Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 8, No. 3, 2020, pp. 523-536
DOI:10.22034/cmde.2020.28975.1407

Symmetry analysis and exact solutions of acoustic equation

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Abstract The Lie symmetry method for differential equations is applied to study the exact solutions of the acoustic PDE. This study is based on two methods: Kudryashov method and the direct method. Some exact solutions are found by using the similarity variables extracted from symmetries.

Keywords. Lie symmetry, group-invariant solution, acoustic equation.

2010 Mathematics Subject Classification. 35L05, 76M60.

1. Introduction

Symmetry analysis of differential equations is a method based on finding some differential operators (vector fields) which called symmetries. These operators are the largest local group of transformations acting on the independent and dependent variables of the system with the property that they transform solutions of the system to other solutions. When we are confronted with a complicated system of PDEs arising from some physically important problem, the discovery of any explicit solution has great importance. Explicit solutions can be used as a model for physical experiments, as benchmarks for testing numerical methods, etc., and often reflect the asymptotic or dominant behavior of more general types of solutions.

The determination of symmetry group of the geometric object can be regarded as a special case of the general equivalence problem. Indeed, provided it lies in the admissible class of changes of variables, a symmetry is merely a self-equivalence of the object. Thus, for instance, the solution of the equivalence problem for differential equations will include a determination of all symmetries of a given differential equation. Two equivalent objects have isomorphic symmetry group, indeed, conjugating any symmetry of the first object by the equivalence transformation produces a symmetry of the second. Thus, one means of recognizing equivalent objects is by inspecting their symmetry group: If the two symmetry groups are not equivalent, e.g.,

Received: 22 August 2018 ; Accepted: 21 April 2019.
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they have different dimensions, or different structures, then the two objects cannot be equivalent. Of course, having isomorphic symmetry groups is no guarantee that the two objects are equivalent; nevertheless, in many highly symmetric cases, including linearization problems, the existence of a suitable symmetry group is both necessary and sufficient for the equivalence of two objects.

The concept of an invariant proves to be of crucial importance in understanding an equivalence problem. By definition, an invariant is a quantity that is unaffected by the chnage of variables. Consequently, two equivalent objects must necessarily have the same invariants. Conversely, in the regular case, if we know enough invariant functions, we can use them to completely characterize the equivalent objects, and thereby completely solve the equivalence problem. Thus, the construction of invariants and their characterization lies at the heart of most approaches to equivalence problems. In addition, every regular invariant system of equations (algebraic, differential, variational, etc.) can be characterized by functions relationships among the invariant functions, so the invariants form the fundamental building blocks which can be used to construct suitably symmetric objects, a process of immense utility in modern physics. Whereas invariant functions are the most important invariant quantities associated with the equivalence, many other invariant objects such as vector fields, differential forms, differential operators, etc. arise naturally and play important roles.

One of the most important applications of symmetry's method is to reduce the systems of differential equations, i.e., finding equivalent systems of differential equations of simpler form, that is called reduction and the obtained solutions are called group-invariant solutions. This system gives us some explicate solutions of the primary system more easily. This method provides a systematic computational algorithm for determining a large class of special solutions. The solutions to the obtained equivalent system will correspond to solutions of the original system. There are a lot of papers in the literature for this process and one can find the classical reduction method in [3, 4, 7]. As it discussed above, these systems and their solutions give some equivalence quantities. For example, in our study, the emphasis will be on the group-invariant solutions of the acoustic PDE. These exact solutions solve the equivalence problem of wave's propagation and diffusion in three-dimensional space. More precisely, the reduced systems classify the acoustic waves and their invariant physical quantities under their symmetries. Lie symmetries of differential equations, and their applications to find analytic solutions of the equations are described in detail in several monographs on the subject (e.g. [1, 2, 5, 6, 8]) and in numerous papers in the literature (e.g. [7, 9]). A short presentation for Lie symmetry method is introduced in the sequel.

Acoustics is a branch of physics and acoustics dealing with sound waves of sufficiently large amplitudes. Large amplitudes require using full systems of governing equations of fluid dynamics (for sound waves in liquids and gases) and elasticity (for sound waves in solids). These equations are generally non-linear, and their traditional linearization is no longer possible. The solutions to these equations show that sound waves are being distorted as they travel. A sound wave propagates through a material as a localized pressure change. Increasing the pressure of a gas or fluid increases its local temperature. The local speed of sound in a compressible material increases with temperature; as a result, the wave travels faster during the high-pressure phase of the oscillation than during the lower pressure phase. This affects the wave's frequency structure; for example, in an initially plane sinusoidal wave of a single frequency, the peaks of the wave travel faster than the troughs, and the pulse becomes cumulatively more like a sawtooth wave. In other words, the wave self-distorts. In doing so, other frequency components are introduced, which can be described by the Fourier series. This phenomenon is characteristic of a non-linear system, since a linear acoustic system responds only to the driving frequency. This always occurs but the effects of geometric spreading and absorption usually overcome the self distortion, so, linear behavior usually prevails and nonlinear acoustic propagation occurs only for very large amplitudes and only near the source. Additionally, waves of different amplitudes will generate different pressure gradients, contributing to the non-linear effect.

The common acoustic PDE is usually written as

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} - u\frac{\partial u}{\partial t}\right) = -\beta u, \qquad \beta = \text{const.}$$

This equation has a generalization in the form of

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} - P(u)\frac{\partial u}{\partial t}\right) = F(x,u). \tag{1.1}$$

In this paper, the symmetry analysis of the equation (1.1) is considered for finding reduced forms and exact solutions.

The paper is organized as follows: Section 2 is devoted to the Lie algorithm of finding symmetry operators. In section 3 exact solutions for some special cases of (1.1) are found via two methods. Kudryashov methods, which deals with special cases of symmetries and the direct method which is based on similarity variables obtained from the operators. It is noteworthy that both methods have several similarities in common but with little differences. The similarities are in the method of reduction and similarity variables construction. But, the constructed solutions are different in almost everywhere. For example, the solutions obtained from Kudryashov method are very complicated while in direct method we have more tangible solutions.

2. Symmetry operastors of equation (1.1)

Consider a system of differential equations (PDE or ODE) in the dependent variables $u^\alpha (1 \le \alpha \le q)$ and independent variables $x^i (1 \le i \le p)$ of the form:

$$\Delta^s(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, ...) = 0, \quad 1 \le s \le k, \tag{2.1}$$

where the subscripts denote partial derivatives (e.g. $u^\alpha_i = \partial u^\alpha / \partial x^i$). To determine continuous symmetries of (2.1), it is useful to consider infinitesimal Lie transformations of the form:

$$\tilde{x}^i = x^i + \varepsilon \xi^i + O(\varepsilon^2), \quad \tilde{u}^\alpha = u^\alpha + \varepsilon \phi^\alpha + O(\varepsilon^2), \tag{2.2}$$

that leave the equation system invariant to $O(\varepsilon^2)$. Lie point symmetries correspond to the case where the infinitesimal generators $\xi^i = \xi^i(x^i, u^\alpha)$ and $\phi^\alpha = \phi^\alpha(x^i, u^\alpha)$ depend only on the x^i and the u^α and not on the derivatives or integrals of the u^α.

It is noteworthy that when the transformations (2.2) also depend on the derivatives or integrals of the u^α, the obtained symmetries are called generalized symmetries.

The infinitesimal transformations for the first and second derivatives to $O(\varepsilon^2)$ are given by the prolongation formula:

$$\tilde{u}_i^\alpha = u_i^\alpha + \varepsilon\zeta_i^\alpha, \quad \tilde{u}_{ij}^\alpha = u_{ij}^\alpha + \varepsilon\zeta_{ij}^\alpha, \tag{2.3}$$

where

$$\zeta_i^\alpha = D_i\hat{\phi}^\alpha + \xi^s u_{si}^\alpha, \quad \zeta_{ij}^\alpha = D_iD_j\hat{\phi}^\alpha + \xi^s u_{sij}^\alpha. \tag{2.4}$$

Here

$$\hat{\phi}^\alpha = \phi^\alpha - \xi^s u_s^\alpha, \tag{2.5}$$

corresponds to the canonical Lie transformation for which $\tilde{x}^i = x^i$ and $\tilde{u}^\alpha = u^\alpha + \varepsilon\hat{\phi}^\alpha$. The symbol D_i in (2.4) denotes the total derivative operator with respect to x^i. Similar formula to (2.4) applies to the transformation of the higher-order derivatives.

The condition for invariance of the system of differential equations (2.1) to $O(\varepsilon^2)$ under the Lie transformation (2.2) can be expressed in the form:

$$\mathcal{L}_X\Delta^s \equiv \tilde{X}(\Delta^s) = 0, \quad \text{whenever} \quad \Delta^s = 0, \quad 1 \leq s \leq k, \tag{2.6}$$

where

$$\tilde{X} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha} + \cdots, \tag{2.7}$$

is the prolongation of the vector field

$$X = \xi^i \frac{\partial}{\partial x^i} + \phi^\alpha \frac{\partial}{\partial u^\alpha}, \tag{2.8}$$

associated with the infinitesimal transformation (2.2). The symbol $\mathcal{L}_X\Delta^s$ in (2.6) denotes the Lie derivative of Δ^s with respect to the vector field X (i.e. $\mathcal{L}_X\Delta^s = \frac{\mathrm{d}\Delta^s}{\mathrm{d}\varepsilon}|_{\varepsilon=0}$).

According to the Eq. (1.1) has a one-parameter Lie point transformation of the form

$$\begin{array}{rcl} x & \longmapsto & x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \\ t & \longmapsto & t + \varepsilon\tau(x,t,u) + O(\varepsilon^2), \\ u & \longmapsto & u + \varepsilon\phi(x,t,u) + O(\varepsilon^2), \end{array} \tag{2.9}$$

including a special one-parameter transformation

$$\begin{array}{rcl} P & \longmapsto & P + \varepsilon\lambda(x,t,u,u_x,u_t,P) + O(\varepsilon^2), \\ F & \longmapsto & F + \varepsilon\mu(x,t,u,u_x,u_t,F) + O(\varepsilon^2), \end{array} \tag{2.10}$$

for the arbitrary functions in (1.1).

A symmetry operator or an infinitesimal generator corresponding to (2.9) and (2.10) is given by

$$\begin{aligned} X = {} & \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u} \\ & + \lambda(x,t,u,u_x,u_t,P)\frac{\partial}{\partial P} + \mu(x,t,u,u_x,u_t,F)\frac{\partial}{\partial F}. \end{aligned} \tag{2.11}$$

Because of the order of equation (2.9) and the invariance condition (2.6), the affection of the second prolongation of the operator (2.11), i.e.,

$$\begin{aligned} X^{(2)} = X &+ \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}} \\ &+\omega_1 \frac{\partial}{\partial \lambda_x} + \omega_2 \frac{\partial}{\partial \lambda_t} + \omega_3 \frac{\partial}{\partial \lambda_u} + \omega_4 \frac{\partial}{\partial \lambda_{u_t}} + \omega_5 \frac{\partial}{\partial \lambda_{u_{xt}}} + \omega_6 \frac{\partial}{\partial \lambda_{u_{tt}}} \\ &+\eta_1 \frac{\partial}{\partial \lambda_x} + \eta_2 \frac{\partial}{\partial \lambda_t} + \eta_3 \frac{\partial}{\partial \lambda_u} + \eta_4 \frac{\partial}{\partial \lambda_{u_t}} + \eta_5 \frac{\partial}{\partial \lambda_{u_{xt}}} + \eta_6 \frac{\partial}{\partial \lambda_{u_{tt}}}, \end{aligned} \tag{2.12}$$

on the (2.9) must be vanished. The prolongation coefficients of $X^{(2)}$ are obtained by the characteristic linear PDE system constructing from (2.4). A complicated computation shows that the Lie algebra of symmetry operators for the equation (2.9) spanned by the following geometric vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= \frac{\partial}{\partial u}, \\ X_4 &= t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + P\frac{\partial}{\partial P}, \\ X_5 &= u\frac{\partial}{\partial u} + F\frac{\partial}{\partial F}, \\ X_6 &= x\frac{\partial}{\partial t} - \frac{\partial}{\partial P}, \\ X_7 &= x\frac{\partial}{\partial x} - P\frac{\partial}{\partial P} - F\frac{\partial}{\partial F}. \end{aligned}$$

For some special cases of P and F the Eq. (1.1) may be admitted some additional symmetries. This will be shown in the next section.

3. Reduction and Exact Solutions for Eq. (1.1)

When we confronted with a complicated system of partial differential equations in some physically important problems, the discovery of any explicit solutions whatsoever is of great interest. Explicit solution can be used as models for physical experiments, as benchmarks for testing numerical methods, etc., and often reflect the asymptotic or dominant behavior of more general types of solutions. In this section, two illustrative methods are applied for our purpose. First, we will work on Kudryashov method.

3.1. **Exact solutions via Kudryashov method.**

This method is discussed in [9] comprehensively. Here we implement it for the Eq. (1.1). For this goal, four different cases of symmetries are considered.

These cases are listed below:

Case 1:

$$P = F = 0, \quad \text{and} \quad \xi = f_1(x), \tau = f_2(x), \phi = C_1 u + f_3(x) + f_4(t), \tag{3.1}$$

where f_is are arbitrary smooth functions and C_1 is a constant.

Case 2:

$$P = 0, F \neq 0, \quad \text{and} \quad \xi = f_3(x,t) + f_4(t,u), \tau = x + (x+t)f_5(u) + f_6(u),$$
$$\phi = -u + f_2(x) + f_1(x), \tag{3.2}$$

where f_is are arbitrary smooth functions.

Case 3:

$$P = \frac{1}{2}, F = 0, \quad \text{and} \quad \xi = -C_1 x - \frac{1}{2}C_1 x^2 - C_2 x - f_5(t), \tau = 0,$$
$$\phi = ((x+t)C_1 + C_2)u + C_3 t + f_4(x), \tag{3.3}$$

where f_is are arbitrary smooth functions and C_is are real constant.

Case 4:

$$P = 1, F = 0, \quad \text{and} \quad \xi = f_5(t), \tau = 0,$$
$$\phi = \frac{1}{2}((2x+t)C_1 + 2C_2)u + C_3 t + f_4(x), \tag{3.4}$$

where f_is are arbitrary smooth functions and C_is are real constant.

Now for these cases, exact solutions could be found in the rest of the section.

3.1.1. *Group-invariant solution for Case 1.* Two situations for the Case 1 are considered for finding exact solutions.

1) Suppose $f_1 = \alpha, f_2 = \beta, f_3 = \gamma, f_4 = \delta$ and $C_1 = 0$. Thus, the symmetry

$$X = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial t} + (\gamma + \delta)\frac{\partial}{\partial u}, \tag{3.5}$$

yields. An integrating through the group trajectory

$$\frac{dx}{\alpha} = \frac{dt}{\beta} = \frac{du}{\gamma + \delta}, \tag{3.6}$$

gives

$$V = -\frac{\alpha}{\beta}x + t, \qquad Y = u - \left(\frac{\gamma + \delta}{\beta}\right)x, \tag{3.7}$$

as the invariant functions. Applying the chain rule on the equation (2.9) with respect to new variables (3.7) gives the reduced equation (3.8);

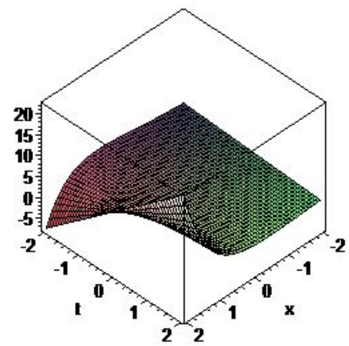
$$V_{YY} = 0. \tag{3.8}$$

This equation has a line solution $Y = k_1 Y + k_2$. By replacing the new variable in this solution with (3.7), the group-invariant solution

$$u = \left(\frac{\gamma + \delta - k_1\alpha}{\beta}\right)x + k_1 t + k_2, \tag{3.9}$$

is obtained.

Figure 1. The solution is plotted for $\alpha = \beta = k_1 = k_2 = 1$



2) Let us $f_1 = \alpha, f_2 = \beta, f_3 = f_4 = 0$ and $C_1 = 0$. The similarity variables for the given symmetry

$$X = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \tag{3.10}$$

are obtained by the characteristic system

$$\frac{dx}{\alpha} = \frac{dt}{\beta} = \frac{du}{u}. \tag{3.11}$$

The solution for the system (3.11) are

$$V = x - \frac{\alpha}{\beta}t, \qquad Y = e^{-\frac{x}{\alpha}}u. \tag{3.12}$$

A similar method gives

$$\frac{\alpha^2}{\beta^2}V_{YY} = 0. \tag{3.13}$$

Thus, the obtained similarity solution is

$$u = \left[k_1\left(x - \frac{\alpha}{\beta}\right)t + k_2\right]\exp\left(\frac{x}{\alpha}\right). \tag{3.14}$$

For some special cases of $\alpha = \beta = k_1 = k_2 = 1$ the solution (3.14) is plotted in the Fig. 1.

3.1.2. *Group-invariant solution for Case 2.* Similarly, two situations will be considered for this case.

1) Suppose $f_1 = f_3 = 0, f_2 = \mu, f_4 = \alpha$ and $f_5 = \rho$. By inserting these variables to the corresponding symmetry we have

$$X = \alpha\frac{\partial}{\partial x} + ((x+t)\rho + x)\frac{\partial}{\partial t} + (\mu - u)\frac{\partial}{\partial u}. \tag{3.15}$$

The similarity variables for this symmetry are obtained via the group trajectory

$$\frac{dx}{\alpha} = \frac{dt}{(x+t)\rho + x} = \frac{du}{\mu - u}. \tag{3.16}$$

The solution for the system (3.16) is

$$\begin{aligned} V &= \left(x + t + \frac{\alpha}{\rho} + \frac{x}{\rho} + \frac{\alpha}{\rho^2}\right) e^{-\frac{\rho}{\alpha}x}, \\ Y &= (u-\mu)e^{\frac{x}{\alpha}}. \end{aligned} \tag{3.17}$$

Applying the chain rule on the equation (2.9) with respect to new variables (3.17) gives the reduced equation (3.18);

$$\left[\frac{(x+t)\rho + x}{\alpha}\right] e^{-\frac{2\rho}{\alpha}x} V_{YY} - F(Y,V) = 0, \tag{3.18}$$

where x is obtained from (3.17). If F is considered as u the, we have the following explicit group-invariant solution:

$$\begin{aligned} &\alpha u e^{\frac{2\rho Y}{\alpha}} \ln((Y+t)\rho + Y)\rho t + \alpha u e^{\frac{2\rho Y}{\alpha}} \ln((Y+t)\rho + Y)\rho Y \\ &+\alpha u e^{\frac{2\rho Y}{\alpha}} \ln((Y+t)\rho + Y)Y - \alpha u e^{\frac{2\rho Y}{\alpha}} \rho t - 2\alpha u e^{\frac{2\rho Y}{\alpha}} Y + C_1\rho^2 Y \\ &+2C_1\rho Y + C_2\rho^2 + C_1 Y + 2C_2\rho + C_2 = 0. \end{aligned}$$

For another example if $F = xu$ then,

$$\begin{aligned} &2\alpha e^{\frac{2\rho Y}{\alpha}} \rho^2 t^2 \ln((Y+t)\rho + Y) + 2\alpha u e^{\frac{2\rho Y}{\alpha}} \rho^2 t \ln((Y+t)\rho + Y)\rho \\ &+2\alpha u e^{\frac{2\rho Y}{\alpha}} \rho t \ln((Y+t)\rho + Y)Y - 2\alpha u e^{\frac{2\rho Y}{\alpha}} \rho^2 t^2 - 2\alpha u e^{\frac{2\rho Y}{\alpha}} \rho^2 t Y - \alpha e^{\frac{2\rho Y}{\alpha}} \rho^2 u Y^2 \\ &-2\alpha u e^{\frac{2\rho Y}{\alpha}} \rho t Y - 2\alpha e^{\frac{2\rho Y}{\alpha}} \rho u Y^2 - \alpha u e^{\frac{2\rho Y}{\alpha}} Y^2 - 2C_1\rho^3 Y - 6C_1\rho^2 Y - 2C_2\rho^3 \\ &-6C_2\rho^2 - 2C_1 Y - 6C_2\rho - 2C_2 = 0. \end{aligned}$$

2) In this case suppose $f_3 = \alpha, f_4 = \beta, f_5 = 0$ and $f_6 = \pi$. Thus, for this symmetry the similarites are:

$$\begin{aligned} Y &= \frac{2\alpha t + 2\beta t - 2\pi x - x^2}{\alpha + \beta}, \\ V &= e^{\frac{x}{\alpha+\beta}}(u - \rho - \mu). \end{aligned} \tag{3.19}$$

Consequently the reduced equation

$$(\pi + 2x)V_{YY} - F(Y,V) = 0. \tag{3.20}$$

The equation (3.19) gives different group-invariant solutions for different cases of F. For example

- For $F = u$ the similarity solution is

$$\begin{aligned} &\frac{1}{2}u\ln(\pi + 2Y)Y + \frac{1}{4}\pi u \ln(\pi + 2Y) \\ &-\frac{1}{4}\pi u - \frac{1}{2}Yu + C_1 Y + C_2 \\ &-e^{\frac{x}{\alpha+\beta}}(u - \rho - \mu) = 0. \end{aligned}$$

- For $F = xu$ we have

$$\begin{aligned}&\frac{1}{4}uY^2 - \frac{1}{4}\pi u \ln(\pi + 2Y)Y\\&-\frac{1}{8}\pi^2 u \ln(\pi + 2Y) + \frac{1}{8}\pi^2 u + \frac{1}{4}\pi uY + C_1 Y + C_2\\&-e^{\frac{x}{\alpha+\beta}}(u - \rho - \mu) = 0.\end{aligned}$$

- For $F = x$ the group-invariant solution is

$$\begin{aligned}&\frac{1}{4}Y^2 - \frac{1}{4}\pi \ln(\pi + 2Y)Y\\&-\frac{1}{8}\pi^2 \ln(\pi_2 Y) + \frac{1}{8}\pi^2 + \frac{1}{4}\pi Y + C_1 Y + C_2\\&-e^{\frac{x}{\alpha+\beta}}(u - \rho - \mu) = 0.\end{aligned}$$

3.1.3. *Group-invariant solution for Case 3.* Let us consider $C_1 = C_2 = C_3 = 0, f_4 = \alpha$ and $f_5 = \beta$. Consequently, the similarity variables are

$$\begin{aligned}Y &= t,\\V &= u - \frac{\beta}{\alpha}x.\end{aligned} \tag{3.21}$$

Variables (3.21) reduce the equation (1.1) to

$$V_{YY} = 0. \tag{3.22}$$

Thus, the group-invariant solution is

$$u = C_1 \frac{\beta}{\alpha}x + C_2.$$

3.1.4. *Group-invariant solution for Case 4.* Suppose $C_1 = C_2 = 1, C_3 = f_4 = 0$ and $f_5 = \beta$. For these changes the similarities are

$$\begin{aligned}Y &= t,\\V &= u \exp\left[-\frac{x(x+t+2)}{\beta}\right].\end{aligned} \tag{3.23}$$

Likewise (3.21), these variables reduce the equation (1.1) to

$$V_{YY} = 0. \tag{3.24}$$

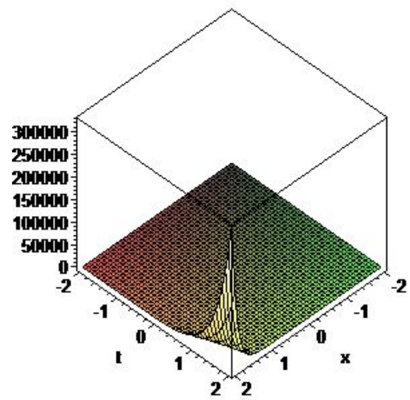
So, the group-invariant solution is

$$u = \exp\left[\frac{x(x+t+2)}{\beta}\right](C_1 t + C_2).$$

For some special cases of $\beta = C_1 = 1, C_2 = 0$ the solution (3.14) is plotted in the Fig. 2.

3.2. **Exact solutions obtained from direct group-invariant method.** In this part, other exact solutions are found via similarity variables extracted from symmetries in some special cases of Eq. (1.1) by using the direct method. Some similarity variables give so complicated reduced form. Thus, numerical simulations are needed for these kinds of equations.

FIGURE 2. The solution is plotted for $\beta = C_1 = 1, C_2 = 0$



3.2.1. *Case 1: $P = u$ and $F = 0$.* For this case the Eq. (1.1) reduces to

$$u_{xt} - u_t^2 - uu_{tt} = 0. \tag{3.25}$$

This equation admits

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t},$$
$$X_4 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \quad X_5 = \frac{x^2}{2}\frac{\partial}{\partial x} + \frac{xt}{2}\frac{\partial}{\partial t} - \frac{1}{2}(xu + t)\frac{\partial}{\partial u},$$

as the basis of symmetries. Now we can reduce the Eq. (3.25) with the similarity variables extracted from X_is.

- Symmetry X_1. This symmetry gives $t = r, u(x,t) = v(r)$ as the similarity variables. Inserting these variables to Eq. (3.25) gives the following reduced equation
$$v'^2 + vv'' = 0, \tag{3.26}$$
where ' means the derivation with respect to r. The solution of this equation is $v = \pm\sqrt{2C_1 r + 2C_2}$. Thus, the group-invariant solution for Eq. (3.25) corresponding to X_1 is
$$u(x,t) = \pm\sqrt{2C_1 t + 2C_2}.$$
This solution is plotted for $\beta = C_1 = C_2 = 1$ in Fig. 3.
- Symmetry X_2. This symmetry treats as same as X_1. It means the reduced equation is like as (3.26) but the similarity variables are $x = r, u(x,t) = v(r)$. Thus, the group-invariant solution is similar for X_1.
- Symmetry X_3. This operator gives $r = \frac{t}{x}, u(x,t) = v(r)$ as the invariants. Thus the reduced equation is
$$v''(v + r) + v'(v' + 1) = 0.$$

FIGURE 3. The solution is plotted for $\beta = C_1 = C_2 = 1$

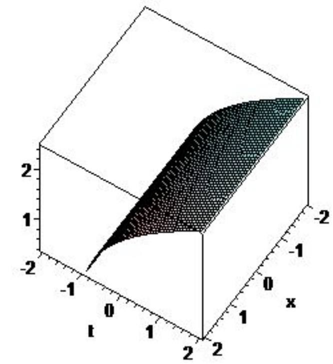
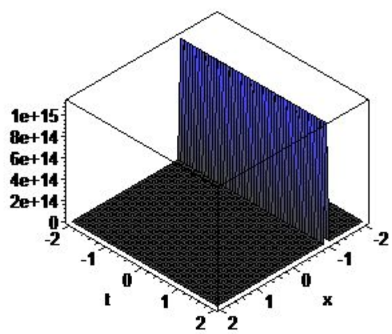


FIGURE 4. The solution is plotted for $C = 1$



So, the group-invariant solution is

$$u(x,t) = -\frac{1}{C_1}\text{LambertW}\left(-\frac{C_1 C_2 e^{-1}}{e^{C_1 \frac{t}{x}}}\right) + \frac{t}{x} + \frac{1}{C_1},$$

where LambertW is the ”Lambert W” function.

- Symmetry X_2. The similarity variables for this operator are $x = r, u(x,t) = v(r)t$. So, the reduced equation is

$$v' - v^2 = 0.$$

Finally the group-invariant solution is

$$u(x,t) = \frac{1}{-x + C}.$$

The solution is plotted in Fig. 4.

- Symmetry X_5. The similarity variables are $t = -\frac{2v(r)}{q}, u(x,t) = -\frac{1}{2}v(r)q - v(r)$ where $q = -\frac{2}{x}$. These variables gives a complicated reduces equation
$$\frac{1}{4}qvv'' - \frac{1}{4}\left(\frac{1}{2}qv' + v\right)qv'' - \frac{1}{2}\left(\frac{1}{2}qv' + 1\right)^2 + \frac{1}{2}(qv' + 1) = 0,$$
where should be solve with numerical approximations.

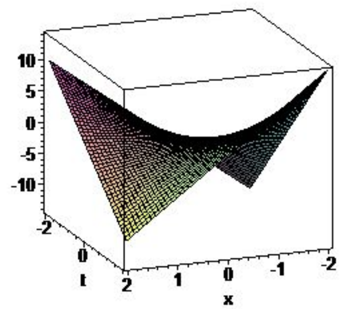
3.2.2. *Case 2: $P = u$ and $F = -\beta u$.* For this case the Eq. (1.1) reduces to
$$u_{xt} - u_t^2 - uu_{tt} = -\beta u. \tag{3.27}$$
The Lie algebra of symmetries for this equation is spanned by
$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}.$$
Now the reduction process is mentioned in the sequel.

- *Reduction for X_1.* The similarity variables are $t = r, u(x,t) = v(r)$. So, the reduced equation is
$$vv'' + v'^2 - \beta v = 0. \tag{3.28}$$
The group-invariant solution for Eq. (3.27) is a smooth function $u(x,t)$ which satisfies the following identity implicitly:
$$\int_0^{u(x,t)} \frac{ds}{\sqrt{6\beta s^3 + 3C}}ds - t - C = 0.$$
- *Reduction for X_2.* This symmetry acts the same as X_1. It means the reduced equation is like (3.28) but the similarity variables are $x = r, u(x,t) = v(r)$. Consequently, the group-invariant solution is similar for X_1.
- *Reduction for X_3.* Computations show that this symmetry gives $r = xt, u(x,t) = \frac{v(r)}{x^2}$. Thus, the reduced equation is
$$v''(v - r) + v'(v' + 1) - \beta v = 0.$$
Numerical approximations are needed here for the solutions.

3.2.3. *Case 3: $P = \frac{1}{u}$ and $F = 0$.* Eq. (1.1) reduces to
$$u_{xt} - \frac{u_{tt}}{u} - \frac{u_t^2}{u^2} = 0, \tag{3.29}$$
by this substitution. This equation admits the following four symmetries:
$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}, \quad X_4 = -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}.$$
Reduced equations are listed below:

- *Reduction for X_1.* The similarity variables $r = t, u(x,t) = v(r)$ reduce the Eq. (3.29) to
$$vv'' + v'^2 = 0.$$
Consequently, the group-invariant solution for Eq. (3.29) is
$$u(x,t) = \pm\sqrt{2C_1 t + 2C_2}.$$

FIGURE 5. $u = -3xt + t$



- *Reduction for* X_2. For the similarity variables $r = x, u(x,t) = r$ we conclude the trivial equation $v' = 0$ with the trivial solution $u = 0$.
- *Reduction for* X_3. The results for this operator is $r = \frac{t}{x}, u(x,t) = v(r)$. So, the reduced equation

$$v''v(rv + 1) + v'(v' + v^2) = 0,$$

is derived. Numerical simulations are needed for the solutions.
- *Reduction for* X_4. The similarity variables are $r = x, u(x,t) = \frac{v(r)}{t}$. If we insert these new variables to Eq. (3.29) we derive the following reduced equation:

$$v' = -3.$$

So, it gives the solution

$$u(x,t) = -3xt + Ct.$$

The solution for $C = 1$ is plotted in Fig. 5

3.2.4. *Case 4:* $P = e^u, F = 0$. The new equation

$$u_{xt} - e^u u_{tt} - e^u u_t^2 = 0, \tag{3.30}$$

is derived by this substitution. This equation has a two-dimensional symmetry algebra spanned by

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}.$$

The similarity variables $r = x, u(x,t) = v(r)$ gives a trivial algebraic equation $u = 0$. Thus, we do not have any group-invariant solution. But for X_2 we have $r = x, u(x,t) = v(r)t$ as the similarities. The reduced equation for this case is

$$v' - v^2 e^v = 0.$$

Thus, the group invariant solution for Eq. (3.30) is the implicit function

$$x + \frac{e^{-u}}{u} - \mathrm{Ei}(1, u) + C = 0,$$

where Ei is the exponential integral

$$\mathrm{Ei}(a, z) = \int_1^{\infty} e^{-sz} s^{-a} ds, \qquad \mathrm{Re}(z) > 0.$$

4. Conclusion

In this paper, an application of Lie symmetries of differential equations is applied in order to find some exact solutions of the acoustic equation. In the first step, the geometric vector fields of symmetries are found then the reduction process is illustrated precisely with two separate methods. This process leads to a class of exact solutions called group-invariant solutions. It is noteworthy that we can extend the obtained solutions to another's by using the flow of symmetries.

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