Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 8, No. 4, 2020, pp. 610-638
DOI:10.22034/cmde.2020.31627.1479

Wavelet-Picard iterative method for solving singular fractional nonlinear partial differential equations with initial and boundary conditions

Amir Mohammadi
Department of Mathematics,
Azarbaijan Shahid Madani University, Tabriz, Iran.
E-mail: amirmohammadi1945@yahoo.com

Nasser Aghazadeh*
Department of Mathematics,
Azarbaijan Shahid Madani University, Tabriz, Iran.
E-mail: aghazadeh@azaruniv.ac.ir

Shahram Rezapour
Department of Mathematics,
Azarbaijan Shahid Madani University, Tabriz, Iran.
E-mail: rezapour@azaruniv.ac.ir

Abstract The present study applies the Picard iterative method to nonlinear singular partial fractional differential equations. The Haar and second-kind Chebyshev wavelets operational matrix of fractional integration will be used to solve problems combining linearization technique with the Picard method. The singular problem will be converted to an algebraic system of equations, which can be easily solved. Numerical examples are provided to illustrate the efficiency and accuracy of the technique.

Keywords. Fractional singular differential equation, Haar wavelets, Second-kind Chebyshev wavelets, Picard iteration.

2010 Mathematics Subject Classification. 35R11, 65T60, 45L05.

1. Introduction

Fractional calculus and fractional differential equations are commonly used in most areas of science, including mathematics, chemistry, physics, and engineering [1, 2, 5]. Many researchers have used fractional calculus to model the nonlinear oscillation of earthquakes [17], fluid dynamics for traffic flow [18], frequency-dependent damping behavior of viscoelastic materials [2], signal processing [27], and control theory [5]. This increase in applications have led to schemes being proposed to solve fractional differential equations. The most frequently used methods are Adomian decomposition method (ADM)[25], Homotopy Perturbation Method (HPM)[13], homotopy analysis method[16], and Variational Iteration Method (VIM)[12]. Moreover, the operational matrices of fractional order integration for Haar wavelets [9, 22, 29] have been developed to solve fractional differential equations. In the last two decades, the use

Received: 20 January 2019 ; Accepted: 11 May 2019.
∗ Corresponding author.

of wavelets for solving fractional differential equations has increased. Of these approaches, Haar wavelets have been more concerned with their simple nature . The matrix of the integral operator was first obtained by Chen and Hsiao [8]. As a branch of mathematical physics, fluid dynamics and astrophysics, it examines singular nonlinear partial differential equations. In fact, these equations have been used to model phenomena in these three fields. Examples of nonlinear singular equations include the equation of motion of a point mass in a central force field, the generalized equation of conventional current flow, Navier-Stokes cylindrical and spherical equation and hydrodynamic equations of cylindrical and spherical fluid instability. Despite their importance, singular nonlinear partial differential equations (PDEs) are difficult to solve [10, 11, 33]. These equations have recently been solved using methods such as ADM [14, 32], modified HPM (MHPM) [30], HPM [13], and MDM [11]. Nevertheless, a literature survey makes it clear that the Haar and second-kind Chebyshev wavelets collocation methods are never attempted when solving singular nonlinear PDEs. The goal of the current research is to solve a class of singular nonlinear differential equations using the Haar and second-kind Chebyshev wavelets collocation methods in combination with the Picard technique. The Picard technique was used to convert a fractional nonlinear singular equation into a system of linear equations to obtain an approximate solution using the Haar and second-kind Chebyshev wavelets collocation methods.

The model is:

$$\begin{aligned} &\frac{\partial^\alpha u}{\partial t^\alpha} - a(x)\frac{\partial^\beta u}{\partial x^\beta} + d(x)u^p = f(x,t), \\ &0 < \alpha \le 1, \quad 1 < \beta \le 2, \quad p > 1, \end{aligned} \tag{1.1}$$

with initial and boundary conditions,

$$\begin{aligned} &u(x,0) = g_1(x), \quad 0 \le x \le 1. \\ &u(0,t) = Y_1(t), \qquad u(1,t) = Y_2(t), \qquad t \in [0,1], \end{aligned}$$

such that $a(x)$ or $d(x)$ maybe have singularity at the point $x = 0$, $\frac{\partial^\alpha u}{\partial t^\alpha}$ donates the Caputo fractional derivative to time and $g_1(x), Y_1(t), Y_2(t), f(x,t)$ are the given functions. It must be noticed that till now, no one has attempted the wavelets (Haar and Chebyshev) collocation Picard methods on solving fractional singular nonlinear partial differential equations.

2. Wavelets and operational matrix of general order integration

2.1. **Haar wavelet and operational matrix of general order integration.** The lth Haar wavelet $h_l(x), x \in [0,1)$ is defined as

$$h_l(x) = \begin{cases} 1 & a(l) \le t < b(l) \\ -1 & b(l) \le t < c(l) \\ 0 & otherwise, \end{cases} \tag{2.1}$$

where $a(l) = \frac{k}{m}, b(l) = \frac{k+0.5}{m}, c(l) = \frac{k+1}{m}, l = 2^j + k + 1, j = 0, 1, 2, 3, \ldots, J$ are dilation parameters, $m = 2^j$ and $k = 0, 1, 2, \ldots 2^j - 1$ are translation parameters. When $k = 0$, $j = 0$, we have $l = 2$, which is the minimal value of l, and the maximal value of l is $2M$ where $M = 2^j$, J is maximal level of resolution. For the uniform Haar wavelet, the wavelet-collocation method is applied. The collocation points for uniform Haar wavelets are usually taken as $x_j = \frac{j-0.5}{2M}, j = 1, 2, 3, \ldots, 2M$.

The Riemann-Liouville fractional integral of the Haar scaling function and the Haar wavelets are given as [28]

$$
\begin{aligned}
P_{\alpha,1}(x) &= I^{\alpha}_{a(1)} h_1(x) = \frac{1}{\Gamma(\alpha)} \int_{a(1)}^{x} (x-s)^{\alpha-1} ds, \quad \alpha > 0 \qquad (2.2)\\
P_{\alpha,i}(x) &= I^{\alpha}_{a} h_i(x)\\
&= \frac{1}{\Gamma(\alpha)} \begin{cases} \int_{a(i)}^{x} (x-s)^{\alpha-1} ds, & a(i) \le x < b(i)\\ \int_{a(i)}^{b(i)} (x-s)^{\alpha-1} ds - \int_{b(i)}^{x} (x-s)^{\alpha-1} ds, & b(i) \le x < c(i)\\ \int_{a(i)}^{b(i)} (x-s)^{\alpha-1} ds - \int_{b(i)}^{c(i)} (x-s)^{\alpha-1} ds, & c(i) \le x. \end{cases}
\end{aligned}
$$

Each function $y \in L_2[0,1]$ can be expressed in terms of the Haar wavelets as

$$y(x) = \sum_{l=1}^{\infty} b_l h_l(x),$$

where b_l's are the Haar wavelets coefficients given by $b_l = \int_0^1 y(x) h_l(x) dx$.

The function $y(x)$ can be approximated by the truncated Haar wavelets series:

$$y(x) \approx y_m(x) = \sum_{l=1}^{m} b_l h_l(x). \qquad (2.3)$$

where $l = 2^j + k + 1, j = 0, 1, \ldots, J, k = 0, 1, \ldots 2^j - 1$.

In order to find the numerical approximation of a function, we put the Haar into a discrete form. For this purpose, we utilized the collocation method. The collocation points for the Haar wavelets are taken as $x_c(i) = \frac{i-0.5}{2M}$, where $i = 1, 2, \ldots 2M$. Each function of two variables $u(x,t) \in L_2([a,b] \times [a,b])$ can be approximated as

$$u(x,t) \approx \sum_{l=1}^{m} \sum_{i=1}^{m} c_{l,i} h_l(x) h_i(t) = H^T(x) C H(t),$$

where C is a $m \times m$ coefficients matrix which can be determined by the inner product $c_{l,i} = \langle h_l(x), \langle u(x,t), h_i(t) \rangle \rangle$.

Taking the collocation points as $x(i) = \frac{i-0.5}{m}$ where $i = 1, 2, \ldots, m$, we define the Haar matrix as

$$H_{m \times m} = \begin{pmatrix} h_2(x_c(1)) & h_1(x_c(2)) & \ldots & h_1(x_c(m))\\ h_2(x_c(1)) & h_2(x_c(2)) & \ldots & h_2(x_c(m))\\ \vdots & \vdots & \ddots & \vdots\\ h_m(x_c(1)) & h_m(x_c(2)) & \ldots & h_m(x_c(m)) \end{pmatrix}.$$

We can represent Eq. (2.3) in vector form as $y = cH$, where $c = [c_1, c_2, \ldots, c_m]^T$. The Haar coefficients b_l can be determined by matrix inversion

$$b = yH^{-1}, \qquad (2.4)$$

where H^{-1} is the inverse of H. Eq. (2.4) gives the Haar coefficients b_l which are used in Eq. (2.3) to get the solution $y(x)$. Similarly, we can obtain the fractional order integration matrix P of Haar functions by substituting the collocation points in Eq. (2.2) $P(l,i) = p_{\alpha,l}(x_c(i)))$, as

$$^{H}P_{2M\times 2M} = \begin{pmatrix} p_1(x_c(1)) & p_1(x_c(2)) & \dots & p_1(x_c(m)) \\ p_2(x_c(1)) & p_2(x_c(2)) & \dots & p_2(x_c(m)) \\ \vdots & \vdots & \ddots & \vdots \\ p_m(x_c(1)) & p_m(x_c(2)) & \dots & p_m(x_c(m)) \end{pmatrix}.$$

For instance, with $\alpha = 0.25,\ J = 2\ (m = 8)$, we get the Haar wavelets operational matrix of fractional integration

$$^{H}P_{8\times 8} = \begin{pmatrix} 0.5516 & 0.7259 & 0.8248 & 0.8972 & 0.9554 & 1.0046 & 1.0474 & 1.0856 \\ 0.5516 & 0.7259 & 0.8248 & 0.8972 & -0.1478 & -0.4473 & -0.60239 & -0.7089 \\ 0.5516 & 0.7259 & -0.2783 & -0.5547 & -0.14267 & -0.0639 & -0.0385 & -0.0263 \\ 0 & 0 & 0 & 0 & 0.5516 & 0.7259 & -0.2783 & -0.5547 \\ 0.5516 & -0.3772 & -0.0754 & -0.0265 & -0.0142 & -0.0090 & -0.0063 & -0.0046 \\ 0 & 0 & 0.5516 & -0.3772 & -0.0754 & -0.0265 & -0.01420 & -0.0090 \\ 0 & 0 & 0 & 0 & 0.5516 & -0.3772 & -0.0754 & -0.0265 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5516 & -0.3772 \end{pmatrix}$$

We derive another operational matrix of fractional integration to solve the fractional boundary value problems. Let $\zeta > 0$ and $z : [0, \zeta] \to R$ be a continuous function and assume that Haar function has $[0, \zeta]$ as compact support, then

$$z(x) I_0^{\alpha} h_1(\zeta) = z(x) \int_0^{\zeta} (\zeta - s)^{\alpha} ds, \quad v^{\alpha,\zeta,1} = z(x) C_{\alpha,1}, \tag{2.5}$$

and

$$z(x) I_0^{\alpha} h_l(\zeta) = z(x) [\int_{a(l)}^{b(l)} (\zeta - s)^{\alpha-1} ds - \int_{b(l)}^{a(l)} (\zeta - s)^{\alpha-1} ds], v^{\alpha,\zeta,1} = z(x) C_{\alpha,l}, \tag{2.6}$$

where $C_{\alpha,1} = \frac{\zeta^{\alpha}}{\Gamma(\alpha+1)}, C_{\alpha,l} = \frac{1}{\Gamma(\alpha+1)} \Big[(\zeta - a(i))^{\alpha} - 2(\zeta - b(i))^{\alpha} + (\zeta - c(i))^{\alpha} \Big]$.

In particular, for $\zeta = 1, z(x) = x, \alpha = 1.25, m = 8$, we get

$$^{H}V_{8\times 8}^{1.25,x,1} = \begin{pmatrix} 0.8826 & 0.8826 & 0.8826 & 0.8826 & 0.8826 & 0.8826 & 0.8826 & 0.8826 \\ 0.1404 & 0.1404 & 0.1404 & 0.1404 & 0.1404 & 0.1404 & 0.1404 & 0.1404 \\ 0.0216 & 0.0216 & 0.0216 & 0.0216 & 0.0216 & 0.0216 & 0.0216 & 0.0216 \\ 0.0590 & 0.0590 & 0.0590 & 0.0590 & 0.0590 & 0.0590 & 0.0590 & 0.0590 \\ 0.0047 & 0.0047 & 0.0047 & 0.0047 & 0.0047 & 0.0047 & 0.0047 & 0.0047 \\ 0.0061 & 0.0061 & 0.0061 & 0.0061 & 0.0061 & 0.0061 & 0.0061 & 0.0061 \\ 0.0091 & 0.0091 & 0.0091 & 0.0091 & 0.0091 & 0.0091 & 0.0091 & 0.0091 \\ 0.0248 & 0.0248 & 0.0248 & 0.0248 & 0.0248 & 0.0248 & 0.0248 & 0.0248 \end{pmatrix}.$$

2.2. **Second-kind Chebyshev wavelets and operational matrix of general order integration.** The second-kind Chebyshev wavelets $\psi_{n,m}(t) = \psi(k, n, m, t)$ have four arguments k, m, n, t, which k can assume any positive integer, $n = 1, 2, 3, \dots, 2^{k-1}$, m is the degree of the second-kind Chebyshev polynomials and t is the normalized time. They are defined on the interval $[0, 1)$ as

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}U_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}} \\ 0, & otherwise. \end{cases} \tag{2.7}$$

$U_m(t)$'s are the second-kind Chebyshev polynomials of degree m which are orthogonal with respect to the weight function $w(t) = \sqrt{1-t^2}$ on the interval $[-1, 1]$ and satisfy the following recursive formula

$$\begin{aligned} &U_0(t) = 1, \quad U_1(t) = 2t, \\ &U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), \qquad m = 1, 2, 3, \dots. \end{aligned}$$

The weight function $\tilde{w}(t) = w(2t - 1)$ has to be dilated and translated as $w_n(t) = w(2^k t - 2n + 1)$. A function $f(x) \in L_2(R)$ defined over $[0, 1)$ can be expanded by the second-kind Chebyshev wavelets as

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{n,m}\psi_{n,m}(x), \tag{2.8}$$

where

$$c_{n,m} = \langle f(x), \psi_{n,m}(x)\rangle.$$

If the infinite series in Eq.(2.8) is truncated, then

$$f(x) \cong \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) = C^T\Psi(x), \tag{2.9}$$

which the coefficients vector C and the second kind Chebyshev wavelet function vectors $\Psi(x)$ are $m^{'} = 2^{k-1}M$ column vectors. For simplicity, we rewrite Eq. (2.9) as

$$f(x) \cong \sum_{i=1}^{m^{'}} c_i\psi_i = C^T\Psi(x), \tag{2.10}$$

where $c_i = c_{n,m}, \psi_i(t) = \psi_{n,m}(t)$. The index i can be determined by the relation $i = M(n-1) + m + 1$. Thus, we have

$$C = [c_1, c_2, c_3, \dots, c_{m'}]^T,$$

$$\Psi(t) = [\psi_1, \psi_2, \psi_3, \dots, \psi_{m'}]^T.$$

By taking the collocation points as $t_i = \frac{2i-1}{2^k M}, i = 1, 2, 3, \dots, 2^{k-1}M$, we define the second-kind Chebyshev wavelets matrix $\Phi(x)_{m'\times m'}$ as

$$\Phi_{m'\times m'} = \Big[\Psi(\frac{1}{2m'}), \Psi(\frac{3}{2m'}), \dots, \Psi(\frac{2m'-1}{2m'})\Big],$$

where $m' = 2^{k-1}M$.

For example, when $M = 4$ and $k = 2$, the second-kind Chebyshev wavelets matrix is expressed as

$$\Phi_{8\times 8} = \begin{pmatrix} 1.5958 & 1.5958 & 1.5958 & 1.5958 & 0 & 0 & 0 & 0 \\ -2.3937 & -2.3937 & -2.3937 & -2.3937 & 0 & 0 & 0 & 0 \\ 1.9947 & 1.9947 & 1.9947 & 1.9947 & 0 & 0 & 0 & 0 \\ -0.5984 & -0.5984 & -0.5984 & -0.5984 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.5958 & 1.5958 & 1.5958 & 1.5958 \\ 0 & 0 & 0 & 0 & -2.3937 & -2.3937 & -2.3937 & -2.3937 \\ 0 & 0 & 0 & 0 & 1.9947 & 1.9947 & 1.9947 & 1.9947 \\ 0 & 0 & 0 & 0 & -0.5984 & -0.5984 & -0.5984 & -0.5984 \end{pmatrix}.$$

In the same way, a function $u(x,t) \in L_2([0,1]\times[0,1])$ can be also approximated as

$$u(x,t) = \Psi^T(x)U\Psi(t), \tag{2.11}$$

which U is a $m' \times m'$ matrix with $u_{ij} = \langle \psi_i(x), \langle u(x,t), \psi_j(t)\rangle\rangle$. We use the wavelet collocation method to determine the coefficients $u_{i,j}$.

Fractional integral formula of the Chebyshev wavelets in the Riemann-Liouville sense is derived by means of the shifted second-kind Chebyshev polynomials U^*_m, which play an important role in dealing with the time fractional equations.

Theorem 2.1. [34] *The fractional integral of a Chebyshev wavelet defined on the interval* $[0,1]$ *with compact support* $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}]$ *is given by*

$$I^\alpha \psi_{n,m}(x) = \tag{2.12}$$
$$\begin{cases} 0, & x < \frac{n-1}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} \sum\limits_{r=0}^{m} \sum\limits_{i=r}^{m} \sum\limits_{j=0}^{r} T^{m,n,k}_{i,i-r} \frac{(-1)^j}{j+\alpha} \times & \\ C^j_r x^{r-j} \left(x - \frac{n-1}{2^{k-1}}\right)^{j+\alpha}, & \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)} 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} \sum\limits_{r=0}^{m} \sum\limits_{i=r}^{m} \sum\limits_{j=0}^{r} T^{m,n,k}_{i,i-r} \frac{(-1)^j}{j+\alpha} \times & \\ C^j_r x^{r-j} \left(\left(x - \frac{n-1}{2^{k-1}}\right)^{j+\alpha} - \left(x - \frac{n}{2^{k-1}}\right)^{j+\alpha}\right), & x > \frac{n}{2^{k-1}} \end{cases}$$

where $T^{m,n,k}_{i,i-r} = (-1)^{m-r} 2^{2i} 2^{r(k-1)} (n-1)^{i-r} \left(\frac{\Gamma(m+i+2)}{\Gamma(m-i+1)\Gamma(2i+2)}\right)\left(\frac{i!}{(n-r)!r!}\right)$, $C^j_r = \frac{r!}{j!(j-r)!}$.

For instance, in the case of $k=2, M=4, x=0.6, \alpha=0.9$, we obtain

$$I^{0.9}\Psi_{8\times 1}(0.6)=\begin{pmatrix} 0.838817891721642\\ 0.045706956934399\\ 0.290734994150959\\ 0.021626272477045\\ 0.208881853762857\\ -0.329813453309774\\ 0.323368822612918\\ -0.217309447751042 \end{pmatrix}.$$

where $\Psi_{8\times 1}=(\psi_{1,0}(x),\psi_{1,1}(x),\psi_{1,2}(x),\psi_{1,3}(x),\psi_{2,0}(x),\psi_{2,1}(x),\psi_{12,2}(x),\psi_{2,3}(x))^T$.

We can obtain the fractional order integration matrix ${}^{C}P^{\alpha}_{m'\times m'}=I^{\alpha}\psi_{n,m}(x)$ by substituting the collocation points in Eq. (2.12) as:

$$^{C}P^{\alpha}_{2^{k-1}M\times 2^{k-1}M}=$$
$$\begin{pmatrix} I^{\alpha}\psi_{1,0}(x(1)) & I^{\alpha}\psi_{1,0}(x(2)) & \dots & I^{\alpha}\psi_{1,0}(x(2^{k-1}M))\\ I^{\alpha}\psi_{1,1}(x(1)) & I^{\alpha}\psi_{1,1}(x(2)) & \dots & I^{\alpha}\psi_{1,1}(x(2^{k-1}M))\\ \vdots & \vdots & \ddots & \vdots\\ I^{\alpha}\psi_{2^{k-1},M}(x(1)) & I^{\alpha}\psi_{2^{k-1},M}(x(2)) & \dots & I^{\alpha}\psi_{2^{k-1},M}(x(2^{k-1}M)) \end{pmatrix}.$$

In particular, we fix $k=2, M=4$ and $\alpha=0.9$, then

$$^{C}P^{0.9}_{8\times 8}=$$
$$\begin{pmatrix} 0.1368 & 0.3678 & 0.5825 & 0.7885 & 0.8517 & 0.8165 & 0.7939 & 0.7771\\ -0.2377 & -0.4452 & -0.3985 & -0.1245 & 0.0545 & 0.0337 & 0.0246 & 0.0194\\ 0.2789 & 0.2423 & 0.0032 & 0.0615 & 0.2996 & 0.2783 & 0.2680 & 0.2612\\ -0.2570 & -0.0232 & -0.0530 & -0.2259 & 0.0274 & 0.0148 & 0.0104 & 0.0081\\ 0 & 0 & 0 & 0 & 0.1368 & 0.3678 & 0.5825 & 0.7885\\ 0 & 0 & 0 & 0 & -0.2377 & -0.4452 & -0.3985 & -0.1245\\ 0 & 0 & 0 & 0 & 0.2789 & 0.2423 & 0.0032 & 0.0615\\ 0 & 0 & 0 & 0 & -0.2570 & -0.0232 & -0.0530 & -0.2259 \end{pmatrix}.$$

We derive another operational matrix of fractional integration to solve the fractional boundary value problems. Let $\eta>0$ and $g:[0,\eta]\to R$ be a continuous function, put

$$g(x)I^{\alpha}\psi_{n,m}(\eta)=v^{\alpha,\eta}. \tag{2.13}$$

We define a matrix V by substituting the collocation points $x_i=\frac{2i-1}{2^kM}, i=1,2,\dots,2^{k-1}M$ in Eq. (2.13),

$$^{C}V^{\alpha,\eta,g(x)}_{2^{k-1}M\times 2^{k-1}M}=$$
$$\begin{pmatrix} g(x_1)I^{\alpha}\psi_{1,0}(\eta) & g(x_2)I^{\alpha}\psi_{1,0}(\eta) & \dots & g(x_{2^{k-1}M})I^{\alpha}\psi_{1,0}(\eta)\\ g(x_1)I^{\alpha}\psi_{1,1}(\eta) & g(x_2)I^{\alpha}\psi_{1,1}(\eta) & \dots & g(x_{2^{k-1}M})I^{\alpha}\psi_{1,1}(\eta)\\ \vdots & \vdots & \ddots & \vdots\\ g(x_1)I^{\alpha}\psi_{2^{k-1},M-1}(\eta) & g(x_2)I^{\alpha}\psi_{2^{k-1},M-1}(\eta) & \dots & g(x_{2^{k-1}M})I^{\alpha}\psi_{2^{k-1},M-1}(\eta) \end{pmatrix}.$$

In particular, for $\eta = 1, g(x) = x, \alpha = 0.9, k = 2$ and $M = 4$, we get

$$
{}^{C}V_{8\times 8}^{0.9,1,x} = \begin{pmatrix}
0.0374 & 0.1112 & 0.1870 & 0.2618 & 0.3366 & 0.4114 & 0.4862 & 0.5610 \\
-0.0083 & -0.0249 & -0.0416 & -0.0582 & -0.0748 & -0.0914 & -0.1080 & -0.1247 \\
0.0125 & 0.0374 & 0.0623 & 0.0873 & 0.1112 & 0.1371 & 0.1621 & 0.1870 \\
-0.0033 & -0.01007 & -0.0166 & -0.0233 & -0.0299 & -0.0366 & -0.0432 & -0.0499 \\
0.0125 & 0.0374 & 0.0623 & 0.0873 & 0.1122 & 0.1371 & 0.1621 & 0.1870 \\
-0.0083 & -0.0249 & -0.0416 & -0.0582 & -0.0748 & -0.0914 & -0.1080 & -0.1247 \\
0.0042 & 0.0125 & 0.0208 & 0.0291 & 0.0374 & 0.0457 & 0.0540 & 0.0623 \\
-0.0033 & -0.0100 & -0.0166 & -0.0233 & -0.0299 & -0.0366 & -0.0432 & -0.0499
\end{pmatrix}.
$$

3. Convergence

3.1. Haar wavelets.

Theorem 3.1. *Consider the functions $u_m(x,t)$ obtained by the Haar wavelet with approximation $u(x,t)$. Then, $\|u(x,t) - u_m(x,t)\|_E \leq \frac{K}{\sqrt{3m}}$, where $\|u(x,t)\|_E = \left(\int_0^1 \int_0^1 u^2(x,t)dxdt\right)^{\frac{1}{2}}$.*

Proof. Suppose $u_m(x,t)$ is the following approximation of $u(x,t)$,

$$u_m(x,t) = \sum_{n=0}^{m-1} \sum_{l=0}^{m-1} u_{nl} h_n(x) h_l(t).$$

Then we have

$$u(x,t) - u_m(x,y) = \sum_{n=m}^{\infty} \sum_{l=m}^{\infty} u_{nl} h_n(x) h_l(t) = \sum_{n=2^{p+1}}^{\infty} \sum_{l=2^{p+1}}^{\infty} u_{nl} h_n(x) h_l(t).$$

The orthogonality of the sequence $h_i(x)$ on $[0,1)$ implies that

$$h_l(.) = 2^{\frac{j}{2}} h(2^j(.) - k). \tag{3.1}$$

Therefore

$$
\begin{aligned}
||u(x,t) - u_m(x,t)||_E^2 &= \int_0^1 \int_0^1 (u(x,t) - u_m(x,t))^2 dxdt \qquad (3.2)\\
&= 2^j \sum_{n=2^{p+1}}^{\infty} \sum_{l=2^{p+1}}^{\infty} \sum_{n'=2^{p+1}}^{\infty} \sum_{l'=2^{p+1}}^{\infty} u_{nl} u_{n'l'} \left(\int_0^1 h_n(x) h_{n'}(x) dx\right) \\
&\times \left(\int_0^1 h_l(t) h_{l'}(t) dt\right) = 2^j \sum_{n=2^{p+1}}^{\infty} \sum_{l=2^{p+1}}^{\infty} u_{nl}^2,
\end{aligned}
$$

where $u_{nl} = \Big\langle h_n(x), \langle u(x,t), h_l(t)\rangle \Big\rangle$.
According to Eq. (2.1) and definition inner production, we have

$$\Big\langle u(x,t), h_l(t) \Big\rangle = \int_0^1 u(x,t)h_l(t)dt = 2^{\frac{j}{2}}\left(\int_{\frac{k-1}{2^j}}^{\frac{k-0.5}{2^j}} u(x,t)dt - \int_{\frac{k-0.5}{2^j}}^{\frac{k}{2^j}} u(x,t)dt\right).$$

By using mean value theorem of integrals

$$\exists t_1, t2: \quad \frac{k-1}{2^j} \le t_1 < \frac{k-0.5}{2^j}, \quad \frac{k-0.5}{2^j} \le t_2 < \frac{k}{2^j},$$

so that

$$\begin{aligned}\Big\langle u(x,t), h_l(t) \Big\rangle &= 2^{\frac{j}{2}}\left((\frac{k-0.5}{2^j} - \frac{k-1}{2^j})u(x,t_1) - (\frac{k}{2^j} - \frac{k-0.5}{2^j})u(x,t_2)\right) \\ &= \frac{2^{\frac{j}{2}}}{2^{j+1}}\Big(u(x,t_1) - u(x,t_2)\Big)\end{aligned}$$

$$\begin{aligned}u_{nl} &= \Big\langle h_n(x), \frac{1}{2^{\frac{j}{2}+1}}(u(x,t_1)-u(x,t_2))\Big\rangle = \frac{1}{2^{j+1}}\int_0^1 h_n(x)(u(x,t_1)-u(x,t_2))dx \\ &= \frac{2^{\frac{j}{2}}}{2^{\frac{j}{2}+1}}\left(\int_{\frac{k-1}{2^j}}^{\frac{k-0.5}{2^j}} u(x,t_1)dx - \int_{\frac{k-0.5}{2^j}}^{\frac{k}{2^j}} u(x,t_1)dx - \int_{\frac{k-1}{2^j}}^{\frac{k-0.5}{2^j}} u(x,t_2)dx + \int_{\frac{k-0.5}{2^j}}^{\frac{k}{2^j}} u(x,t_2)dx\right).\end{aligned}$$

By using mean value theorem of integrals again

$$\exists x_1, x_2, x_3, x_4 :: \quad \frac{k-1}{2^j} \le x_1, x_2 < \frac{k-0.5}{2^j}, \quad \frac{k-0.5}{2^j} \le x_3, x_4 < \frac{k}{2^j}$$

$$\begin{aligned}u_{nl} &= \frac{1}{2}\Big\{(\frac{k-0.5}{2^j} - \frac{k-1}{2^j})u(x_1,t_1) - (\frac{k-}{j} - \frac{k-0.5}{2^j})u(x_2,t_1) - \\ &\quad (\frac{k-0.5}{2^j} - \frac{k-1}{2^j})u(x_3,t_2) + (\frac{k}{2^j} - \frac{k-0.5}{2^j})u(x_4,t_2)\Big\} \\ &= \frac{1}{2^{j+2}}\Big\{(u(x_1,t_1) - u(x_2,t_1)) - (u(x_3,t_2) - u(x_4,t_2))\Big\}\end{aligned}$$

$$u_{nl}^2 = \frac{1}{2^{2j+4}}\Big\{(u(x_1,t_1) - u(x_2,t_1)) - (u(x_3,t_2) - u(x_4,t_2))\Big\}^2.$$

By using mean value theorem of derivatives

$$\exists \xi_1, \xi_2: \quad x_1 \le \xi_1 < x_2, \quad x_3 \le \xi_2 < x_4$$

so that

$$\begin{aligned}u_{nl}^2 &\le \tfrac{1}{2^{2j+4}} \quad \Big\{(x_2-x_1)^2[\frac{\partial u(\xi_1,t_1)}{\partial x}]^2 + (x_4-x_3)^2[\frac{\partial u(\xi_1,t_1)}{\partial x}]^2 + \\ &\quad 2(x_2-x_1)(x_4-x_3)\left|\frac{\partial u(\xi_1,t_1)}{\partial x}\right|\left|\frac{\partial u(\xi_2,t_2)}{\partial x}\right|\Big\}.\end{aligned}$$

We assume that $\frac{\partial u(x,t)}{\partial x}$ is continuous and bounded on $(0,1)\times(0,1)$, then

$$\exists K>0, \forall x,t \in (0,1)\times(0,1), \qquad \left|\frac{\partial u(x,t)}{\partial x}\right| \leq K.$$

$$u_{nl}^2 \leq \left(\frac{1}{2^{2j+4}}\right)\frac{4K^2}{2^{2j}} = \frac{4K^2}{2^{4j+4}} \tag{3.3}$$

substituting Eq. (3.3) into Eq. (3.2), we have

$$\begin{aligned} ||u(x,t)-u_m(x,t)||_E^2 &= \sum_{j=p+1}^{\infty}\left(\sum_{n=2^j}^{2^{j+1}-1}\sum_{n=2^j}^{2^{j+1}-1} u_{nl}^2\right) \leq \sum_{j=p+1}^{\infty}\left(\sum_{n=2^j}^{2^{j+1}-1}\sum_{n=2^j}^{2^{j+1}-1}\frac{4K^2}{2^{4j+4}}\right) \\ &= 4K^2\sum_{j=p+1}^{\infty}\left(\sum_{n=2^j}^{2^{j+1}-1}\sum_{n=2^j}^{2^{j+1}-1}\frac{1}{2^{4j+4}}\right) \\ &= \frac{K^2}{3}\frac{1}{4^{p+1}} = \frac{K^2}{3m^2}. \end{aligned}$$

Thus, $\|u(x,t)-u_m(x,t)\|_E \leq \frac{K}{\sqrt{3}m}$ and so $\|u(x,t)-u_m(x,t)\|_E \to 0$ when $m\to\infty$. By using a similar procedure, we can show that $\|u_{r+1}(x,t)-u_{r+1}^m(x,t)\|_E \leq \frac{K}{\sqrt{3}m}$ which implies that error between the exact and approximate solution at the (r+1)-th iteration is inversely proportional to the maximal level of resolution. This shows that $u_{r+1}^m(x,t)$ converges to $u_{r+1}(x,t)$ as $m\to\infty$. Since $u_{r+1}(x,t)$ is obtained at $(r+1)$-th iteration of Picard technique, we conclude that $u_{r+1}(x,t)$ converges to $u(x,t)$ as $r\to\infty$. Thus, $\lim_{m,r\to\infty} u_{r+1}^m(x,t) = u(x,t)$. □

3.2. **Second-kind Chebyshev wavelets.**

Theorem 3.2. [35] *Let $f(x)$ be a second-order derivative square-integrable function defined on $[0,1)$ with bounded second order derivative, say $|f''(x)| \leq B$ for some constant B, then*
(i) $f(x)$ can be expanded as an infinite sum of the second kind Chebyshev wavelets and the series converges to $f(x)$ uniformly, that is

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\psi_{n,m}(x),$$

where $c_{nm} = \langle f(x), \psi_{n,m}(x)\rangle$.
(ii)

$$\sigma_{f,k,M} \leq \frac{\sqrt{\pi}B}{2^3}\left(\sum_{n=2^{k-1}+1}^{\infty}\frac{1}{n^5}\sum_{m=M}^{\infty}\frac{1}{(m-1)^4}\right)^{\frac{1}{2}},$$

where $\sigma_{f,k,M} = \left(\int_0^1 |f(x) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{n,m}(x)|^2 w_n(x)dx\right)^{\frac{1}{2}}.$

Theorem 3.3. [34] *Supposing that $u(x,t) \in \mathcal{L}^2(\mathbb{R}^2)$ is a continuous function defined on $[0,1) \times [0,1)$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^4 u}{\partial x^2 \partial t^2}$ are bounded with some positive constant B. Then for any positive integer k:*

i) the series

$$\sum_{n=1}^{2^{k-1}} \sum_{n'=1}^{2^{k-1}} \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} d_{n,m,n',m'} \psi_{n,m}(x) \psi_{n',m'}(t)$$

convergences to $u(x,t)$ uniformly in $\mathcal{L}^2(\mathbb{R}^2)$, where

$$d_{n,m,n',m'} = < u(x,t), \psi_{n,m}(x)\psi_{n',m'}(t) >_{L^2_w([0,1]\times[0,1])} .$$

ii)

$$\begin{aligned} \sigma^2_{\mu,k,M} &< \frac{(B\pi)^2}{2^6} \sum_{n=1}^{2^{k-1}} \frac{1}{n^3} \sum_{n'=1}^{2^{k-1}} \frac{1}{n'^3} \sum_{m=M}^{\infty} \frac{1}{m^2(m-1)^2} \\ &+ \frac{(B\pi)^2}{2^{15}} \sum_{n=1}^{2^{k-1}} \frac{1}{n^5} \sum_{n'=1}^{2^{k-1}} \frac{1}{n'^5} \sum_{m=2}^{\infty} \frac{1}{m^2(m-1)^2} \sum_{m'=M}^{\infty} \frac{1}{m'^2(m'-1)^2}, \end{aligned}$$

where

$$\sigma_{\mu,k,M} = \left(\int_0^1 \int_0^1 \left| u(x,t) - \sum_{n=1}^{2^{k-1}} \sum_{n'=1}^{2^{k-1}} \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} d_{n,m,n',m'} \psi_{n,m}(x) \psi_{n',m'}(t) \right|^2 w_n(x) w_{n'}(t) dx dt \right)^{\frac{1}{2}}.$$

Since k, n and m are positive finite constants, from Theorems (3.2), (3.3), we conclude that $u_{r+1}(x,t) \to u(x,t)$ as $r \to \infty$.

4. Description of the proposed methods

4.1. **The Haar wavelets method.** In this section, we describe the procedure of implementing the method for solving singular nonlinear fractional partial differential equation (SPDE). First, we convert SPDE into a discrete fractional PDE by the Picard technique. Thus, we solve it to obtain the solution of the problem by the Haar wavelet operational matrix method.

Consider the following nonlinear singular fractional partial differential equation:

$$\begin{aligned} &\frac{\partial^\alpha u}{\partial t^\alpha} - a(x) \frac{\partial^\beta u}{\partial x^\beta} + d(x) u^p = f(x,t), \\ &0 < \alpha \le 1, \quad 1 < \beta \le 2, \quad p > 1, \end{aligned} \tag{4.1}$$

with initial and boundary conditions:

$$\begin{aligned} &u(x,0) = g_1(x), \quad 0 \le x \le 1, \\ &u(0,t) = Y_1(t), \qquad u(1,t) = Y_2(t), \qquad t \in [0,1], \end{aligned}$$

where the $a(x)$ or $d(x)$ maybe have singularity at the point $x = 0$.

TABLE 1. Absolute error of $|u_5(x,t) - u(x,t)|$, for different values of α, β when they tend to $1, 2$ respectively at the 5th iteration, $J = 4$ in Example (5.1) using HWCM

(x,t)	$\alpha = 0.7$	$\alpha = 0.95$	$\alpha = 0.95$	$\alpha = 1$	$\alpha = 1$ method[30]
	$\beta = 1.7$	$\beta = 1.95$	$\beta = 2$	$\beta = 2$	$\beta = 2$
$(\frac{1}{64}, \frac{1}{64})$	4.7655e-03	4.3028e-04	1.7837e-05	1.7260e-06	7.7254e-03
$(\frac{5}{64}, \frac{5}{64})$	4.1936e-03	2.6200e-04	1.1640e-04	3.1004e-06	7.0041e-03
$(\frac{9}{64}, \frac{9}{64})$	9.9047e-03	5.4967e-04	2.5370e-04	1.2849e-06	6.5127e-03
$(\frac{13}{64}, \frac{13}{64})$	1.1744e-02	8.7121e-04	3.9266e-04	2.5508e-07	1.2543e-03
$(\frac{17}{64}, \frac{17}{64})$	1.3552e-02	1.1836e-03	5.1850e-04	1.4435e-07	2.2215e-04
$(\frac{21}{64}, \frac{21}{64})$	1.5329e-02	1.4910e-03	6.3004e-04	2.9942e-07	3.0219e-04
$(\frac{25}{64}, \frac{25}{64})$	1.7075e-02	1.7820e-03	7.2762e-04	3.8982e-07	4.0546e-04
$(\frac{29}{64}, \frac{29}{64})$	2.4886e-02	2.0631e-03	8.1290e-04	4.8944e-07	3.7491e-04
$(\frac{33}{64}, \frac{33}{64})$	2.2391e-02	2.3308e-03	8.8798e-04	6.4271e-07	2.7741e-04
$(\frac{37}{64}, \frac{37}{64})$	1.9748e-02	2.5897e-03	9.5451e-04	8.8990e-07	2.5123e-04
$(\frac{41}{64}, \frac{41}{64})$	1.6960e-02	2.8387e-03	1.3726e-03	1.2652e-06	2.1752e-04
$(\frac{45}{64}, \frac{45}{64})$	1.4027e-02	3.0805e-03	1.3434e-03	1.7799e-06	2.0514e-04
$(\frac{49}{64}, \frac{49}{64})$	1.0951e-02	3.3151e-03	1.2763e-03	2.3957e-06	1.7517e-05
$(\frac{53}{64}, \frac{53}{64})$	7.7348e-03	1.2613e-03	6.2277e-04	2.9841e-06	1.4721e-05
$(\frac{57}{64}, \frac{57}{64})$	4.3814e-03	7.3475e-04	1.3665e-04	3.1817e-06	1.9541e-06
$(\frac{61}{64}, \frac{61}{64})$	8.9485e-03	1.8182e-04	4.6175e-05	1.9165e-06	1.1449e-07

Applying the Picard technique [4] to Eq. (4.1), we get

$$\frac{\partial^\alpha u_{r+1}}{\partial t^\alpha} - a(x)\frac{\partial^\beta u_{r+1}}{\partial x^\beta} = f(x,t,u_r), \qquad (4.2)$$
$$0 < \alpha \leq 1, \quad 1 < \beta \leq 2, \quad , r \geq 1$$

with the initial and boundary conditions:

$$u_{r+1}(x,0) = g_1(x), \quad 0 \leq x \leq 1,$$
$$u_{r+1}(0,t) = Y_1(t), \qquad u_{r+1}(1,t) = Y_2(t), \qquad t \in [0,1],$$

where $f(x,t,u_r) := f(x,t) - d(x)u_r^p$.

Applying the Haar wavelet method to Eq. (4.2), we approximate the higher order term by the Haar wavelet series as

$$\frac{\partial^\beta u_{r+1}}{\partial x^\beta} = \sum_{l=1}^{m}\sum_{p=1}^{m} c_{lp}^{r+1} h_l(x)h_p(t) = H^T(x)C^{r+1}H(t), \qquad (4.3)$$

TABLE 2. Absolute error of $|u_5(x,t) - u(x,t)|$, for different values of α, β when they tend to $1, 2$ respectively at the 5th iteration, $k = 2, M = 8$ in Example (5.1), using CWCM

(x,t)	$\alpha = 0.5$ $\beta = 1.75$	$\alpha = 0.95$ $\beta = 1.95$	$\alpha = 1$ $\beta = 2$	$\alpha = 1$ method[30] $\beta = 2$
$(\frac{1}{32}, \frac{1}{32})$	1.0235e-03	2.9795e-04	9.8797e-13	4.2154e-02
$(\frac{3}{32}, \frac{3}{32})$	2.7733e-03	7.6429e-04	6.3487e-12	4.0041e-02
$(\frac{5}{32}, \frac{5}{32})$	3.8470e-03	8.8657e-04	3.7582e-11	6.1784e-03
$(\frac{7}{32}, \frac{7}{32})$	4.4533e-03	8.4230e-04	2.3422e-10	1.9223e-03
$(\frac{9}{32}, \frac{9}{32})$	4.5295e-03	5.9850e-04	9.6780e-10	2.7205e-04
$(\frac{11}{32}, \frac{11}{32})$	4.0359e-03	1.4470e-04	2.9476e-09	2.3709e-04
$(\frac{13}{32}, \frac{13}{32})$	2.8440e-03	5.9145e-04	7.3681e-09	2.6057e-04
$(\frac{15}{32}, \frac{15}{32})$	9.8309e-04	1.5515e-03	1.6114e-08	4.0174e-04
$(\frac{17}{32}, \frac{17}{32})$	2.0000e-03	3.0606e-03	3.1917e-08	3.4774e-04
$(\frac{19}{32}, \frac{19}{32})$	5.7003e-03	4.7506e-03	5.8314e-08	3.6823e-04
$(\frac{21}{32}, \frac{21}{32})$	1.0550e-02	6.9530e-03	9.8903e-08	1.7064e-04
$(\frac{23}{32}, \frac{23}{32})$	1.6528e-02	9.6129e-03	1.5538e-07	3.7981e-04
$(\frac{25}{32}, \frac{25}{32})$	2.3668e-02	1.2753e-02	2.2322e-07	2.9751e-04
$(\frac{27}{32}, \frac{27}{32})$	3.1955e-02	1.6371e-02	2.8391e-07	1.4021e-04
$(\frac{29}{32}, \frac{29}{32})$	4.1299e-02	2.0420e-02	2.9268e-07	1.0691e-05
$(\frac{31}{32}, \frac{31}{32})$	5.1541e-02	2.4899e-02	1.6170e-07	1.0037e-05

applying the fractional integral operator I_x^β on Eq. (4.3) gives

$$u_{r+1}(x,t) = ({}^H P_x^2)^T C^{r+1} H(t) + p(t)x + q(t), \tag{4.4}$$

where $p(t)$ and $q(t)$ are functions of t. Using the boundary conditions and Eqs. (4.3),(4.4), we get

$$\begin{aligned} q(t) &= Y_1(t) \\ p(t) &= -\sum_{l=1}^{m}\sum_{p=1}^{m} c_{lp}^{r+1}(I_x^\beta h_l(x))h_p(t) + Y_2(t) - Y_1(t). \end{aligned}$$

Eq. (4.4) can be written as

$$\begin{aligned} u_{r+1}(x,t) &= ({}^H P_x^\beta)^T C^{r+1} H(t) - x\Big[({}^H P_x^\beta(1))^T C^{r+1} H(t) \\ &+ Y_2(t) - Y_1(t)\Big] + Y_1(t). \end{aligned} \tag{4.5}$$

TABLE 3. Comparison of the absolute error $|u_5(x,t) - u(x,t)|$, for $\alpha = 1, \beta = 2$ at the 5th iteration, $k = 2, M = 8, J = 3$ in Example (5.1)

(x,t)	E_{HWCM}	E_{CWCM}	E_{MHPM}[30]
$(\frac{1}{32},\frac{1}{32})$	1.1842e-05	9.8797e-13	4.2154e-02
$(\frac{3}{32},\frac{3}{32})$	1.3220e-05	6.3487e-12	4.0041e-02
$(\frac{5}{32},\frac{5}{32})$	2.2711e-07	3.7582e-11	6.1784e-03
$(\frac{7}{32},\frac{7}{32})$	8.1053e-07	2.3422e-10	1.9223e-03
$(\frac{9}{32},\frac{9}{32})$	7.2534e-07	9.6780e-10	2.7205e-04
$(\frac{11}{32},\frac{11}{32})$	1.5311e-06	2.9476e-09	2.3709e-04
$(\frac{13}{32},\frac{13}{32})$	7.2014e-07	7.3681e-09	2.6057e-04
$(\frac{15}{32},\frac{15}{32})$	1.3386e-06	1.6114e-08	4.0174e-04
$(\frac{17}{32},\frac{17}{32})$	1.0314e-06	3.1917e-08	3.4774e-04
$(\frac{19}{32},\frac{19}{32})$	8.2611e-07	5.8314e-08	3.6823e-04
$(\frac{21}{32},\frac{21}{32})$	1.6654e-06	9.8903e-08	1.7064e-04
$(\frac{23}{32},\frac{23}{32})$	5.4611e-07	1.5538e-07	3.7981e-04
$(\frac{25}{32},\frac{25}{32})$	2.7450e-06	2.2322e-07	2.9751e-04
$(\frac{27}{32},\frac{27}{32})$	1.6417e-06	2.8391e-07	1.4021e-04
$(\frac{29}{32},\frac{29}{32})$	6.1593e-07	2.9268e-07	1.0691e-05
$(\frac{31}{32},\frac{31}{32})$	3.8078e-06	1.6170e-07	1.0037e-05

For simplicity, let

$$S(x,t) = f(x,t,u_r) = \sum_{l=1}^{m}\sum_{p=1}^{m} m_{lp} h_l(x) h_p(t) = H^T(x) M H(t), \tag{4.6}$$

where $m_{lp} = \langle h_l(x), \langle S(x,t), h_p(t)\rangle\rangle$. Substituting Eqs. (4.6),(4.3) in Eq. (4.1) we obtain

$$\frac{\partial^\alpha u_{r+1}}{\partial t^\alpha} = a(x) H^T(x) C^{r+1} H(t) + H^T(x) M H(t), \tag{4.7}$$

We apply fractional integral operator I_t^α to Eq. (4.7) and use the initial conditions to obtain

$$u_{r+1}(x,t) = a(x) H^T(x) C^{r+1} ({}^H P_t^\alpha) + H^T(x) M ({}^H P_t^\alpha) + g_1(x). \tag{4.8}$$

Let $K(x,t) = -g_1(x) + x(Y_2(t) - Y_1(t)) + Y_1(t)$. From Eqs. (4.8),(4.5)

$$\begin{aligned} &({}^H P_x^\beta)^T C^{r+1} H(t) - x\Big[({}^H P_x^\beta(1))^T C^{r+1} H(t)\Big] + K(x,t) \\ &= a(x) H^T(x) C^{r+1} ({}^H P_t^\alpha) + H^T(x) M ({}^H P_t^\alpha). \end{aligned} \tag{4.9}$$

TABLE 4. Absolute error of $|u_6(x,t) - u(x,t)|$, for different values of α, β when they tend to $1, 2$ respectively at the 6th iteration, $J = 4$ in Example (5.2), using HWCM

(x,t)	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 1$	$\alpha = 1$	$\alpha = 1$ method[30]
	$\beta = 1.7$	$\beta = 1.8$	$\beta = 1.95$	$\beta = 2$	$\beta = 2$
$(\frac{1}{64}, \frac{1}{64})$	3.6794e-03	1.9186e-03	2.6858e-04	6.5242e-06	4.1257e-02
$(\frac{5}{64}, \frac{5}{64})$	1.6448e-02	1.8094e-02	3.2705e-03	1.2905e-05	4.1007e-02
$(\frac{9}{64}, \frac{9}{64})$	2.9277e-02	2.6402e-02	5.2280e-03	4.1986e-06	3.0574e-02
$(\frac{13}{64}, \frac{13}{64})$	4.1427e-02	3.3969e-02	7.0957e-03	1.1843e-05	2.9743e-02
$(\frac{17}{64}, \frac{17}{64})$	5.2605e-02	4.0971e-02	8.9773e-03	1.2676e-05	2.7714e-02
$(\frac{21}{64}, \frac{21}{64})$	1.1756e-01	4.7314e-02	1.0664e-03	1.1320e-05	2.0468e-02
$(\frac{25}{64}, \frac{25}{64})$	1.2149e-01	5.3088e-02	1.2272e-03	9.6613e-06	1.9724e-02
$(\frac{29}{64}, \frac{29}{64})$	2.9076e-01	5.8305e-02	1.3684e-03	8.1439e-06	1.7752e-02
$(\frac{33}{64}, \frac{33}{64})$	2.4825e-01	8.1434e-02	9.1821e-03	6.7756e-06	1.4029e-02
$(\frac{37}{64}, \frac{37}{64})$	2.0626e-01	7.9743e-02	8.4079e-03	5.4982e-06	9.0051e-03
$(\frac{41}{64}, \frac{41}{64})$	1.6487e-02	7.7926e-02	7.6241e-03	4.2648e-06	7.8057e-03
$(\frac{45}{64}, \frac{45}{64})$	1.2416e-02	7.5991e-02	6.8316e-03	3.1086e-06	4.6721e-03
$(\frac{49}{64}, \frac{49}{64})$	8.4215e-03	7.3946e-02	6.0310e-03	2.3351e-06	2.2147e-03
$(\frac{53}{64}, \frac{53}{64})$	4.5107e-03	7.1799e-02	1.9388e-03	2.6056e-06	1.6858e-03
$(\frac{57}{64}, \frac{57}{64})$	5.0256e-03	3.2819e-03	1.1079e-03	3.1908e-06	9.6858e-04
$(\frac{61}{64}, \frac{61}{64})$	6.9186e-04	6.1649e-04	2.7489e-04	3.2617e-06	7.6858e-04

In discrete form, using Eq. (4.9) and collocation points , we have the matrix form

$$\begin{aligned} &({}^H P_x^{\beta})^T C^{r+1} H -{}^H V^{\beta,1,x} C^{r+1} H \\ &-AH^T C^{r+1}({}^H P_t^{\alpha}) - H^T M({}^H P_t^{\alpha}) + K = 0, \end{aligned} \tag{4.10}$$

where H is a $m \times m$ Haar matrix, ${}^H V^{\beta,1,x} = xI_1^{\beta} H^T$ is $m \times m$ fractional integration matrix for boundary value problem and ${}^H P_x^{\beta} = I_x^{\beta} H^T, {}^H P_t^{\alpha} = I_t^{\alpha} H$ are $m \times m$ matrices of fractional integration of the Haar wavelets. M is a $m \times m$ coefficients matrix determined by inner products $m_{lp} = \langle h_l(x), \langle S(x,t), h_p(t)\rangle\rangle$.

Let $Q := (AH^T)^{-1}$ is a $m \times m$ matrix, where A is a diagonal matrix and is given by

$$A = \begin{pmatrix} a(x(1)) & 0 & \dots & 0 \\ 0 & a(x(2)) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a(x(m)) \end{pmatrix}.$$

Table 5. Comparison of the u_{CWCM}(at the 5th iteration), $k = 2, M = 8, \alpha = 1$ and $\beta = 2$, the $u_{HWCM}(J = 3)$, the exact solution and the MHPM [30] approximate solution in Example (5.2)

(x,t)	u_{HWCM}	u_{CWCM}	u_{Exact}	E_{HWCM}	E_{CWCM}	E_{MHPM}[30]
$(\frac{1}{32},\frac{1}{32})$	1.0905e-04	5.7490e-05	5.7445e-05	5.1606e-05	4.5476e-08	3.6472e-04
$(\frac{3}{32},\frac{3}{32})$	1.4831e-03	1.3878e-03	1.3877e-03	9.5304e-05	1.4538e-08	1.5483e-03
$(\frac{5}{32},\frac{5}{32})$	5.7102e-03	5.8129e-03	5.8129e-03	1.0267e-04	1.0483e-08	1.2287e-03
$(\frac{7}{32},\frac{7}{32})$	1.4540e-02	1.4564e-02	1.4564e-02	2.3910e-05	4.4076e-09	1.0073e-03
$(\frac{9}{32},\frac{9}{32})$	2.8419e-02	2.8477e-02	2.8477e-02	5.7525e-05	8.4568e-09	4.5571e-03
$(\frac{11}{32},\frac{11}{32})$	4.8097e-02	4.8141e-02	4.8141e-02	4.3717e-05	6.3251e-09	7.1737e-03
$(\frac{13}{32},\frac{13}{32})$	7.3950e-02	7.3983e-02	7.3983e-02	3.2961e-05	1.8287e-08	7.0063e-03
$(\frac{15}{32},\frac{15}{32})$	1.0629e-01	1.0632e-01	1.0632e-01	3.1537e-05	5.2295e-08	4.2133e-03
$(\frac{17}{32},\frac{17}{32})$	1.4536e-01	1.4539e-01	1.4539e-01	2.8429e-05	4.9514e-07	2.8067e-04
$(\frac{19}{32},\frac{19}{32})$	1.9137e-01	1.9138e-01	1.9138e-01	1.2920e-05	1.6857e-07	5.2674e-04
$(\frac{21}{32},\frac{21}{32})$	2.4440e-01	2.4443e-01	2.4443e-01	3.0522e-05	3.6552e-08	5.1135e-04
$(\frac{23}{32},\frac{23}{32})$	3.0467e-01	3.0466e-01	3.0466e-01	4.6925e-06	8.0461e-09	2.3357e-04
$(\frac{25}{32},\frac{25}{32})$	3.7214e-01	3.7217e-01	3.7217e-01	2.7440e-05	7.9465e-09	1.9736e-05
$(\frac{27}{32},\frac{27}{32})$	4.4701e-01	4.4702e-01	4.4702e-01	4.1784e-06	8.3986e-09	1.6483e-05
$(\frac{29}{32},\frac{29}{32})$	5.2930e-01	5.2928e-01	5.2928e-01	2.8561e-05	3.4844e-08	1.0247e-05
$(\frac{31}{32},\frac{31}{32})$	6.1894e-01	6.1900e-01	6.1900e-01	5.2938e-05	1.3780e-07	9.7433e-06

So Eq (4.10) can be written as

$$\begin{aligned} &Q((^HP_x^{\beta})^T -^H V^{\beta,1,x})C^{r+1} \\ &-C^{r+1}(^HP_t^{\alpha})H^{-1} + Q(K - H^TM(^HP_t^{\alpha})) = 0, \end{aligned} \tag{4.11}$$

which is Sylvester equation. Solving Eq. (4.11) for C^{r+1}, and substituting in Eq. (4.4 or 4.8), we get solution u_{r+1} at the collocation points.
In particular, given an initial approximation $u_0(x,t)$, we get a linear fractional singular problem in $u_1(x,t)$ by substituting $r = 0$ in Eq. (4.1), which is solved by above procedure to get $u_1(x,t)$ at the collocation points.

4.2. **The second-kind Chebyshev wavelets method.** Similarly, we describe the method of solving the singular fractional nonlinear partial differential equations using the second-kind Chebyshev wavelets. The first step is converting the singular fractional nonlinear partial differential equations to a system of linear equations by the Picard technique. Then, we solve the obtained linear system by the second-kind Chebyshev wavelets operational matrix method. We apply the Picard iteration technique to Eq. (1.1):

$$\frac{\partial^\alpha u_{r+1}}{\partial t^\alpha} - a(x)\frac{\partial^\beta u_{r+1}}{\partial x^\beta} = f(x,t,u_r), \tag{4.12}$$
$$0 < \alpha \leq 1, \quad 1 < \beta \leq 2, \quad , r \geq 1$$

with the initial and boundary conditions:

$$u_{r+1}(x,0) = g_1(x), \quad 0 \leq x \leq 1,$$
$$u_{r+1}(0,t) = Y_1(t), \qquad u_{r+1}(1,t) = Y_2(t), \qquad t \in [0,1],$$

where $f(x,t,u_r) = f(x,t) - d(x)u_r^p$.

For applying the second-kind Chebyshev wavelets collocation method to Eq. (4.12), we suppose

$$\frac{\partial^\beta u_{r+1}}{\partial x^\beta} = \sum_{l=1}^{m'}\sum_{p=1}^{m'} c_{lp}^{r+1}\psi_l(x)\psi_p(t) = \Psi^T(x)C^{r+1}\Psi(t). \tag{4.13}$$

Applying the fractional integral operator I_x^β on Eq. (4.13), we get

$$u_{r+1}(x,t) = ({}^C P_x^2)^T C^{r+1}\Psi(t) + p(t)x + q(t), \tag{4.14}$$

where $p(t)$ and $q(t)$ are functions of t. Using the boundary conditions and Eqs. (4.14, 4.13), we get

$$q(t) = Y_1(t),$$
$$p(t) = -({}^C P^2(1))^T C^{r+1}\Psi(t) + Y_2(t) - Y_1(t).$$

Eq. (4.14) can be rewritten as:

$$\begin{aligned} u_{r+1}(x,t) &= ({}^C P^\beta)^T C^{r+1}H(t) - x\Big[({}^C P^\beta(1))^T C^{r+1}H(t) \\ &+ Y_2(t) - Y_1(t)\Big] + Y_1(t). \end{aligned} \tag{4.15}$$

For simplicity, let

$$S(x,t) = f(x,t,u_r) = \sum_{l=1}^{m'}\sum_{p=1}^{m'} m_{lp}\psi_l(x)\psi_p(t) = \Psi^T(x)M\Psi(t), \tag{4.16}$$

where $m_{lp} = \langle h_l(x), \langle S(x,t), h_p(t)\rangle\rangle$. Substituting Eqs. (4.16, 4.13) in Eq. (4.12) we obtain

$$\frac{\partial^\alpha u_{r+1}}{\partial t^\alpha} = a(x)\Psi^T(x)C^{r+1}\Psi(t) + \Psi^T(x)M\Psi(t). \tag{4.17}$$

Apply fractional integral operator I_t^α to Eq. (4.17) and use the initial conditions to obtain

$$u_{r+1}(x,t) = a(x)\Psi^T(x)C^{r+1}({}^C P_t^\alpha) + \Psi^T(x)M({}^C P_t^\alpha) + g_1(x). \tag{4.18}$$

Let $K(x,t) = -g_1(x) + x(Y_2(t) - Y_1(t)) + Y_1(t)$. From Eqs. (4.18, 4.15) we have

$$\begin{aligned} &({}^C P^\beta)^T C^{r+1}\Psi(t) - x\Big[({}^C P^\beta(1))^T C^{r+1}\Psi(t)\Big] + K(x,t) \\ &= a(x)\Psi^T(x)C^{r+1}({}^C P_t^\alpha) + \Psi^T(x)M({}^C P_t^\alpha). \end{aligned} \tag{4.19}$$

In discrete form, using Eq. (4.19) and collocation points, we have the matrix form

$$\begin{aligned}&({}^{C}P^{\beta})^{T}C^{r+1}\Psi - {}^{C}V^{\beta,1,f(x)}C^{r+1}\Psi - \\ &A\Psi^{T}C^{r+1}({}^{C}P_t^{\alpha}) - \Psi^{T}M({}^{C}P_t^{\alpha}) + K = 0,\end{aligned} \tag{4.20}$$

where Ψ is a $2^{k-1}M \times 2^{k-1}M$ second kind Chebyshev matrix, ${}^{C}V^{\beta,1,x} = xI^{\beta}(\Psi(1))^{T}$ is $2^{k-1}M \times 2^{k-1}M$ fractional integration matrix for boundary value problems and ${}^{C}P^{\beta} = I^{\beta}(\Psi), {}^{C}P^{\alpha} = I^{\alpha}(\Psi)$ are $2^{k-1}M \times 2^{k-1}M$ matrices of fractional integration of the second-kind Chebyshev wavelets. M is a $2^{k-1}M \times 2^{k-1}M$ coefficients matrix determined by inner products $m_{lp} = \langle h_l(x), \langle S(x,t), h_p(t)\rangle\rangle$. Let $Q := (A\Psi^{T})^{-1}$ is a $2^{k-1}M \times 2^{k-1}M$ matrix, where A is a diagonal matrix and is given by

$$A = \begin{pmatrix} a(x(1)) & 0 & \dots & 0 \\ 0 & a(x(2)) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a(x(2^{k-1}M)) \end{pmatrix}.$$

So we rewrite Eq (4.20) as

$$\begin{aligned}&Q(({}^{C}P^{\beta})^{T} - {}^{C}V^{\beta,1,f(x)})C^{r+1} - \\ &C^{r+1}({}^{C}P_t^{\alpha})\Psi^{-1} + Q(K - \Psi^{T}M({}^{C}P_t^{\alpha})) = 0,\end{aligned} \tag{4.21}$$

which is the Sylvester equation. Solving Eq. (4.21) for C^{r+1}, and substituting in Eqs. (4.14 or 4.18), we get the solution u_{r+1} at the collocation points.

In particular, given an initial approximation $u_0(x,t)$, we get a linear fractional singular problem in $u_1(x,t)$ by substituting $r = 0$ in Eq. (4.12), which is solved by above procedure to get $u_1(x,t)$ at the collocation points.

5. Experiments and results

In this section we use the HWCM(Haar Wavelets Collocation Method) and CWCM(Chebyshev Wavelets Collocation Method) for solving the nonlinear singular fractional differential equations. We provide two examples to illustrate the methods.

Example 5.1. Consider the singular fractional nonlinear partial differential equation:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} - (\frac{1}{sin(x)} + \frac{x}{3})\frac{\partial^{\beta}u}{\partial x^{\beta}} - u^2 = f(x,t) \tag{5.1}$$

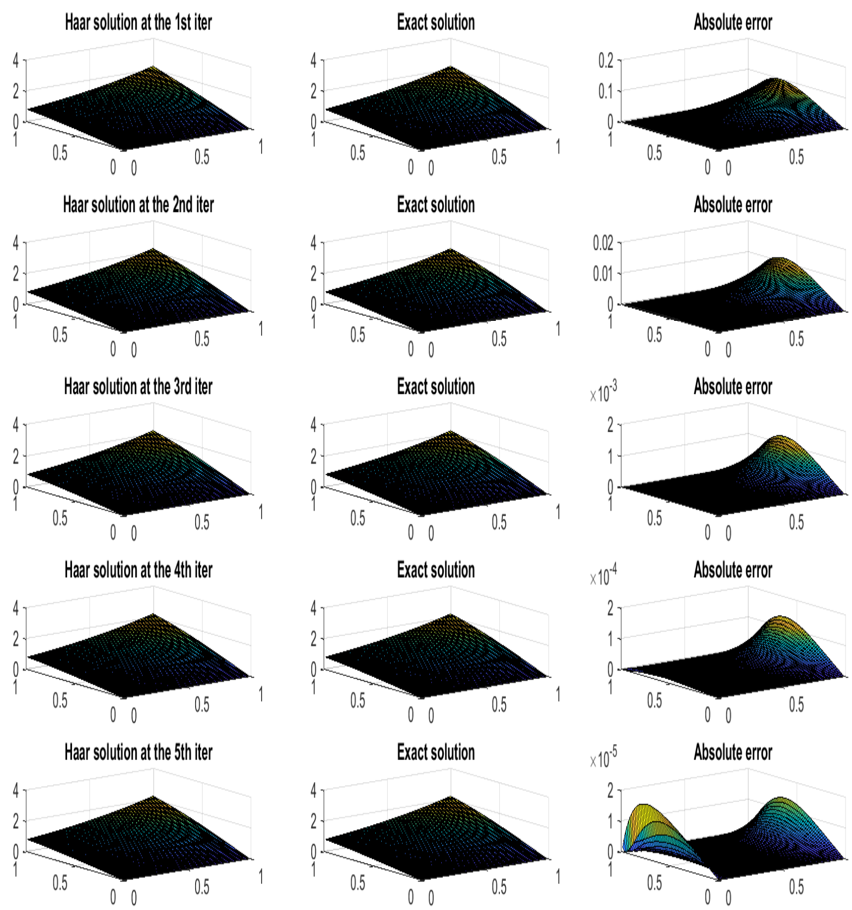
with initial and boundary condition:

$$\begin{aligned}&u(x,0) = sin(x), \quad u(0,t) = 0,\\ &u(1,t) = e^{t}sin(1), \quad 0 \le x, t \le 1.\end{aligned}$$

Exact solution for $\alpha = 1, \beta = 2$ is $u(x,t) = e^{t}sin(x)$ and $f(x,t) = \frac{1}{3}e^{t}(3-3e^{t}(sin(x))^{2} + (x+3)sin(x))$.

We use $u_0 = sin(x)$ as an initial approximation and apply our techniques to get the approximate solution of this singular problem and describe the procedure of implementation in more details, which enable the readers to understand the method more efficiently.

FIGURE 1. The Haar wavelets collocation approximate solutions, exact solution and absolute errors for $\alpha = 1, \beta = 2, J = 5$ in Example (5.1)



Step 1. Applying the Picard technique to Eq. (5.1), we get

$$\frac{\partial^{\alpha} u_{r+1}(x,t)}{\partial t^{\alpha}} - (\frac{1}{sin(x)} + \frac{x}{3})\frac{\partial^{\beta} u_{r+1}(x,t)}{\partial x^{\beta}} \\ -u_r^2(x,t) = \frac{1}{3}e^t(3 - 3e^t(sin(x))^2 + (x+3)sin(x)), \tag{5.2}$$

FIGURE 2. The Haar wavelets collocation approximate solution, exact solution and absolute error for $\alpha = 1, \beta = 2, J = 2$ in Example (5.1)

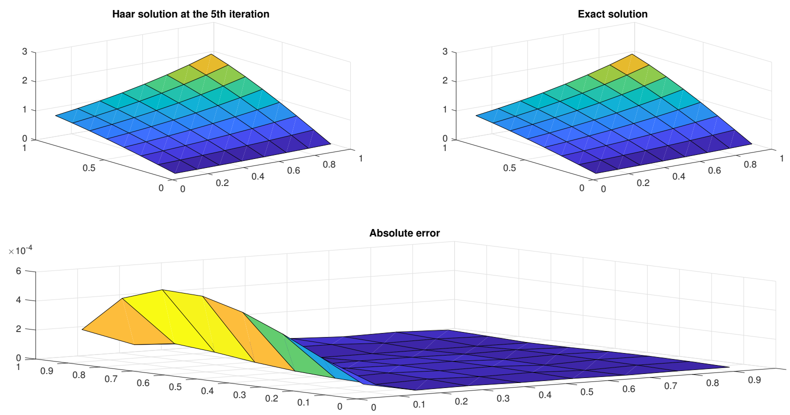
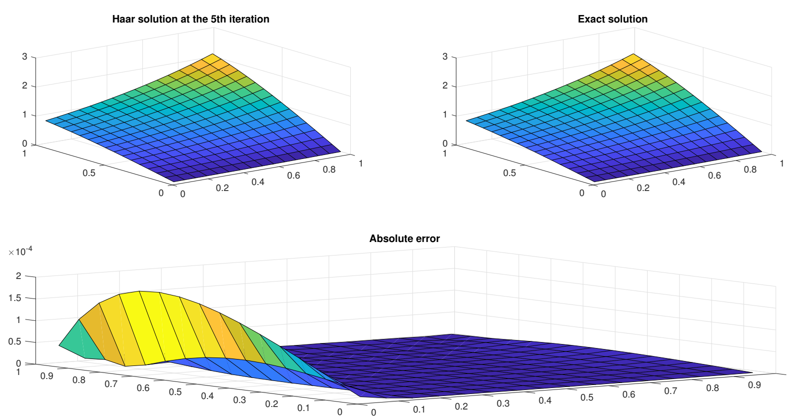


FIGURE 3. The Haar wavelets collocation approximate solution, exact solution and absolute error for $\alpha = 1, \beta = 2, J = 3$ in Example (5.1)



with initial and boundary conditions:

$$u_{r+1}(x, 0) = sin(x), u_{r+1}(0, t) = 0,$$
$$u_{r+1}(1, t) = e^t sin(1), \quad 0 \leq x, t \leq 1.$$

FIGURE 4. The Haar wavelets collocation approximate solution, exact solution and absolute error for $\alpha = 1, \beta = 2, J = 4$ in Example (5.1)

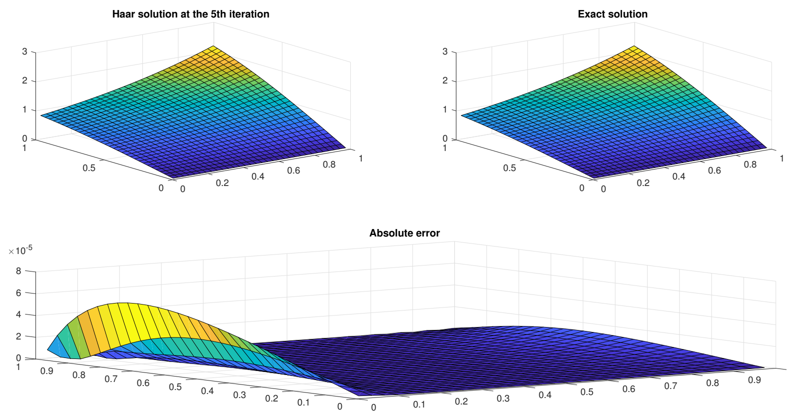
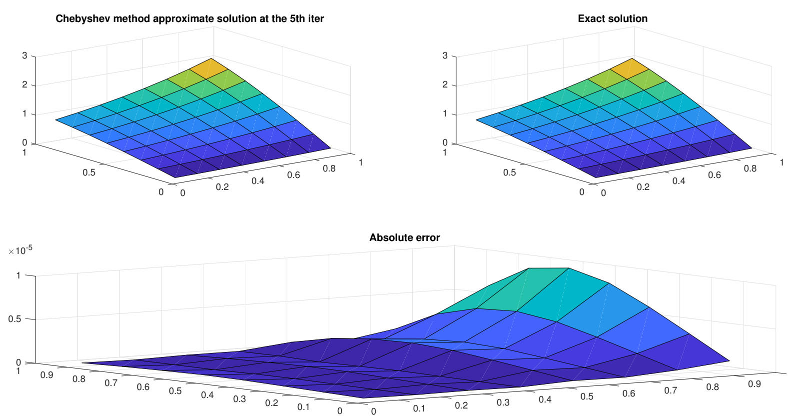


FIGURE 5. The second-kind Chebyshev wavelets approximate solution, exact solution and absolute error for $\alpha = 1, \beta = 2, k = 2, M = 4$ in Example (5.1)



Step 2. Applying the second-kind Chebyshev wavelets method to Eq. (5.2), we have

$$\frac{\partial^\beta u_{r+1}}{\partial x^\beta} = \sum_{l=1}^{m'} \sum_{p=1}^{m'} c_{lp}^{r+1} \psi_l(x)\psi_p(t) = \Psi^T(x) C^{r+1} \Psi(t), \tag{5.3}$$

Figure 6. The second-kind Chebyshev wavelets approximate solution, exact solution and absolute error for $\alpha = 1, \beta = 2, k = 2, M = 8$ in Example (5.1)

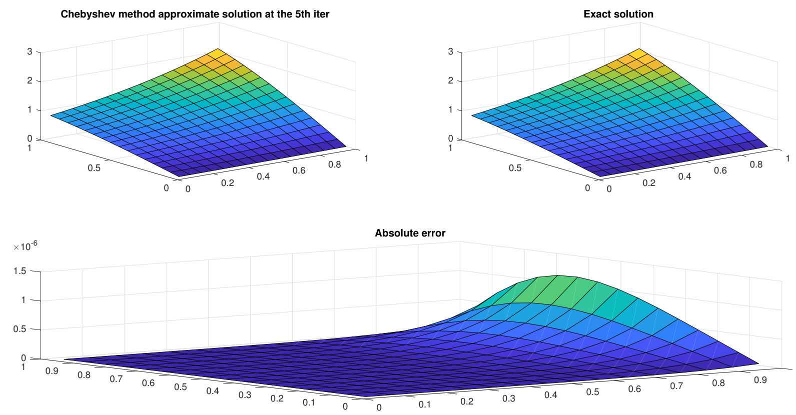
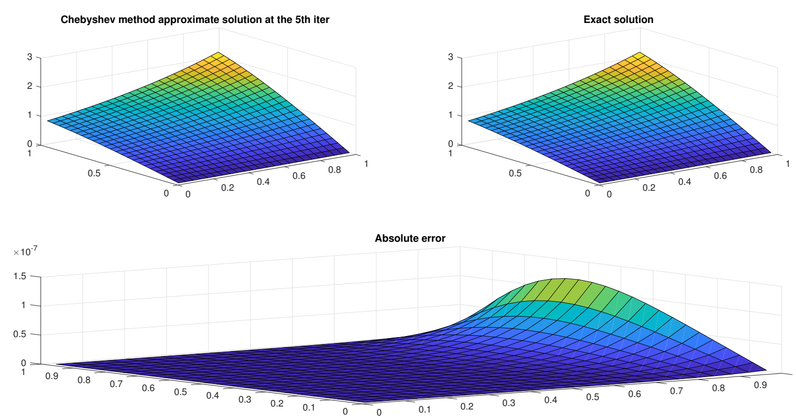


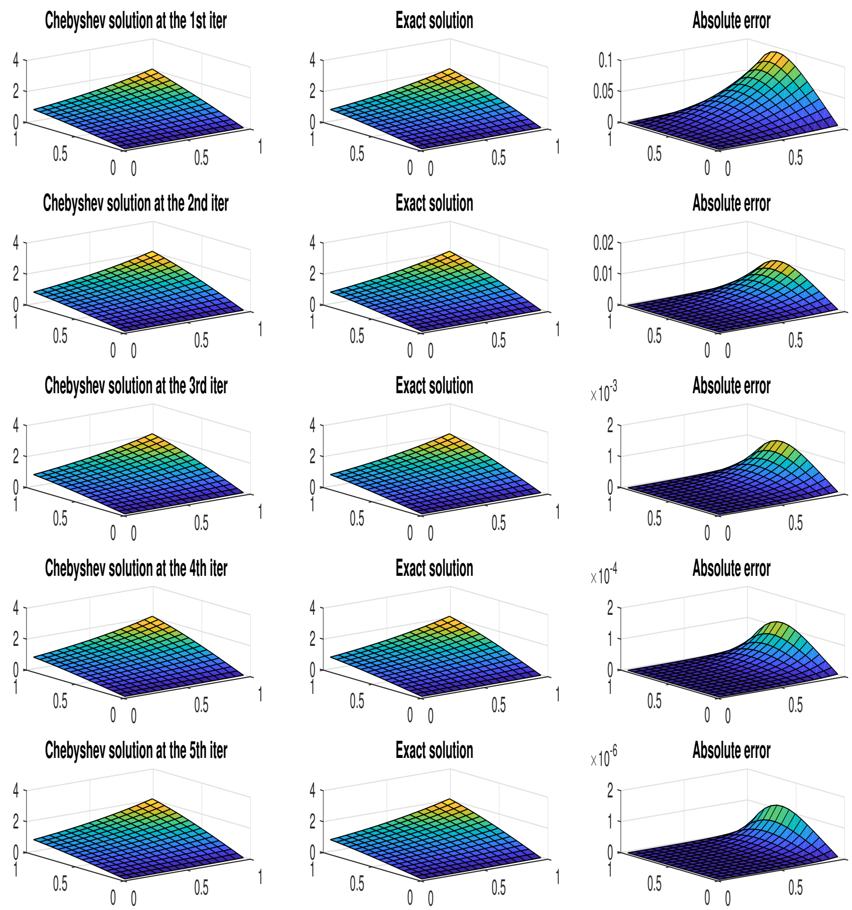
Figure 7. The second-kind Chebyshev wavelets approximate solution, exact solution and absolute errors for $\alpha = 1, \beta = 2, k = 2, M = 12$ in Example (5.1)



Applying the fractional integration operator I_x^{β} to Eq. (5.3) and using initial and boundary conditions, we have

$$u_{r+1}(x,t) = ({}^{C}P_x^{\beta})^T C^{r+1}\Psi(t) - x\Big[({}^{C}P^{\beta}(1))^T C^{r+1}\Psi(t) + e^t sin(1)\Big], \qquad (5.4)$$

FIGURE 8. The second-kind Chebyshev wavelets approximate solutions, exact solution and absolute errors for $\alpha = 1, \beta = 2, k = 2, M = 8$ in Example (5.1)



Rewriting Eq. (5.4)

$$u_{r+1}(x,t) = \left[({}^{C}P_x^{\beta})^T - x({}^{C}P^{\beta}(1))^T\right] C^{r+1}\Psi(t) + xe^t sin(1), \tag{5.5}$$

where $x({}^{C}P^{\beta}(1))$ is the fractional integration matrix for boundary value problems introduced in Eq. (2.13).

Step 3. We set $S(x,t) = u_r^2(x,t) + \frac{1}{3}e^t(3 - 3e^t(sin(x))^2 + (x+3)sin(x))$, so

$$S(x,t) = \sum_{l=1}^{m'}\sum_{p=1}^{m'} m_{lp}\psi_l(x)\psi_p(t) = \Psi^T(x)M\Psi(t), \tag{5.6}$$

where $m_{lp} = \langle h_l(x), \langle S(x,t), h_p(t)\rangle\rangle$. Substituting Eqs. (5.6),(5.3) in Eq. (5.2), we have

$$\frac{\partial^\alpha u_{r+1}(x,t)}{\partial t^\alpha} = (\frac{1}{sin(x)} + \frac{x}{3})\Big(\Psi^T(x)C^{r+1}\Psi(t)\Big) + \Psi^T(x)M\Psi(t). \tag{5.7}$$

Step 4. Applying the fractional integral operator I_t^α to Eq. (5.7), and using $u_{r+1}(x,0) = sin(x)$, we have

$$u_{r+1}(x,t) = (\frac{1}{sin(x)} + \frac{x}{3})\Big(\Psi^T(x)C^{r+1}(^CP_t^\alpha)\Big) + \Psi^T(x)M(^CP_t^\alpha) + sin(x). \tag{5.8}$$

For simplicity, let

$$A(x) = (\frac{1}{sin(x)} + \frac{x}{3}), K(x,t) = sin(x) - xe^t sin(1).$$

Using Eqs. (5.8),(5.5), we get

$$\begin{aligned}&\Big[(^CP_x^\beta)^T - x(^CP^\beta(1))^T\Big]C^{r+1}\Psi(t) - \\ &A(x)\Big[\Psi^T(x)C^{r+1}(^CP_t^\alpha)\Big] = \Psi^T(x)M(^CP_t^\alpha) + K(x,t).\end{aligned} \tag{5.9}$$

In discrete form, from Eq. (5.9) and using collocation points, we have the matrix form

$$\Big[(^CP_x^\beta)^T - (^CV^{\beta,1,x})^T\Big]C^{r+1}\Psi - A\Big[\Psi^T C^{r+1}(^CP_t^\alpha)\Big] = \Psi^T M(^CP_t^\alpha) + K. \tag{5.10}$$

Step 5. Let $Q := (A\Psi^T)^{-1}$, and some calculations we get

$$\begin{aligned}&Q\Big[(^CP_x^\beta)^T - (^CV^{\beta,1,x})^T\Big]C^{r+1} - \\ &C^{r+1}(^CP_t^\alpha)(\Psi)^{-1} = Q\Big[\Psi^T M(^CP_t^\alpha) + K\Big](\Psi)^{-1},\end{aligned} \tag{5.11}$$

which is the Sylvester equation.

Step 6. Solving Eq. (5.11), we get the coefficients C^{r+1}. Replacing $r = 0$ and using $u_0(x,t) = sin(x)$ in **Step 3**, solving Eq. (5.11), we get C^1 coefficients and substituting these coefficients in Eqs. (5.5 or 5.8), we obtain the approximate solution $u_1(x,t)$. By doing this process, replacing $r = 1$ and using $u_1(x,t)$ in **Step 3**, solving Eq. (5.11), we obtain u_2 and so on.

The approximate solutions from the HWCM and CWCM and the Picard technique were plotted at $\alpha = 1, \beta = 2, J = 4, k = 2$ and $M = 8$ and the solutions at different iterations were compared with the exact solution (Figures (1,8)). This revealed that the absolute error decreased as the number of iterations increased. Furthermore, Figures (2, 3, 4, 5, 6 and 7) compare the exact and approximate solutions for Example (5.1) at the 5th iteration for different values of J and M and showed that the absolute errors decreased as J for HWCM and M for CWCM increased. Tables (1,2) show the different values of α, β. When $\alpha = 1, \beta = 2$, the absolute errors showed that the solution of the proposed method converged to the exact solution. Table (3) compares the absolute error between the HWCM and CWCM and the MHPM[30]. As seen, the absolute errors of the CWCM and HWCM in comparison with the MHPM demonstrate the high accuracy and efficiency of the proposed methods.

Example 5.2. Consider the following nonhomogeneous singular partial differential equation of fractional order

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - (\frac{1}{x} + sin(x))\frac{\partial^{\beta} u}{\partial x^{\beta}} - \frac{1}{x^2}u^2 = f(x,t). \tag{5.12}$$

with initial and boundary condition:

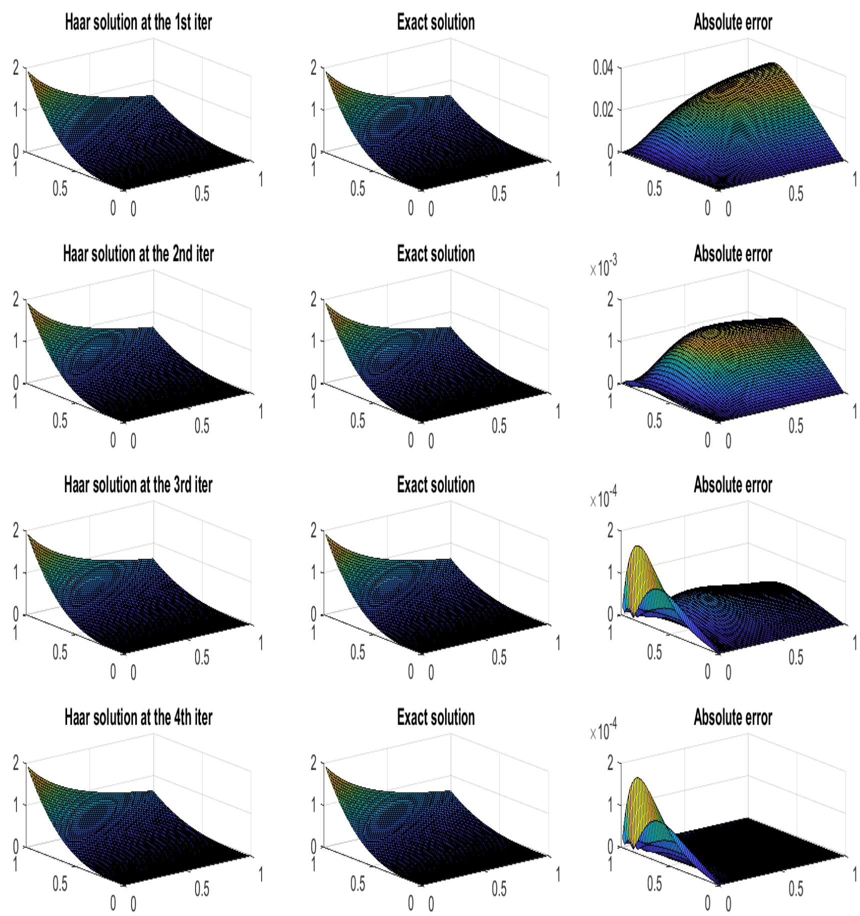
$$u(x,0) = 2x^3, \quad u(0,t) = 0, \quad u(1,t) = \frac{2}{1+2t}, \quad 0 \leq x, t \leq 1. \tag{5.13}$$

Exact solution for $\alpha = 1, \beta = 2$ is $u(x,t) = \frac{2x^3}{1+2t}$, and

$$f(x,t) = \frac{-12(2t+1)xsin(x) - 4x^4 - 4x^3 - 24t - 12}{(1+2t)^2}.$$

The initial approximation of $u_0(x,t) = 2x^3$ was used when applying the proposed methods to arrive at an approximate solution to this singular problem. The solutions were plotted using the HWCM and CWCM and the Picard technique subject to $\alpha = 1, \beta = 2, J = 5, k = 2, M = 8$. The solutions obtained at different iterations were compared with the exact solution, as shown in Figures (1,8). It was revealed that absolute errors decreased as the number of iterations increased. Moreover, comparison of the Haar and Chebyshev Picard methods and MHPM [30] solutions with the exact solution by computation of the absolute errors of these methods clearly revealed that the proposed techniques were much more efficient and accurate with equal collocation points than the MHPM [30] as shown in Tables (4,5). Table (5) compares the approximate solutions obtained from the HWCM and CWCM with the exact solution and the approximate solution obtained from the MHPM [30]. As seen, the solutions obtained from the wavelets methods(HWCM and CWCM) are more accurate that of the MHPM [30].

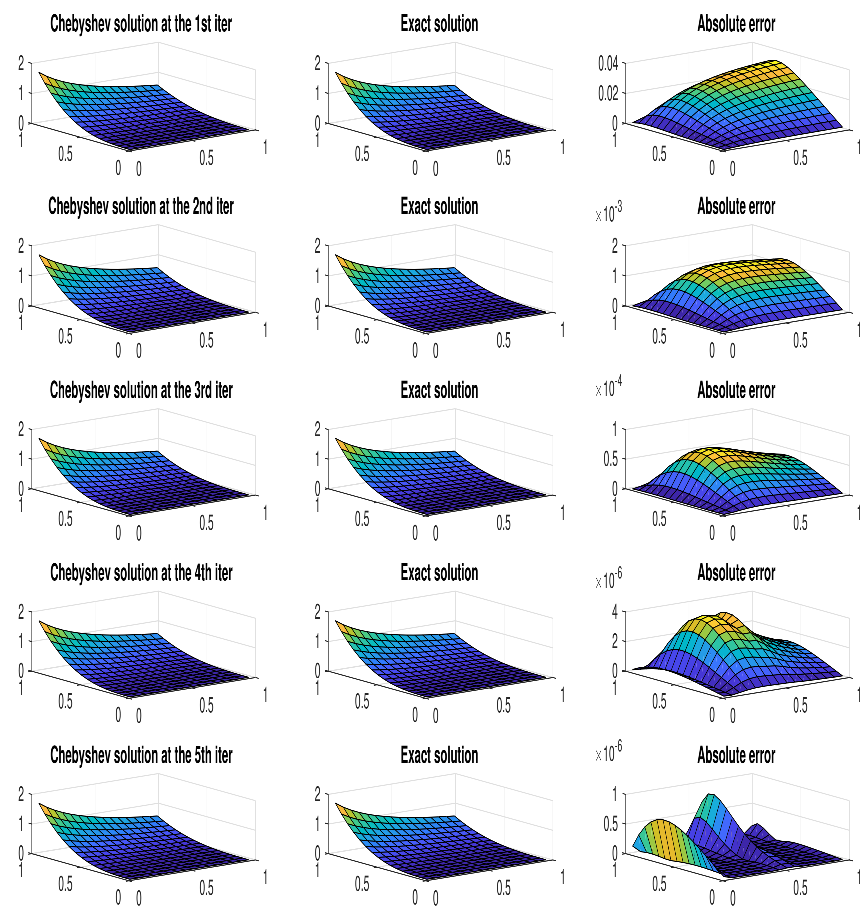
FIGURE 9. The Haar wavelets collocation approximate solutions, exact solution and absolute errors for $\alpha = 1, \beta = 2, J = 5$ in Example (5.2) at various iterations



6. CONCLUSION

The Haar and Chebyshev wavelet collocation methods have been employed to solve singular nonlinear fractional differential equations, which are used to model various types of problems in fluid dynamic and mathematical physics. The solution is substantiated using illustrative examples and the numerical solutions are presented in the tables and figures. It was observed that the Haar Wavelets Collocation Method

FIGURE 10. The second-kind Chebyshev wavelets collocation approximate solutions, exact solution and absolute errors for $\alpha = 1, \beta = 2, k = 2, M = 8$ in Example (5.2) at various iterations



(HWCM) and Chebyshev Wavelets Collocation Method (CWCM) provide more accurate approximate solutions in comparison with the exact solution than does the Modified Homotopy Perturbation Method (MHPM). It is clear that these methods are more accurate when computing approximate solutions and the results show the efficiency of the proposed method. This is the first time that these singular fractional equations have been solved by the HWCM and CWCM and the Picard technique.

The advantages of the present methods over other methods are as follows:

(1) Instead of an operational derivative, an operational integral matrix was used.
(2) The boundary and initial conditions were derived automatically. By using the derivative matrix, we usually have to choose the base to satisfy the initial conditions, but we do not have this restriction for the integral operator.
(3) Instead of approximating the integral operation with the use of black-pulse functions, the fractional integral operation was directly obtained for better approximation.
(4) By using the wavelet bases and transforming the nonlinear and singular problem into a Sylvester equation, good results were obtained with little calculation and few iterations.
(5) The main advantage of these methods is the conversion of singular nonlinear PDEs to a system of algebraic equations. These equations can be easily solved by a computer.

References

[1] A. Arikoglu and I. Ozkol, *Solution of fractional differential equations by using differential transform method*, Chaos, Solitons Fract, *34*(5) (2007), 1473-1481.
[2] R. L. Bagley and P.J. Torvik, *A theoretical basis for the application of fractional calculus to viscoelasticity*, J Rheol, *27*(3) (1983), 201-210.
[3] R. T. Baillie, *Long memory processes and fractional integration in econometrics*, Econometrics, *73*(1) (1996), 55-59.
[4] R. E. Bellman and R. E. Kalaba, *Quasilinearization and nonlinear boundary value problems*, American Elsevier Publishing Company, 1965.
[5] G. W. Bohannan, *Analog fractional order controller in temperature and motor control applications*, J. Vib. Control, *14*(9) (2008), 1487-1498.
[6] C. Cattani, *Shannon Wavelets Theory*, Mathematical Problems in Engineering, *2008*, 1-24, DOI:10.1155/2008/164808.
[7] Z. Chen and Q. Gou, *Piecewise Picard iteration method for solving nonlinear fractional differential equation with proportional delays*, Applied Mathematics and Computation, *348*(C)(2019), 465-478.
[8] C. F. Chen and C. H. Hsiao, *Haar wavelet method for solving lumped and distributed-parameter systems*, IEE Proc. Control Theory Applic, *144*(1) (1997).
[9] Y. M. Chen, Y. B. Wu, and et al, *Wavelet method for a class of fractional convection-diffusion equation with variable coefficients*, J. Comput. Sci, *1*(3) (2010), 146-149.
[10] F. C. Cirstea, *Nonlinear methods in the study of singular partial differential equations (Ph.D. thesis)*, Victoria University of Technology, Victoria, Australia, 2004.
[11] N. Dalir, *Improvement of the Modified Decomposition Method for Handling Third-Order Singular Nonlinear Partial Differential Equations with Applications in Physics*, International Journal of Partial Differential Equations, *2014*(1), 1-6.
[12] S. Das, *Analytical solution of a fractional diffusion equation by variational iteration method*, Comput. Math. Appl, *57*(3) (2009), 483-487.
[13] A. A. Elbeleze, A. Kilicman, and B. M. Taib, *Numerical Methods for Fractional Order Singular Partial Differential Equations with Variable Coefficients*, Mathematical Problems in Engineering, *2014*(1), 1-8.
[14] S. A. El-Wakil, A. Elhanbaly, and M. Abdou, *Adomian decomposition method for solving fractional nonlinear differential equations*, Appl. Math. Comput, *182*(1)(2006), 313-324.
[15] R. Garra, *Analytic solution of a class of fractional differential equations with variable coefficients by operatorial methods*, Commun. Nonlinear Sci. Numer. Simul, *17*(4) (2012), 1549-1554.

[16] I. Hashim, O. Abdulaziz, and S. Momani, *Homotopy analysis method for fractional ivps*, Commun. Nonlinear Sci. Numer. Simul, *14*(3) (2009), 674-684.
[17] J. H. He, *Nonlinear oscillation with fractional derivative and its applications*, International conference on vibrating engineering, China: Dalian, *98*(1) (1998), 288-291.
[18] J. H. He, *Some applications of nonlinear fractional differential equations and their approximations*, Bull Sci Technol, *15*(2)(1999), 86-90.
[19] M. H. Heydari, M. R. Hooshmandasl, F. M. Maalek Ghaini, and F. Fereidouni, *Two-dimensional Legendre wavelets for solving fractional poisson equation with dirichlet boundary conditions*, Engineering Analysis with Boundary Elements, *37*(11)(2013), 1331-1338.
[20] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, 2006.
[21] Y. L. Li and N. Sun, *Numerical solution of fractional differential equation using the generalized block puls operational matrix*, Comput. Math. Appl, *62*(3)(2011), 1046-1054.
[22] Y. Li and W. Zhao, *Haar wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations*, Appl. Math. Comput, *216*(8) (2010), 2276-2285.
[23] A. Mainardi, *Fractals and fractional calculus in continuum mechanics*. New York: Springer, Verlag, *378*(1) (1997), 291-348.
[24] M. Meerschaert and C. Tadjeran, *Finite difference approximations for two-sided space-fractional partial differential equations*, Appl. Numer. Math, *56*(1) (2006), 80-90.
[25] Z. Odibat and S. Momani, *Numerical methods for nonlinear partial differential equations of fractional order*, Appl. Math. Model, *32*(1) (2008), 28-39.
[26] Z. Odibat and N. Shawagfeh, *Generalized taylor's formula*, Appl. Math. Comput, *186*(1) (2007), 286-293.
[27] R. Panda and M. Dash, *Fractional generalized splines and signal processing*, Signal Process, *86*(9) (2006), 2340-2350.
[28] U. Saeed and M. Rehman, *Haar wavelet Picard method for fractional nonlinear partial differential equations*, Applied Mathematics and Computation, *264*(1) (2015), 310-322.
[29] S. Saha Ray and A. Patra, *Haar wavelet operational methods for the numerical solutions of fractional order nonlinear oscillatory Van der Pol system*, Applied Mathematics and Computation, *220*(1) (2013), 659-667.
[30] R. Singh, S. Singh, and A. M. Wazwaz, *A modied homotopy perturbation method for singular time dependent Emden-Fowler equations with boundary conditions*, J Math Chem, *54*(2) (2016), 918-931.
[31] N. Sweilam, M. Khader, and R. Al-Bar, *Numerical studies for a multi-order fractional differential equation*, Phys. Lett. A, *371*(1-2)(2007), 26-33.
[32] A. M. Wazwaz, *Adomian decomposition method for a reliable treatment of the Emden-Fowler equation*, Applied Mathematics and Computation, *161*(2)(2005), 543-560.
[33] A. M. Wazwaz, *Partial Differential Equations and Solitary Wave Theory*, Springer, New York, NY, USA, 2009.
[34] F. Zhou and X. Xu, *Numerical Solution of Time-Fractional Diffusion-Wave Equations via Chebyshev Wavelets Collocation Method*, Advances in Mathematical Physics, *2017*(1), 1-17, DOI:/10.1155/2017/2610804.
[35] F. Zhou and X. Xu, *Numerical solution of the convection diffusion equations by the second kind Chebyshev wavelets*, Applied Mathematics and Computation, *247*(3) (2014), 353-367.