial derivative with a boundary value scheme and then we combined the difference scheme with cubic $C^1$-spline collocation technique, for time integration. Both these joined methods have fourth-order accuracy in time and space variables. The numerical results confirm the validity of the methods. In the spline method, we should solve $N$ linear systems of $(n-1)$ equations. But in the boundary value method, we need to solve a linear system of $N \times (n-1)$ equations and we get all the grid points at once. It makes it possible to have more time and more memory. The BVM gives more accurate pointwise approximations than the solutions of another method. But, using the spline method in each space step a closed-form approximation is obtained.

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