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Asymptotic decomposition method for fractional order Riccati differential equation

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Abstract In this study, the asymptotic Adomian decomposition method (AADM) is implemented to solve fractional order Riccati differential equations. The product integration method is used to solve the singular integrals, resulted from fractional derivative. Some fractional order Riccati differential equations are presented as examples to illustrate the ability and efficiency of the proposed approach. The approximate solutions of AADM are compared with the results of the Laplace Adomian Pade method (LAPM). Generalizing AADM for solving fractional Riccati differential equations by the far-field approximation indicates the novelty of the paper.

Keywords. Fractional order Riccati differential equations, Asymptotic Adomian decomposition method, Singular integrals, Product integration, Caputo fractional derivative.

1991 Mathematics Subject Classification.

1. Introduction

The introduction of the fractional calculus goes back to more than 300 years ago. The fractional calculus appears in many natural phenomena [17], so this branch of mathematics is very popular for scientists. One of the important branches of fractional calculus is the fractional differential equations. In recent years, many fractional differential equations have been investigated and various approaches have been adopted to solve such equations, in applied mathematics [17, 49]. All of the numerical methods used to solve the fractional differential equations are the same as those are applied to solve differential equations [4, 15, 20, 21, 24, 28, 33, 34, 41, 47, 59]. Adomian introduced a different interpretation of his decomposition method called asymptotic Adomian decomposition method [6, 33, 34]. In this approach, the canonical form has

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been changed, but the iteration is the same as in the main approach. AADM leads to a steady state solution of the equation. In fact, instead of nested integrations, which are used in the decomposition approach, we are dealing with nested differentiation (see [5, 6, 7, 8, 11, 50]). It should be mentioned that the convergence of Adomian decomposition series has been proved by several articles. In [14], Cherruault presented the first proof of the convergence of the ADM in accordance with the fixed point theorem. Furthermore, the convergence of the ADM for solving linear and nonlinear differential equations and integral equations were discussed by Cherruault and his collaborators [1, 2, 13, 25].

The order of convergence of the ADM was introduced by Babolian and Biazar [10]. Abdelrazec and Pelinovsky used the Cauchy Kowalevskaya theorem to demonstrate a new proof of convergence of the ADM [3]. Many other papers such as [26, 27, 38, 51, 52] previously discussed such a problem. AADM is rarely used to achieve the approximate solution of fractional differential equation problems. In this paper, AADM has been stated for solving fractional order Riccati differential equations, by the far-field approximation, for the first time. AADM is used for solving Riccati differential equations [11]. One of the most important and well-known among fractional order differential equations is the Riccati, that finds many applications in physical phenomena, mechanics and other fields of sciences (see [12, 35, 39, 53, 58]). For example, the Riccati equation appears in super symmetry theories [16], quantum chemistry [22], nonlinear physics [46], and thermodynamics [54], see [11] for more examples. The general form of these equations is as follows:

$$D_x^{\alpha}u(x) = r(x)u^2(x) + q(x)u(x) + p(x), \quad x > 0,\ 0 < \alpha \leq 1, \tag{1.1}$$

with the initial condition

$$u(0) = k, \tag{1.2}$$

where $p(x), q(x)$ and $r(x)$ are known functions, D_x^{α} is the Caputo fractional derivative operator that has been defined in the preliminaries, next section. For $\alpha = 1$, the fractional order Riccati differential equation is the same as the Riccati differential equation of order one. There are many approaches used for solving differential equations that can be generalized for solving fractional order equations. Some of these methods are as the following:
Differential transform [20], Adomian decomposition [28], homotopy analysis [21, 24], homotopy perturbation [4], optimal homotopy asymptotic [30, 42, 43, 44, 45], finite difference [15], variational iteration [18, 32, 48, 60, 61], Taylor matrix [23], Runge-Kutta [40], B-spline functions [36, 37], Chebyshev cardinal operational matrix [31], first integral [29], and Tau [56].
The rest of this paper is organized as follows; in section 2, some preliminaries such as product integration approach and Hermite interpolation is introduced. In section 3, AADM is described. Section 4 is devoted to applying AADM for solving fractional Riccati differential equations and four examples will be presented to illustrate the ability and the efficiency of the proposed method. Eventually, discussions and conclusions are presented in section 5.

2. Preliminaries

The purpose of this section is to recall some prefaces for the proposed method.

2.1. **Caputo fractional derivatives.** The Caputo fractional derivative of a function $f(t)$ is defined as follows

$$D_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \int_{x_0}^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, & m-1<\alpha<m,\ m \in \mathbb{N}, \\ \dfrac{d^m}{dx^m} f(x) & \alpha = m,\ m \in \mathbb{N}, \end{cases} \tag{2.1}$$

where Γ is the Gamma function. Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation

$$D_x^\alpha \left(rf(x) + kg(x)\right) = rD_x^\alpha f(x) + kD_x^\alpha g(x), \tag{2.2}$$

where r and k are real constants [9, 55].

2.2. **Product integration.** We now consider the numerical solution of a Volterra integral equation,

$$x(s) = y(s) + \int_a^s k\left(s,t,x(t)\right) dt, \quad a \le s \le b, \tag{2.3}$$

where $k\left(s,t,x(t)\right)$ is a singular kernel. It is presumed that the kernel function badly behaves, and

$$k\left(s,t,x(t)\right) = p(s,t)\bar{k}\left(s,t,x(t)\right), \tag{2.4}$$

where $p(s,t)$ and $\bar{k}\left(s,t,x(t)\right)$ are respectively singular and well behaved functions of their arguments. Then the product integration can be used to solve equation (2.3). The interval $[a,b]$ is subdivided into M subintervals of equal lengths, h_i where

$$h_i = s_{i+1} - s_i, \quad i = 0,1,\cdots,M-1 \tag{2.5}$$

and

$$a = s_0 < s_1 < \cdots < s_M = b. \tag{2.6}$$

Then the method continues by approximating the integral term in (2.3). For $s = s_i,\ i = 1,\cdots,M$, by a quadrature rule of the following form

$$\int_a^{s_i} p(s_i,t)\bar{k}\left(s_i,t,x(t)\right) dt \approx \sum_{j=0}^{i} w_{ij}\bar{k}\left(s_i,t_j,x(t_j)\right), \tag{2.7}$$

where $t_i = s_i,\ i = 0,1,\cdots,M$ the weights are constructed by insisting that the rule in (2.7) be exact when $\bar{k}\left(s_i,t,x(t)\right)$ is a polynomial in t of degree less than or equal to r. For each value of i, the existence of $(r+1)$ moments is needed.

$$\mu_{ij} = \int_a^{s_i} t^j p(s_i,t)dt, \quad j = 0,1,\cdots,r. \tag{2.8}$$

By interpolating well behaved part of the kernel function at the points s_j, s_{j+1} and using numerical integration, it implies that

$$\int_a^{s_i} p(s_i,t)\bar{k}\left(s_i,t,x(t)\right)dt = \sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} p(s_i,t)\bar{k}(s_i,t,x(t))dt \tag{2.9}$$

$$\begin{aligned}&\approx \sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} p(s_i,t)\{\frac{(t_{j+1}-t)}{h_j}\bar{k}(s_i,t_j,x(t_j))\\ &+\frac{(t-t_j)}{h_j}\bar{k}(s_i,t_{j+1},x(t_{j+1}))\}dt\end{aligned} \tag{2.10}$$

$$= \sum_{j=0}^{i} w_{ij}\bar{k}\left(s_i,t_j,x(t_j)\right). \tag{2.11}$$

Hence

$$\int_a^{s_i} p(s_i,t)\bar{k}(s_i,t,x(t))dt = \sum_{j=0}^{i} w_{ij}\bar{k}\left(s_i,t_j,x(t_j)\right)+O(h^2). \tag{2.12}$$

where the weights are calculated as follows

$$w_{i0} = \frac{1}{h_0}\int_{t_0}^{t_1} p(s_i,t)(t_1-t)dt, \tag{2.13}$$

$$\begin{aligned}w_{ij} &= \frac{1}{h_j}\int_{t_j}^{t_{j+1}} p(s_i,t)(t_{j+1}-t)dt\\ &+\frac{1}{h_{j-1}}\int_{t_{j-1}}^{t_j} p(s_i,t)(t-t_{j-1})dt \quad j=1,2,\cdots,i-1,\end{aligned} \tag{2.14}$$

$$w_{ii} = \frac{1}{h_{i-1}}\int_{t_{i-1}}^{t_i} p(s_i,t)(t-t_{i-1})dt. \tag{2.15}$$

See [19] for more details and examples.

2.3. **Hermite interpolation.** Now, we need to explain Hermite interpolation [57]. Consider $z_i, f_i^{(k)}$, $i=0,1,\cdots,n$, $k=0,1,\cdots,m_i-1$, are real numbers, in which $z_0<z_1<\cdots<z_n$. The Hermite interpolation is determining a polynomial P whose degree is less than or equal to m, $P\in\Pi_m$ where

$$m+1=\sum_{i=0}^{n} m_i, \tag{2.16}$$

and which satisfies the bellow conditions

$$P^{(k)}(z_i)=f_i^{(k)}, \quad i=0,1,\cdots,n,\ k=0,1,\cdots,m_i-1. \tag{2.17}$$

Hermite interpolating polynomials can be expressed according to Lagrange interpolation formula. $P \in \Pi_m$ can be defined as follows

$$P(z) = \sum_{i=0}^{n} \sum_{k=0}^{m_i-1} f_i^{(k)} L_{ik}(z), \tag{2.18}$$

(2.18) satisfies in (2.17). The polynomials $L_{ik}(z) \in \Pi_m$ are generalized Lagrange polynomials. These polynomials are defined as follows: by helping auxiliary polynomials

$$l_{ik}(z) = \frac{(z-z_i)^k}{k!} \Pi_{\substack{j=0 \\ j\neq i}}^{n} \left(\frac{z-z_j}{z_i-z_j}\right)^{m_j}, \quad 0 \le i \le n,\ 0 \le k \le m_i, \tag{2.19}$$

put

$$L_{i,m_i-1}(z) = l_{i,m_i-1}(z), \quad i = 0, 1, \cdots, n, \tag{2.20}$$

and for $k = m_i - 2, m_i - 3, \cdots, 0$,

$$L_{ik}(z) = l_{ik}(z) - \sum_{\theta=k+1}^{m_i-1} l_{ik}^{\theta}(z_i) L_{i\theta}(z). \tag{2.21}$$

By strong induction

$$L_{ik}^{\alpha}(z_j) = \begin{cases} 1 & if\ i = j\ and\ k = \alpha \\ 0 & otherwise \end{cases} \tag{2.22}$$

So, P in (2.18) is called Hermite interpolating polynomial.

Theorem 2.1. For arbitrary real numbers $z_i, f_i^{(k)},\ i = 0, 1, \cdots, m_i - 1$, with $z_0 < z_1 < \cdots < z_n$, there exists exactly one polynomial $P \in \Pi_m,\ m + 1 = \sum_{i=0}^{n} m_i$ which it satisfies in equation (2.17).

Proof. (See [57]). □

3. Asymptotic decomposition method

In this section, first AADM approach is introduced and then its application in solving fractional Riccati differential equations will be presented.

3.1. **Description of AADM.** In this section, we explain Adomian asymptotic decomposition method for solving a nonlinear differential equation as, $Fu = g(x)$, where Fu is the summation of a linear term, Lu and a nonlinear term Nu, in the other words

$$Lu + Nu = g. \tag{3.1}$$

It must be noted that Nu is represented by $\sum_{n=0}^{\infty} A_n$, where $A_n = A_n(u_0, u_1, \cdots, u_n)$ is called Adomian polynomial, defined as the following [7]

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{k=0}^{\infty} u_k \lambda^k \right) \right]_{\lambda=0}. \tag{3.2}$$

Then, by inserting $u = \sum_{n=0}^{\infty} u_n$ in (3.2), we have

$$\sum_{n=0}^{\infty} A_n = g(x) - L \sum_{n=0}^{\infty} u_n. \tag{3.3}$$

We identify $A_0 = g(x),\ A_1 = -Lu_0,\ A_2 = -Lu_1$, etc. Then A_0 derive from (3.2) and u_0 will be determined, by the following process we determine $u_1, u_2, \cdots$ and hence we consider $\varphi_v(x) = \sum_{k=0}^{v-1} u_k(x)$ as an approximate solution of nonlinear differential equation, that converges rapidly to $u = \sum_{k=0}^{\infty} u_k$ when $v \to \infty$. It must be noted that one of the advantages of this approach for solving differential equations is that any initial condition is not required to gain an asymptotic solution. In this approach for solving differential equations, nested differentiations are used instead of nested integrations which are used in the decomposition method. In fact we do not inverting L to obtain the solution, but instead, we decompose the nonlinear operator, Nu, and determine an asymptotic solution.

3.2. **Application of AADM for solving fractional Riccati differential equations.** In this section, we apply the Adomian asymptotic decomposition method for solving fractional order Riccati differential equations. We rewrite (1.1) as

$$r(x)u^2(x) = -p(x) - q(x)u(x) + D_x^{\alpha} u(x). \tag{3.4}$$

When the coefficient $r(x) \neq 0$, we have

$$u^2(x) = -\frac{p(x)}{r(x)} - \frac{q(x)}{r(x)} u(x) + \frac{1}{r(x)} D_x^{\alpha} u(x) \tag{3.5}$$

According to subsection 3.1 by substituting $u(x) = \sum_{n=0}^{\infty} u_n(x),\ u^2(x) = \sum_{n=0}^{\infty} A_n(x)$, and $D_x^{\alpha} u(x) = \sum_{n=0}^{\infty} D_x^{\alpha} u_n(x)$ respectively, we obtain

$$\sum_{n=0}^{\infty} A_n(x) = -\frac{p(x)}{r(x)} - \frac{q(x)}{r(x)} \sum_{n=0}^{\infty} u_n(x) + \frac{1}{r(x)} \sum_{n=0}^{\infty} D_x^{\alpha} u_n(x). \tag{3.6}$$

Hence

$$A_0(x) = -\frac{p(x)}{r(x)}, \tag{3.7}$$

$$A_{n+1}(x) = -\frac{q(x)}{r(x)} u_n(x) + \frac{1}{r(x)} D_x^{\alpha} u_n(x), \quad n \geq 0. \tag{3.8}$$

It must be noted that Adomian polynomials A_n for $f(u) = u^2$ are $A_n = \sum_{i=0}^{n} u_i u_{n-i}$. As a result, by substituting $A_0(x) = u_0^2(x)$ and $A_{n+1}(x) = \sum_{i=0}^{n+1} u_i(x) u_{n+1-i}(x)$ in (3.8), we rewrite the recursion scheme (3.8) as

$$\sum_{i=0}^{n+1} u_i(x) u_{n+1-i}(x) = -\frac{q(x)}{r(x)} u_n(x) + \frac{1}{r(x)} D_x^{\alpha} u_n(x), \quad n \geq 0. \tag{3.9}$$

After appropriate manipulations, we obtain

$$u_0(x) = \sqrt{-\frac{p(x)}{r(x)}}, \tag{3.10}$$

$$u_{n+1}(x) = \frac{1}{2u_0(x)}(-\frac{q(x)}{r(x)}u_n(x) + \frac{1}{r(x)}D_x^{\alpha}u_n(x) - \sum_{i=1}^{n} u_i(x)u_{n+1-i}(x)), \quad n \geq 0. \tag{3.11}$$

Thus, by summation of the obtained v-terms of u_i's, we have $\varphi_v(x) = \sum_{k=0}^{v-1} u_k(x)$. This is the asymptotic solution of fractional order Riccati differential equations, where x is far from the initial point $x = x_0$.

4. Numerical Examples

In this section, the application of AADM in solving fractional Riccati differential equations has been illustrated by four examples.

Example 4.1. Consider the Riccati differential equation

$$D_x^{\alpha}u(x) = 1 + 2u(x) - u^2(x), \quad 0 < \alpha \leq 1, \tag{4.1}$$

with the initial condition

$$u(0) = 0. \tag{4.2}$$

The exact solution for $\alpha = 1$ is

$$u(x) = 1 + \sqrt{2}\tanh\left[\sqrt{2}x + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]. \tag{4.3}$$

By Adomian asymptotic decomposition method for solving fractional order Riccati differential equation according to recursion schemes (3.10) and (3.11), the solution components of the far-field approximation $\varphi_v(x)$ are computed as

$$u_0(x) = 1, \; u_1(x) = 1, \; u_2(x) = \frac{1}{2}, \cdots \tag{4.4}$$

Thus, by calculating ten terms of asymptotic approximation, we gain $\varphi_{10}(x) = \sum_{k=0}^{9} u_k(x)$, as a result $u(x) \approx \varphi_{10}(x) = 2.4258$ for $\alpha = 1$ which is an approximation of the horizontal asymptote of the exact solution with absolute error 0.0116 when the independent variable x in (4.3) approaches infinity. The curves of far-field approximation $\varphi_{10}(x)$ obtained by AADM, far field approximation achieved by Laplace Adomian Pade method (LAPM) [35] and the exact solution $u(x)$ in example 4.1 are plotted in figure 1. Comparison between the exact solution, the AADM, and the LADM solutions shows that the AADM solution converges to the exact solution when $x \to \infty$. So the far-field curve of example 4.1 gained by AADM overlaps with the exact solution when the independent variable x approaches infinity.

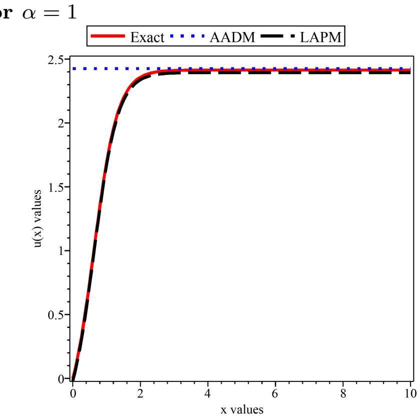
Example 4.2. Consider the following Riccati differential equation:

$$D_x^{\alpha}u(x) = 1 - u^2(x), \quad 0 < \alpha \leq 1, \tag{4.5}$$

with the initial condition

$$u(0) = 0. \tag{4.6}$$

FIGURE 1. **The approximate solutions solved by different methods in example 4.1 for $\alpha = 1$**



The exact solution for $\alpha = 1$ is

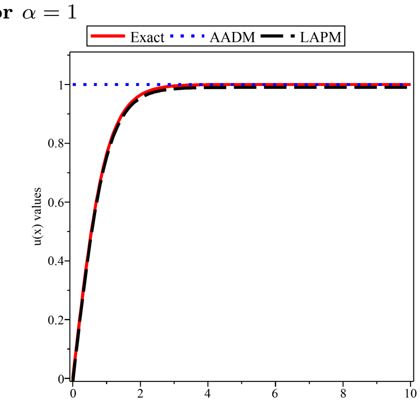
$$u(x) = \frac{e^{2x} - 1}{e^{2x} + 1}. \tag{4.7}$$

By Adomian asymptotic decomposition method for solving fractional order Riccati differential equation based on recursion schemes (3.10) and (3.11) the solution components of the far-field approximation $\varphi_v(x)$ are computed as

$$u_0(x) = 1, \; u_1(x) = 0, \; u_2(x) = 0, \; \cdots \tag{4.8}$$

Thus, by calculating three terms of asymptotic an approximation, we gain $\varphi_3(x) = \sum_{k=0}^{2} u_k(x)$, as a result $u(x) = \varphi_3(x) = 1$ for $\alpha = 1$ which is approximation of the horizontal asymptote of the exact solution with absolute error 0.0000 when the independent variable x in (4.7) approaches infinity. The curves of far-field approximation

FIGURE 2. **The approximate solutions solved by different methods in example 4.2 for $\alpha = 1$**



$\varphi_3(x)$ obtained by AADM, far field approximation achieved by LAPM [35] and the exact solution $u(x)$ in example 4.2 are plotted in figure 2. Comparison between the exact solution, the AADM, and the LADM solutions show that the AADM solution converges to the exact solution when $x \to \infty$. So the far-field curve of example 4.2 gained by AADM overlaps with the exact solution when the independent variable x approaches infinity.

Example 4.3. Consider the following Riccati differential equation:

$$D_x^\alpha u(x) = 1 + x^2 - u^2(x), \quad 0 < \alpha \leq 1, \tag{4.9}$$

with the initial condition

$$u(0) = 1. \tag{4.10}$$

The exact solution for $\alpha = 1$ is

$$u(x) = x + \frac{e^{-x^2}}{1 + \int_0^x e^{-t^2} dt}. \tag{4.11}$$

If $\alpha = 1$, the Caputo fractional derivative is reduced to the ordinary derivative operator in (4.9), so the solution components of the far-field approximation $\varphi_v(x)$ according to (3.10) and (3.11) can be calculated as follows (see [11])

$$u_0(x) = \sqrt{1 + x^2},\ u_1(x) = -\frac{x}{2(1+x^2)},\ u_2(x) = \frac{2 - 3x^2}{8(1+x^2)^{\frac{5}{2}}},$$
$$u_3(x) = \frac{3x(3 - 2x^2)}{8(1+x^2)^4}, \cdots \tag{4.12}$$

Thus, by calculating four terms of asymptotic approximation, we gain $\varphi_4(x) = \sum_{k=0}^{3} u_k(x)$, as a result $u(x) \approx \varphi_4(x) = \sqrt{1+x^2} - \frac{x}{2(1+x^2)} + \frac{2-3x^2}{8(1+x^2)^{\frac{5}{2}}} + \frac{3x(3-2x^2)}{8(1+x^2)^4}$, which is the far-field approximation of the exact solution. If, $x \to \infty$ then $|u(x) - \varphi_4(x)| \to 0$.

FIGURE 3. **The approximate solutions solved by different methods in example 4.3 for $\alpha = 1$**

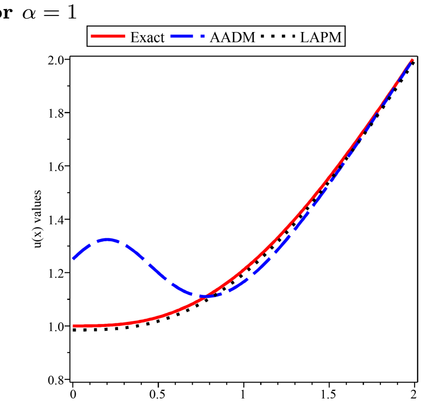
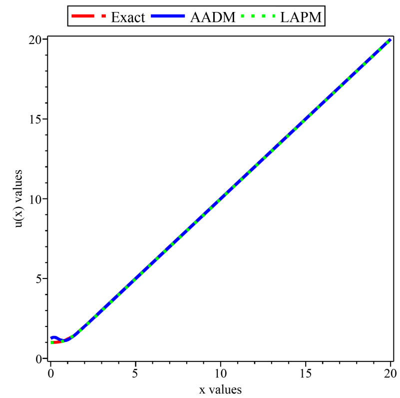


FIGURE 4. **The approximate solutions solved by different methods in example 4.3 for** $\alpha = 1$



Figures 3 and 4 show a comparison between the exact solution, the AADM and the LADM approximate solutions for example 4.3. From figure 3, the solution of AADM is not in agreement with the exact solution in initial values, but from figure 4, the far-field curve of example 4.3 gained by AADM overlaps with the exact solution when the independent variable x approaches infinity.

For obtaining $u_{n+1}(x)$ from equation (3.11), the Caputo fractional derivative of $u_n(x)$, $D_x^\alpha(x)$, is needed. Product integration approach is implemented, which in this process $u_1(x)$ appears in a complicated form when $0 < \alpha < 1$. To proceed, $u_1(x)$ is approximated by Hermite interpolation. So when $0 < \alpha < 1$, the solution components of the far-field approximation $\varphi_v(x)$ are computed as

$$u_0(x) = \sqrt{1+x^2}, \tag{4.13}$$

$$u_1(x) = \frac{-1}{2\sqrt{1+x^2}} D_x^\alpha u_0(x) \tag{4.14}$$

$$= \frac{-1}{2\sqrt{1+x^2}} \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} \frac{2t}{\sqrt{1+t^2}} dt. \tag{4.15}$$

According to subsection 2.2

$$u_1(x) \cong \frac{-1}{2\sqrt{1+x^2}} \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{i} w_{ij} \frac{2t_j}{\sqrt{1+t_j^2}} \tag{4.16}$$

where $h_i = s_{i+1} - s_i,\ i = 0, 1, \cdots, N-1$ and $a = s_0 < s_1 < \cdots < s_M = b$. It must be noted that $t_i = s_i,\ i = 0, 1, \cdots, N$. We consider $h_i = h,\ i = 0, 1, \cdots, M-1$. Suppose that $s_i = ih,\ i = 0, 1, \cdots, M$, we gain

$$u_1(x) \cong \frac{-1}{2\sqrt{1+x^2}} \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{i} w_{ij} \frac{2jh}{\sqrt{1+j^2h^2}}, \tag{4.17}$$

where the weights are calculated from the expressions below:

$$w_{i0} = \frac{1}{h}\int_0^h (ih-t)^{-\alpha}(h-t)dt, \tag{4.18}$$

$$w_{ij} = \frac{1}{h}\int_{jh}^{(j+1)h} (ih-t)^{-\alpha}\left((j+1)h-t\right)dt+$$
$$\frac{1}{h}\int_{(j-1)h}^{jh} (ih-t)^{-\alpha}(t-(j-1)h)dt, \tag{4.19}$$

$$w_{ii} = \frac{1}{h}\int_{(i-1)h}^{ih} (ih-t)^{-\alpha}(t-(i-1)h)dt. \tag{4.20}$$

Putting an example like $\alpha = 0.98$ in $u_1(x)$ can help us to follow the behavior of approximate solution $u(x)$ according to α. We reconstruct the polynomial of $u_1(x)$ by using Hermite interpolation in arbitrary interval. All calculations have been done in Maple.

$$\begin{aligned}
&u_1(x) = -0.9961 + (x-1)(-0.0264) + (x-1)^2(0.2102)\\
&+(x-1)^2(x-2)(-0.1614) + (x-1)^2(x-2)^2(0.0599)\\
&+(x-1)^2(x-2)^2(x-3)(-0.0194) + (x-1)^2(x-2)^2(x-3)^2(0.0047)\\
&+(x-1)^2(x-2)^2(x-3)^2(x-4)(-0.0010)\\
&+(x-1)^2(x-2)^2(x-3)^2(x-4)^2(0.0004)\\
&+(x-1)^2(x-2)^2(x-3)^2(x-4)^2(x-5)(-0.0002)
\end{aligned} \tag{4.21}$$

Hence, according to the recursion scheme (3.11), we gain

$$u_2(x) \cong \frac{1}{2\sqrt{1+x^2}}\left(-D_x^{\alpha}u_1(x) - (u_1(x))^2\right), \quad \alpha = 0.98 \tag{4.22}$$

Finally

$$\begin{aligned}
&u_2(x) = \frac{1}{2\sqrt{1+x^2}}(21.337x^{51/50} - 0.4248x^{301/50} + 2.4278x^{251/50}\\
&-26.4247x^{101/50} - 7.4887x^{1/50} + 0.0415x^{351/50} - 8.5210x^{201/50}\\
&+18.9598x^{151/50} - 0.0017x^{251/50} - 0.0017x^{401/50}\\
&-(-0.9697 - 0.0264x + 0.2102(x-1)^2 - 0.1614(x-1)^2(x-2)\\
&+0.0594(x-1)^2(x-2)^2 - 0.0194(x-1)^2(x-2)^2(x-3)\\
&+0.0047(x-1)^2(x-2)^2(x-3)^2 - 0.0010(x-1)^2(x-2)^2(x-3)^2(x-4)\\
&+0.0004(x-1)^2(x-2)^2(x-3)^2(x-4)^2\\
&-0.002(x-1)^2(x-2)^2(x-3)^2(x-4)^2(x-5))^2).
\end{aligned} \tag{4.23}$$

From which we obtain approximate solution of example 4.3 such as $u(x) \cong \varphi_3(x) = \sum_{k=0}^{2} u_k(x)$ when $\alpha = 0.98$. The curve of far-field approximation $\phi_3(x)$ is plotted in

FIGURE 5. **The approximate solutions solved by AADM methods in example 4.3 for $\alpha = 0.98$**

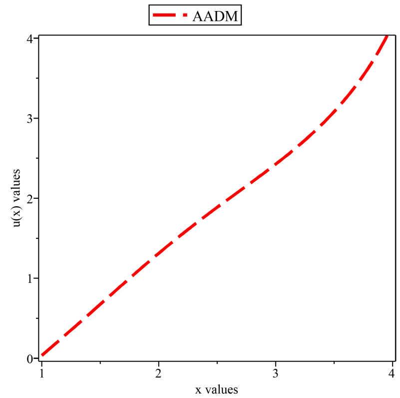


figure 5 when $\alpha = 0.98$, so we can obtain approximate solution of $u(x)$ by AADM, for arbitrary values of $0 < \alpha < 1$.

Example 4.4. Consider the following Riccati differential equation:

$$D_x^{\alpha} u(x) = x + 3u(x) - u^2(x), \quad 0 < \alpha \leq 1, \tag{4.24}$$

with the initial condition

$$u(0) = 1. \tag{4.25}$$

If $\alpha = 1$, the solution components of the far-field approximation $\phi_v(x)$ according to (3.10) and (3.11) are calculated as

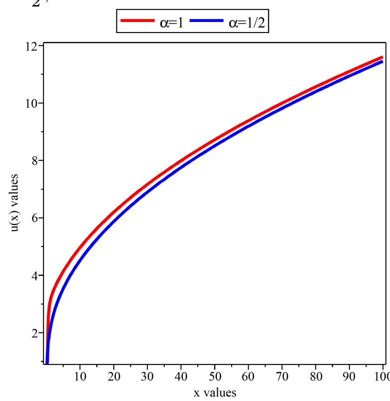
$$u_0(x) = \sqrt{x},\ u_1(x) = \frac{3}{2} - \frac{1}{2x},\ u_2(x) = \frac{1}{2\sqrt{x}}\left(\frac{9}{4} - \frac{1}{6x^2}\right)\cdots \tag{4.26}$$

Thus, by calculating three terms of asymptotic approximation, we gain $\phi_3(x) = \sum_{k=0}^{2} u_k(x)$, as a result $u(x) \approx \phi_3(x) = \sqrt{x} + \frac{3}{2} - \frac{1}{2x} + \frac{1}{2\sqrt{x}}\left(\frac{9}{4} - \frac{1}{6x^2}\right)$, which is far-field approximate solution. If $\alpha = \frac{1}{2}$, the far-field approximation $\phi_2(x)$ will be achieved as

$$u(x) \approx \phi_2(x) = \sqrt{x} + \frac{3}{2} - \frac{\sqrt{\pi}}{4}\frac{1}{\sqrt{x}}. \tag{4.27}$$

Figure 6 shows a comparison between the far field approximate solutions solved by AADM when α is $\frac{1}{2}$ and 1. All calculations have been done using maple on a computer with Intel Core i5-2430M CPU 2.400 GHz, 4.00 GB of RAM and 64-bit operating system (windows 7).

FIGURE 6. **The far-field approximate solutions of example 4.4 solved by AADM for** $\alpha = \frac{1}{2}, 1$



5. Conclusion

In this work, the asymptotic Adomian decomposition method has been implemented for solving fractional Riccati differential equations successfully. Instead of nested fractional integrations, in decomposition method, nested fractional differentiations have been used. In the four examples for fractional Riccati differential equations, an acceptable horizontal asymptotic approximate solution has been achieved. In the example of three, product integration is used to overcome the singularity in the integrand. Implemented product integration leads to a good approximate solution for arbitrary interval $[a, b]$, especially when x approaches to infinity. Using Hermite interpolation, for a complicated form of $u_n(x)$, is another novelty of this research. The application of the proposed approach to singular equations is under study in our research group. As a direction for future research, the application of AADM for asymptote solution or traditional solution of other fractional functional equations can be proposed.

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