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Optimal homotopy asymptotic and multistage optimal homotopy asymptotic methods for Abel Volterra integral equation of the second kind

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Abstract In this paper, optimal homotopy asymptotic method (OHAM) and multistage optimal homotopy asymptotic (MOHAM) method are applied to find an approximate solution to Abel's integral equation, that is in fact a weakly singular Volterra integral equation. To illustrate these approaches one example is presented. The results confirm the efficiency and ability of these methods to such equations. The results will be compared with the exact solution to find out that which method of these two is more accurate.

Keywords. Abel integral equation, Weakly singular Volterra equations, Optimal homotopy asymptotic method, Multistage optimal homotopy asymptotic method, Series solutions.

2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

Abel's integral equations appear in mathematical modeling of many phenomena in different branches of science and engineering such as heat conduction scattering theory, seismology, metallurgy, chemical reactions, mechanics, fluid flow, electronics and population dynamics, [6, 7, 9, 13]. Abel integral equations of the second kind are as follows

$$u(t) = f(t) + \int_0^t \frac{u(s)}{\sqrt{t-s}} ds, \tag{1.1}$$

where $f(t)$ is a known continuous function over the closed interval $[0, 1]$ and $u(t)$ is an unknown function. In recent years, Abel integral equation (1.1) has been solved by many authors such as Alipour and Rostamy via Bernstein polynomials [2], Cameron

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and McKee used Product integration methods [4], Khan and Gondal used a method called two-step Laplace decomposition algorithm [10], Kumar and et al. via homotopy perturbation transform method [11], Yousefi applied Legendre wavelets [16], and some others.

One of the powerful and efficient methods for solving integral equations is OHAM. In this paper, we apply OHAM to solve Abel's integral equation of the second kind. We will also consider a modified version of OHAM, that is called multistage optimal homotopy asymptotic method (MOHAM). This approach was introduced for the first time by Anakira et al. to approximate the solutions of differential equations with initial-values [3].

The organization of this research is as the following: in section 2, OHAM and MOHAM are introduced. In sections 3 and 4, applications of OHAM and MOHAM to Abel integral equation of the second kind are explained, respectively. Section 5 is devoted to proving the convergence of OHAM. In section 6, an illustrative example presented, and conclusion is appeared in the last section.

2. OHAM and MOHAM

These two approaches are usually applied to solve boundary value functional equations, say;

$$L\left(u(t)\right)+f(t)+N\left(u(t)\right)=0,\ B\left(u,\frac{du}{dt}\right)=0. \tag{2.1}$$

where L and N are linear and nonlinear functional operators, respectively. $f(t)$ is a known function, $u(t)$ is an unknown function, and B is a boundary operator [5, 8, 12, 13, 14].

According to OHAM we construct a homotopy $\varphi\left(u(t,P);P\right):\mathbb{R}\times[0,1]\to\mathbb{R}$ for (2.1), as the following

$$\begin{aligned}&(1-P)\left[L\left(u(t,P)\right)+f(t)\right]=H(P)\left[L\left(u(t,P)\right)+f(t)+N\left(u(t,P)\right)\right],\\&B\left(u(t,P),\frac{du(t,P)}{dt}\right)=0,\end{aligned} \tag{2.2}$$

where $P\in[0,1]$ is an embedding parameter, $H(P)$, for $P\neq 0$ is a non-zero auxiliary function where $H(0)=0$. For $P=0$, and $P=1$ we have $u(t,0)=u_0(t)$, and $u(t,1)=u(t)$, respectively. Thus, as P increases from 0 to 1, the solution $u(t,P)$ varies from $u_0(t)$ to the solution $u(t)$, where $u_0(t)$ is an initial guess, for the solution, that satisfies the linear operator which is obtained from (2.2), for $P=0$;

$$L\left(u_0(t)\right)+f(t)=0,\ B\left(u_0,\frac{du_0}{dt}\right)=0. \tag{2.3}$$

The auxiliary function $H(P)$ is considered as the following power series in P;

$$H(P)=C_1P+C_2P^2+\cdots, \tag{2.4}$$

where $C_1,C_2,\cdots$ are constants that will be determined shortly.

Approximate solution, $u(t;P,C_1,\ldots,C_m)$, is also considered as a power series about P

$$u(t;P,C_1,\ldots,C_m)=u_0(t)+\sum_{k\geq 1}u_k(t,C_1,\ldots,C_m)P^k. \tag{2.5}$$

Substitution in (2.2) from (2.5) and equating the coefficients of the terms with identical powers of P, leads to governing equations of $u_0(t),u_1(t),\ldots,u_k(t)$, which starts from (2.3) and followed by;

$$L\left(u_1(t)\right)=C_1N_0\left(u_0(t)\right),\ B\left(u_1,\frac{du_1}{dt}\right)=0, \tag{2.6}$$

$$\begin{aligned}&L\left(u_k(t)-u_{k-1}(t)\right)=C_kN_0\left(u_0(t)\right)\\&\qquad+\sum_{i=1}^{k-1}C_i\left[L\left(u_{k-i}(t)\right)+N_{k-i}\left(u_0(t),u_1(t),\ldots,u_{k-i}(t)\right)\right],\\&B\left(u_k,\tfrac{du_k}{dt}\right)=0,k=2,3,\ldots,\end{aligned} \tag{2.7}$$

where $N_m\left(u_0(t),u_1(t),\ldots,u_m(t)\right)$ is the coefficient of P^m in the expansion of $N\left(u(t;P,C_1,\ldots,C_m)\right)$ about the embedding parameter P

$$N\left(u(t;P,C_1,\ldots,C_m)\right)=N_0\left(u_0(t)\right)+\sum_{k\geq 1}N_k(u_0,u_1,\ldots,u_k)P^k, \tag{2.8}$$

where $u(t;P,C_1,\ldots,C_m)$ is given by Eq. (2.5).

Study of the rate of convergence of the series (2.5) depends upon the auxiliary constants C_i, $i=1,2,\ldots$. If the series (2.5) converges for $P=1$, one has

$$u(t,C_1,C_2,\ldots,C_m)=u_0(t)+\sum_{k\geq 1}u_k(t,C_1,C_2,\ldots). \tag{2.9}$$

Then the mth order approximation can be denoted as follows

$$\tilde{u}(t,C_1,C_2,\ldots,C_m)=u_0(t)+\sum_{k=1}^{m}u_k(t,C_1,C_2,\ldots,C_m). \tag{2.10}$$

Substitution of (2.10) into Eq. (1.1), results in the following expression for the residual

$$R(t,C_1,\ldots,C_m)=L\left(\tilde{u}(t,C_1,\ldots,C_m)\right)+f(t)+N\left(\tilde{u}(t,C_1,\ldots,C_m)\right). \tag{2.11}$$

If $R(t,C_1,\ldots,C_m)=0$, then $\tilde{u}(t,C_1,\ldots,C_m)$ will be the exact solution and this, in general, does not happen especially in nonlinear problems. In order to find the optimal values of C_i, $i=1,2,\ldots,m$, we apply least squares minimization approach

$$\frac{\partial J}{\partial C_1}=\frac{\partial J}{\partial C_2}=\cdots=\frac{\partial J}{\partial C_m}=0, \tag{2.12}$$

where

$$J(C_1,C_2,\ldots,C_m)=\int_a^b R^2(t,C_1,C_2,\ldots,C_m)dt \tag{2.13}$$

and a and b are two values, depending on the given problem. Knowing $C_i, i=1,2,\ldots,m$, from (2.12), the approximate solution of order m will be determined easily.

If the interval of changes of the time variable is long, then OHAM fails to reach accurate solutions. MOHAM overcomes this shortcoming by partitioning the time interval, $[t_0, T]$, into N subintervals $[t_0, t_1), \ldots, [t_{\gamma-1}, t_\gamma]$, where $t_\gamma = T$ and OHAM will be applied over each subinterval. The solution at the last point, in each subinterval, denotes an initial approximation to the solution, over the next interval. The process will continue until we achieve the pre-assigned time, T.

Implementation of MOHAM is almost the same as OHAM, with some minor changes:

Equations (2.4), (2.10), (2.11), (2.12), and (2.13), change to, (2.16), (2.17), (2.18), (2.20), and (2.19), respectively. Also, initial approximation in $[t_\gamma, t_{\gamma+1})$, $\gamma = 0, 1, \ldots, N-1$ will be considered as

$$u_{0,j}(t_j) = \alpha_j, \; j = 1, \ldots, N, \tag{2.14}$$

In addition, deformation equation in each subinterval changes to the following (see [3])

$$\begin{aligned}(1-P)\left[L\left(u_j(t,P)\right) - u_{0,j}(t)\right] = H(P)\left[L\left(u_j(t,P)\right)\right.\\ \left.+f(t) + N\left(u_j(t,P)\right)\right].\end{aligned} \tag{2.15}$$

Moreover, the auxiliary function $H(P, t)$ can be generalized as follows,

$$H_j(P,t) = \left(C_{1,j} + C_{2,j}t + C_{3,j}t^2 + \cdots\right) P, \; j = 1, \ldots, N. \tag{2.16}$$

For $i = 1, 2, \ldots, m$, $j = 1, \ldots, N$, we have

$$\tilde{u}_j(t, C_{ij}) = u_{0,j}(t) + \sum_{k=1}^{m} u_{k,j}(t, C_{ij}), \tag{2.17}$$

$$R_j(t, C_{ij}) = L\left(\tilde{u}(t, C_{ij})\right) + f(t) + N\left(\tilde{u}(t, C_{ij})\right), \tag{2.18}$$

$$J_j(C_{ij}) = \int_{t_\gamma}^{t_\gamma + h} R_j^2(s, C_{ij}) ds, \; \gamma = 0, 1, \ldots, N-1. \tag{2.19}$$

The length of the subinterval $[t_\gamma, t_{\gamma+1})$ is apparently h, and the number of subintervals is $N = \lfloor T/h \rfloor$ Now, we consider derivatives of (2.19), with respect to $C_{ij}, (i = 1, \ldots, m, \; j = 1, \ldots, N)$, to zero. In fact, we define $\alpha_j = \tilde{u}_j(t_j)$, in each subinterval $[t_\gamma, t_{\gamma+1})$. Therefore, the convergence control parameters can be determined from the solution of the following system of equations

$$\frac{\partial J_j}{\partial C_{1,j}} = \frac{\partial J_j}{\partial C_{2,j}} = \cdots = \frac{\partial J_j}{\partial C_{m,j}} = 0. \tag{2.20}$$

Approximate solutions, on each subinterval, are as follows

$$\tilde{u}(t) = \begin{cases} \tilde{u}_1(t), \; t_0 \le t < t_1, \\ \tilde{u}_2(t), \; t_1 \le t < t_2, \\ \vdots \\ \tilde{u}_N(t), \; t_{N-1} \le t \le T. \end{cases} \tag{2.21}$$

3. Application of OHAM to Abel integral equation of the second kind

In this section, we apply OHAM to the following Abel integral equation of the second kind, with weakly singular kernel [1, 11, 15].

$$u(t) = f(t) + \int_0^t \frac{u(s)}{\sqrt{t-s}} ds, t \in [0,1]. \tag{3.1}$$

Using aforementioned OHAM, results in the following sequential equations.

$$\begin{aligned}
&P^0 : u_0(t) = f(t),\\
&P^1 : u_1(t) = -C_1 \textstyle\int_0^t \frac{u_0(s)}{\sqrt{t-s}} ds,\\
&P^2 : u_2(t) = (1+C_1)u_1(t) - C_1 \textstyle\int_0^t \frac{u_1(s)}{\sqrt{t-s}} ds - C_2 \int_0^t \frac{u_0(s)}{\sqrt{t-s}} ds,\\
&\vdots\\
&P^j : u_j(t) = (1+C_1)u_{j-1}(t) + \sum_{i=2}^{j-1} C_i u_{j-i}(t) - \sum_{k=1}^{j} C_k \textstyle\int_0^t \frac{u_{j-k}(s)}{\sqrt{t-s}} ds.
\end{aligned} \tag{3.2}$$

Using (2.11, 2.12, 2.13) we find C_i, $i = 1, 2, \ldots, m$.

Knowing these constants, an approximate solution, of order will be determined.

4. Application of MOHAM to Abel integral equation of the second kind

In this section, we apply MOHAM to equation (3.1). This procedure leads to the following sequence of equations,

$$\begin{aligned}
&P^0 : u_{0,j}(t_j) = \alpha_j,\\
&P^1 : u_{1,j}(t) = (C_{1,j} + C_{2,j}t + \cdots)\left(u_{0,j}(t) - f(t) - \textstyle\int_0^t \frac{u_{0,j}(s)}{\sqrt{t-s}} ds\right),\\
&P^2 : u_{2,j}(t) = u_{1,j}(t) + (C_{1,j} + C_{2,j}t + \cdots)\left(u_{1,j}(t) - \textstyle\int_0^t \frac{u_{1,j}(s)}{\sqrt{t-s}} ds\right),\\
&\vdots\\
&P^j : u_{j,j}(t) = u_{j-1,j}(t) + (C_{1,j} + C_{2,j}t + \cdots)\\
&\qquad\qquad \times \left(u_{j-1,j}(t) - \textstyle\int_0^t \frac{u_{j-1,j}(s)}{\sqrt{t-s}} ds\right).
\end{aligned} \tag{4.1}$$

In addition, by using (2.18, 2.19, 2.20) we find C_{ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, N$.

Knowing the values of the parameters, an approximate solution of order will be determined.

5. Convergence of OHAM

There are two proofs presented in [3] and [5], that have some oversights, i.e. in both papers, as $n \to \infty$,

$$\sum_{k=2}^{n} u_k(t, C_1, \ldots, C_k) - \sum_{k=2}^{n} u_{k-1}(t, C_1, \ldots, C_{k-1}).$$

This should be replaced by

$$\sum_{k=2}^{\infty} u_k(t, C_1, \ldots, C_k) - \sum_{k=2}^{\infty} u_{k-1}(t, C_1, \ldots, C_{k-1}).$$

Eq. (2.25) and Eq. (42), in [3] and [5] respectively, as the following;

$$\sum_{i=1}^{k} C_{i-k} \left[L \left(u_{i-1}(t, C_1, \ldots, C_{i-1}) \right) + N_{i-1} \left(u_0(t), u_1(t), \ldots, u_{k-i}(t) \right) \right]$$

This should be changed to

$$\sum_{i=1}^{k} C_{k-i} \left[L \left(u_{i-1}(t, C_1, \ldots, C_{i-1}) \right) + N_{i-1} \left(u_0(t), u_1(t), \ldots, u_{k-i}(t) \right) \right].$$

Proposed proof, in this manuscript, overcomes these shortcomings.

Theorem 5.1. *If the series (2.9), converges to $u(t)$ where $u_k(t, C_1, \ldots, C_k) \in L^2[c, d]$ is produced by Eqs.(2.3), (2.6), and (2.7), then $u(t)$ is the exact solution of equation (2.1).*

Proof. Since the series

$$\sum_{k=0}^{\infty} u_k(t, C_1, \ldots, C_k),$$

is convergent, it holds

$$\lim_{k \to \infty} u_k(t, C_1, \ldots, C_k) = 0.$$

The left hand-side of Eq. (2.7) satisfies

$$\begin{aligned} u_1(t, C_1) &+ \sum_{k=2}^{n} u_k(t, C_1, \ldots, C_k) \\ &- \sum_{k=2}^{n} u_{k-1}(t, C_1, \ldots, C_{k-1}) = u_n(t, C_1, \ldots, C_n). \end{aligned} \tag{5.1}$$

According to Eq. (5.1) we have

$$\begin{aligned} u_1(t, C_1) &+ \sum_{k=2}^{\infty} u_k(t, C_1, \ldots, C_k) \\ &- \sum_{k=2}^{\infty} u_{k-1}(t, C_1, \ldots, C_{k-1}) = \lim_{n \to \infty} u_n(t, C_1, \ldots, C_n) \\ &= 0. \end{aligned} \tag{5.2}$$

Since the operator L is linear, from (5.2), we have

$$\begin{aligned} L\left(u_1(t,C_1)\right) + \sum_{k=2}^{\infty} L\left(u_k(t,C_1,\dots,C_k)\right) \\ - \sum_{k=2}^{\infty} L\left(u_{k-1}(t,C_1,\dots,C_{k-1})\right) &= L\left(u_1(t,C_1)\right) \\ &+ L\left(\sum_{k=2}^{\infty} u_k(t,C_1,\dots,C_k)\right) \\ &- L\left(\sum_{k=2}^{\infty} u_{k-1}(t,C_1,\dots,C_{k-1})\right) = 0. \end{aligned}$$

Then

$$\begin{aligned} L\left(u_1(t,C_1)\right) &+ L\left(\sum_{k=2}^{\infty} u_k(t,C_1,\dots,C_k)\right) \\ &- L\left(\sum_{k=2}^{\infty} u_{k-1}(t,C_1,\dots,C_{k-1})\right) = L\left(u_1(t,C_1)\right) \\ &+ \sum_{k=2}^{\infty}\Bigg(C_k N_0\left(u_0(t)\right) + \sum_{i=1}^{k-1} C_i\big[L\left(u_{k-i}(t)\right) \\ &+ N_{k-i}\left(u_0(t),u_1(t),\dots,u_{k-i}(t)\right)\big]\Bigg) \\ &\qquad\qquad\qquad = 0. \end{aligned} \tag{5.3}$$

Eq. (5.3), can be written as follows

$$\begin{aligned} L\left(u_0(t)\right) + f(t) + C_1 N_0\left(u_0(t)\right) &+ \sum_{k=2}^{\infty}\Bigg(C_k N_0\left(u_0(t)\right) \\ &+ \sum_{i=1}^{k-1} C_i\big[L\left(u_{k-i}(t)\right) + N_{k-i}\left(u_0(t),u_1(t),\dots,u_{k-i}(t)\right)\big] \\ = \sum_{k=1}^{\infty}\sum_{i=1}^{k} C_{k-i}\big[L\left(u_{i-1}(t,C_1,\dots,C_{i-1})\right) \\ + N_{i-1}\left(u_0(t),u_1(t),\dots,u_{i-1}(t)\right)\big] + f(t) \\ = 0. \end{aligned} \tag{5.4}$$

If C_n, $n=0,1,\dots,m$, chosen properly, then Eq. (5.4) leads to Eq. (2.1).

□

6. Numerical Example

In this section, a linear Abel integral equation of the second kind will be solved to show the accuracy of the both OHAM and MOHAM. Matlab package is used to carry out the computations, with double precision.

Example 6.1. Let us consider the following linear Abel integral equation of the second kind, with the exact solution $u(t)=t$.

$$u(t) = t + \frac{4}{3}t^{\frac{3}{2}} - \int_0^t \frac{u(s)}{\sqrt{t-s}}ds,\ 0 \le t \le 1. \tag{6.1}$$

Following aforementioned OHAM, results in;

$$\begin{aligned}
u_0(t) &= t + \tfrac{4}{3}t^{\frac{3}{2}},\\
u_1(t) &= C_1\left(\tfrac{4}{3}t^{\frac{3}{2}} + \tfrac{\pi}{2}t^2\right),\\
u_2(t) &= \tfrac{4}{3}t^{\frac{3}{2}}\left(C_1 + C_2 + C_1^2\right) + \tfrac{\pi}{2}t^2\left(C_1 + C_2 + 2C_1^2\right) + \tfrac{8\pi}{15}C_1^2 t^{\frac{5}{2}},\\
u_3(t) &= \tfrac{4}{3}t^{\frac{3}{2}}\left(C_1 + C_2 + 2C_1^2 + 2C_1C_2 + C_1^3 + C_3\right)\\
&\quad + \tfrac{\pi}{2}t^2\left(C_1 + C_2 + 4C_1^2 + 4C_1C_2 + 3C_1^3 + C_3\right)\\
&\quad + \tfrac{8\pi}{15}t^{\frac{5}{2}}\left(2C_1^2 + 3C_1^3 + 2C_1C_2\right) + \tfrac{\pi^2}{6}C_1^3t^3.
\end{aligned}$$

Replacement of the first, second, third, and forth terms in (2.10) result in;

$$\begin{aligned}
\tilde{u}(t) = t &+ \tfrac{4}{3}t^{\frac{3}{2}}\left(1 + 3C_1 + 2C_2 + 3C_1^2 + 2C_1C_2 + C_1^3 + C_3\right)\\
&+ \tfrac{\pi}{2}t^2(3C_1 + 2C_2 + 6C_1^2 + 4C_1C_2 + 3C_1^3 + C_3)\\
&+ \tfrac{8\pi}{15}t^{\frac{5}{2}}\left(3C_1^2 + 3C_1^3 + 2C_1C_2\right) + \tfrac{\pi^2}{6}C_1^3t^3.
\end{aligned} \tag{6.2}$$

Using the same technique as presented in (2.11)-(2.13), we find C_1, C_2, and C_3 as follows,

$$C_1 = 0, C_2 = 0, C_3 = -0.4967138397.$$

Substituting the values of C_1, C_2, and C_3 into (6.2), yields;

$$\begin{aligned}
\tilde{u}(t) = 4.065163e^{-41}t^3 &- 1.737856e^{-27}t^{\frac{5}{2}}\\
&-0.7802363t^2 + 0.6710482t^{\frac{3}{2}} + t.
\end{aligned}$$

To derive a solution to Eq. (6.2), for $0 \le t \le 1$, by MOHAM, we consider the following initial approximations

$$u_{0,j}(t_j) = \alpha_j,\ j = 1, 2. \tag{6.3}$$

Now, we will consider the auxiliary function $H(P,t)$ as the following

$$H(P,t) = (C_{1,j} + C_{2,j}t)P.$$

where $C_{1,j}$, and $C_{2,j}$ are unknown now.

Regarding (2.17), the first-order approximate solution, for $m = 1$, appears as following

$$\tilde{u}_j(t) = u_{0,j}(t) + u_{1,j}(t), \tag{6.4}$$

where

$$u_{1,j}(t) = (C_{1,j} + C_{2,j}t)\left(u_{0,j}(t) - t - \frac{4}{3}t^{\frac{3}{2}} + \int_0^t \frac{u_{0,j}(s)}{\sqrt{t-s}}ds\right). \tag{6.5}$$

Substituting (6.5) into (6.4), and regarding (6.3), result in

$$\begin{aligned}
\tilde{u}_j(t) = \alpha_j &+ C_{1,j}\alpha_j - C_{1,j}t - \tfrac{4}{3}C_{1,j}t^{\frac{3}{2}} + C_{1,j}\int_0^t \tfrac{u_{0,j}(s)}{\sqrt{t-s}}ds\\
&+C_{2,j}\alpha_j t - C_{2,j}t^2 - \tfrac{4}{3}C_{2,j}t^{\frac{5}{2}} + C_{2,j}\int_0^t \tfrac{u_{0,j}(s)}{\sqrt{t-s}}ds.
\end{aligned} \tag{6.6}$$

Substitution of (6.6) into (2.18)-(2.20), determines the residual and functional $J_j(C_{1,j}, C_{2,j})$, respectively

$$\begin{aligned}
R_j(t, C_{1,j}, C_{2,j}) &= \tilde{u}_j(t) - t - \tfrac{4}{3}t^{\frac{3}{2}} + \int_0^t \tfrac{\tilde{u}_j(s)}{\sqrt{t-s}}ds,\\
J_j(C_{1,j}, C_{2,j}) &= \int_{t_\gamma}^{t_\gamma + h} R_j^2(s, C_{1,j}, C_{2,j})ds.
\end{aligned}$$

Minimization condition leads to

$$\frac{dJ_j}{dC_{1,j}} = \frac{dJ_j}{dC_{2,j}} = 0.$$

Parameters that control convergence, $C_{1,j}$, and $C_{2,j}$ are presented in Table 1, where $\alpha_1 = 0$, $h = 0.5$, for $t_0 = 0$ up to $t_2 = T = 1$.

TABLE 1. values of the control parameters C_{ij}

j	$C_{1,j}$	$C_{2,j}$
1	-0.705537381	0.39914917
2	-0.842296294	0.29022925

By substituting the values of $C_{1,j}$, and $C_{2,j}$ into (6.6), one obtains

$$\tilde{u}(t) = \begin{cases} -0.5321989t^{\frac{5}{2}} - 0.3991492t^2 + 0.9407165t^{\frac{3}{2}} & \\ \qquad -0.7055374t, & 0 \le t < 0.5, \\ -0.1516404t^4 - 0.2780275t^{\frac{7}{2}} + 0.6458906t^3 + 0.4337602t^{\frac{5}{2}} & \\ \qquad -1.081889t^2 + 0.4790542t^{\frac{3}{2}} & \\ \qquad +0.9535622t, & 0.5 \le t \le 1. \end{cases}$$

First-order approximate solution, by MOHAM, and three-order approximate solution, by OHAM, are compared with the exact solution in Table 2, plots are also presented in Figure 1.

Absolute errors for OHAM and MOHAM are plotted in Figure 2.

FIGURE 1. The results of OHAM, MOHAM and exact solution

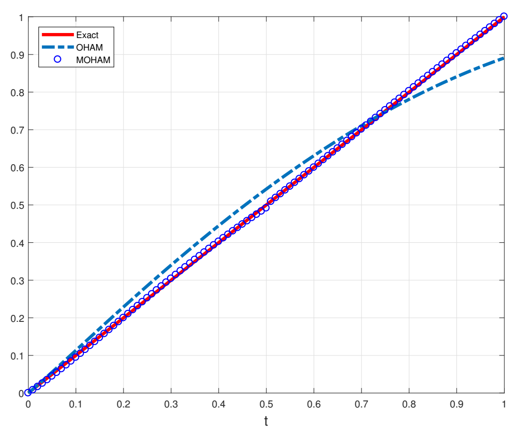

FIGURE 2. Absolute errors of OHAM and MOHAM

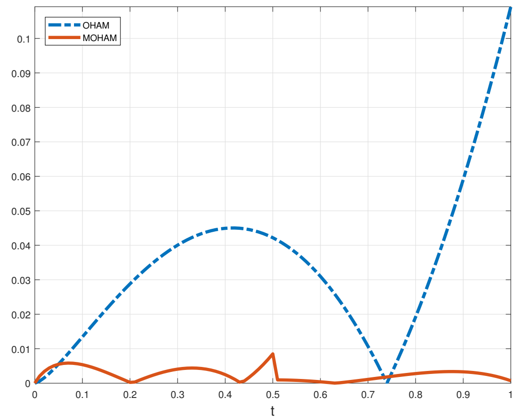

TABLE 2. The results of applying OHAM, MOHAM and the exact solution

t_j	Exact	OHAM	Ab.Error(OHAM)	MOHAM	Ab.Error(MOHAM)
0.0	0.0	0.00000000	0.00000000	0.00000000	0.00000000
0.1	0.1	0.11341804	0.01341804	0.09462735	0.00537265
0.2	0.2	0.22881092	0.02881092	0.19976148	0.00023852
0.3	0.3	0.34004320	0.04004320	0.30407852	0.00407852
0.4	0.4	0.44492545	0.04492545	0.40248088	0.00248088
0.5	0.5	0.54219220	0.04219229	0.49149450	0.00850550
0.6	0.6	0.63098996	0.03098996	0.59959985	0.00040015
0.7	0.7	0.71069167	0.01069167	0.70110243	0.00110243
0.8	0.8	0.78081177	0.01918823	0.80278520	0.00278520
0.9	0.9	0.84095626	0.05904374	0.90329730	0.00329730
1.0	1.0	0.89081193	0.10918807	1.00071030	0.00071030

7. Conclusion and discussion

In this paper, two well known approaches, OHAM, and MOHAM have been applied for solving linear Abel integral equation of the second kind. The results of these two approaches are presented in Tables, and are plotted in Figures. Table 2, shows the absolute errors of OHAM and MOHAM at some selected points. Comparison of the numerical results, reveals that MOHAM is more accurate than OHAM, especially for the points farther from the initial point. Moreover, MOHAM is very efficient and convenient to use for finding approximate solutions of such an integral equation of the second kind.

References

[1] J. Abdalkhani, *A numerical approach to the solution of Abel integral equations of the second kind with nonsmooth solution*, J. Comput. AppI. Math., *29*(3) (1990), 249–255.

[2] M. Alipour and D. Rostamy, *Bernstein polynomials for solving Abel's integral equation*, J. Math. Comput. Sci, *3*(4) (2011), 403–412.

[3] N. R. Anakira, A. K. Alomari, A. F. Jameel, and I. Hashim, *Multistage Optimal homotopy asymptotic method for solving initial–value problems*, J. Nonlinear Sci. Appl, *9*(4) (2016), 1826–1843.

[4] R. F. Cameron and S. McKee, *Product integration methods for second-kind Abel integral equations*, J. Comput. Appl. Math., *11*(1) (1984), 1–10.

[5] H. Daei kasmaei and J. Rashidinia, *Optimal Homotopy Asymptotic and Homotopy perturbation method for linear mixed Volterra–Fredholm Integral Equations*,Nevşehir Bilim ve Teknoloji Dergisi, *5*(2) (2016), 86–103. .

[6] Q. K. Ghori, M. Ahmed, and A. M. Siddiqui, *Application of homotopy perturbation method to squeezing flow of a Newtonian fluid*, Int. J. Nonlinear Sci. Number. Simul., *8*(2) (2007), 179–184.

[7] R. Gorenflo and S. Vessella, *Abel integral equations, Analysis and Applications*, Lecture notes in mathematics, Vol. 1461, Springer, Heidelberg, 1991.

[8] M. S. Hashmi, N. Khan, and S. Iqbal, *Numerical solutions of weakly singular Volterra integral equations using the optimal homotopy asymptotic method*, J. Compute . Math. Appl, *64*(6) (2012), 1567–1574.

[9] N. Herisanu, V. Marinca, T. Dordea, and G. Madescu, *A new analytical approach to nonlinear vibration of an electric machine*, Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci, *9*(3) (2008), 229–236.

[10] M. Khan and M. A. Gondal, *A reliable treatment of Abel's second kind singular integral equations*, Appl. Math. Lett, *25*(11) (2012), 1666–1670.

[11] S. Kumar, A. Kumar, D. Kumar, J. Singh, and A. Singh, *Analytical solution of Abel integral equation arising in astrophysics via Laplace transform*,J. Egypt. Math. Soc, *23*(1) (2015), 102–107.

[12] V. Marinca, N. Herisanu, and I. Nemes, *Optimal homotopy asymptotic method with application to thin film flow*, Cent. Eur. J. ys, *6*(3) (2008), 648–653.

[13] V. Marinca, N. Herisanu, C. Bota, and B. Marinca, *An optimal homotopy asymptotic method applied to the steady flow of fourth–grade fluid past a porous plate*, Appl. Math. Lett, *22*(2) (2009), 245–251.

[14] V. Marinca and N. Herisanu, *An optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer*, Int. Commun. Heat Mass Transfer, *35*(6) (2008), 710–715.

[15] R. K. Pandey, Om P. Sing, and V. K. Singh, *Efficient algorithms to solve singular integral equations of Abel type*, J. Comput. Appl. Math., *57*(4) (2009), 664–676.

[16] S. A. Yousefi, *Numerical solution of Abels integral equation by using Legendre wavelets*, Appl. Math. Comput, *175*(1) (2006), 574–580.