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Haar wavelet iteration method for solving time fractional Fisher's equation

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Abstract In this work, we investigate a fractional version of the Fisher equation and solve it by using an efficient iteration technique based on the Haar wavelet operational matrices. In fact, we convert the nonlinear equation into a Sylvester equation by the Haar wavelet collocation iteration method (HWCIM) to obtain the solution. We provide four numerical examples to illustrate the simplicity and efficiency of the technique.

Keywords. Fractional differential equation, Haar wavelet, Operational matrices, Numerical solution, iterative technique, Sylvester equation.

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1. Introduction

Fractional calculation has different applications in physics and engineering such electro-magnetics, acoustics, viscoelasticity, electrochemistry and material science (see for example [1, 13, 18, 23]). In 1937 Fisher , Kolmogorov, Petrovsky and Piscounov [5, 10] investigated independently the Fisher-KPP equation, after then, it is extensively known as the Fisher's equation. Fisher's equation belong to the class of reaction-diffusion equation, it is one of the simplest nonlinear r.d.e. Fisher proposed this equation in his article, "The wave of advance of advantageous genes" in 1937, which is about of population dynamics to describe the spatial spread of an advantageous allele and explored its trowelling wave solutions.

Distinct numerical methods have been applied for approximate solutions of Fisher's equation including the Adomian decomposition method (ADM) [9], the homotopy perturbation method (HPM) [24], the variational iteration method (VIM) [15], the Haar

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wavelet method [3, 4, 7, 8, 11, 17, 20] and the tanh method [21]. As you know the time fractional Fisher's equation, which is a mathematical model for a wide range of important physical phenomena, is a partial differential equation obtained from the classical Fisher's equation by replacing the time derivative with a fractional derivative of order $0 < \alpha \leq 1$. In fact, the Fisher's equation as a model for the propagation of a mutant gene is encountered in chemical kinetics and population dynamics which includes problems such as nonlinear evolution of a population in a one-dimensional habitat, neutron population in a nuclear reaction. Moreover, the same equation occurs in logistic population growth models, flame propagation, neurophysiology, autocatalytic chemical reactions and branching Brownian motion processes [14, 16]. Due to the occurrence of Fisher's equation in many biological and chemical processes, it is one of the most important classes of nonlinear equations [6].

There are different kinds of wavelets. The simplest orthogonal wavelet with compact support is the Haar wavelet. Since the derivatives do not exist in the points of discontinuity, it is not possible to apply the Haar wavelets for solving differential equations directly. We use Haar wavelet based on the collocation method. In this paper, we use a combination of Haar wavelet and iteration technique for numerical solutions of the modified Fisher's equation with time-fractional derivative of the form

$$\begin{aligned} &D_t^{\alpha} u(x,t) = D_{xx} u(x,t) + \lambda u(x,t)(1 - u^n(x,t)) + q(x,t), \\ &(x,t) \in [0,1] \times [0,1], u(x,0) = f(x), u(0,t) = y_1(t), u(1,t) = y_2(t), \end{aligned} \tag{1.1}$$

where $0 < \alpha \leq 1$, λ is real parameter, D_t^{α} denotes the Caputo fractional derivative in time and $y_1(t)$, $y_2(t)$, $f(x)$ and $q(x,t)$ are known functions. We discretize the nonlinear fractional partial differential equation by iteration technique (Picard iteration) and then convert the obtained discritized equation into a Sylvester equation by the Haar wavelet collocation method to obtain the solution.

2. Operational matrices

2.1. **Integer and fractional integration of operational matrix.** The i-th uniform Haar wavelet $h_i(x)$ is defined by

$$h_i(x) = \begin{cases} 1 & a(i) \leq x < b(i) \\ -1 & b(i) \leq x < c(i) \\ 0 & otherwise, \end{cases} \tag{2.1}$$

where $a(i) = \frac{k-1}{m}$, $b(i) = \frac{k-0.5}{m}$, $c(i) = \frac{k}{m}$, $i = 2^j + k + 1$, $j = 0,1,2,3,\ldots,J$, $x \in [0,1)$, js are dilation parameters, $m = 2^j$ and $k = 0,1,2,\cdots,2^j - 1$ are translation parameter. The Maximum level of resolution is J. In particular $h_1(x) = \chi_{[0,1)}(x)$ is the Haar scaling function, where $\chi_{[0,1)}(x)$ is characteristic function on interval $[0,1)$. Let us define the collocation points $x_l = \frac{l-0.5}{m}$ where $l = 1,2,3,\ldots,m$. We establish an operational matrix for integration via Haar wavelet. The operational matrix of integration of integer order is obtained by integrating Eq. (2.1) as following

$$P_{\alpha,1}(x) = I^{\alpha}_{a(1)} h_1(x) = \frac{1}{\Gamma(\alpha)} \int_{a(1)}^{x} (x-s)^{\alpha-1} ds, \qquad \alpha > 0$$

$$P_{\alpha,i}(x) = I^{\alpha}_{a} h_i(x) = \tag{2.2}$$

$$\frac{1}{\Gamma(\alpha)} \begin{cases} \int_{a(i)}^{x} (x-s)^{\alpha-1} ds & a(i) \leq x < b(i), \\ \int_{a(i)}^{b(i)} (x-s)^{\alpha-1} ds - \int_{b(i)}^{x} (x-s)^{\alpha-1} ds & b(i) \leq x < c(i), \\ \int_{a(i)}^{b(i)} (x-s)^{\alpha-1} ds - \int_{b(i)}^{c(i)} (x-s)^{\alpha-1} ds & x \geq c(i). \end{cases}$$

By simplifying, we get

$$P_{\alpha,1}(x) = \frac{(x-a(1))^{\alpha}}{\Gamma(\alpha+1)} \tag{2.3}$$

and

$$\begin{aligned} P_{\alpha,i}(x) &= I^{\alpha}_{a} h_i(x) \\ &= \frac{1}{\Gamma(\alpha)} \begin{cases} (x-a(i))^{\alpha}, & a(i) \leq x < b(i); \\ (x-a(i))^{\alpha} - 2(x-b(i))^{\alpha}, & b(i) \leq x < c(i); \\ (x-a(i))^{\alpha} - 2(x-b(i))^{\alpha} + (x-c(i))^{\alpha}, & x \geq c(i). \end{cases} \end{aligned} \tag{2.4}$$

Each function $y \in \mathcal{L}_2[0,1]$ can be expressed in terms of the Haar wavelet by:

$$y(x) = \sum_{i=1}^{\infty} c_i h_i(x),$$

where c_is are the Haar wavelet coefficients given by $c_i = \int_0^1 y(x) h_i(x) dx$. We can approximate the function $y(x)$ by the truncated series:

$$y(x) \approx \sum_{i=1}^{m-1} c_i h_i(x). \tag{2.5}$$

By taking the collocation points $x(l) = \frac{l-0.5}{m}$ for $l = 1, 2, \ldots, m$, we define Haar wavelet matrix $H_{m\times m}$ by

$$H_{m\times m} = \begin{pmatrix} h_1(x(1)) & h_1(x(2)) & \cdots & h_1(x(m)) \\ h_2(x(1)) & h_2(x(2)) & \cdots & h_2(x(m)) \\ \vdots & \vdots & \ddots & \vdots \\ h_m(x(1)) & h_m(x(2)) & \cdots & h_m(x(m)) \end{pmatrix}.$$

We can represent Eq. (2.5) in vector form as $y = cH$, where $c = [c_1, c_2, \ldots, c_m]$. The Haar coefficients c_i can be evaluated by $c = yH^{-1}$ where H^{-1} is inverse of H. Similarly, we can obtain the fractional order integration operational matrix P of Haar function by substituting the collocation points in Eq. (2.3) and Eq. (2.4) as following:

$$P^{\alpha}_{m\times m} = \begin{pmatrix} P_{\alpha,1}(x(1)) & P_{\alpha,1}(x(2)) & \cdots & p_{\alpha,1}(x(m)) \\ P_{\alpha,2}(x(1)) & p_{\alpha,2}(x(2)) & \cdots & P_{\alpha,2}(x(m)) \\ \vdots & \vdots & \ddots & \vdots \\ p_{\alpha,m}(x(1)) & p_{\alpha,m}(x(2)) & \cdots & p_{\alpha,m}(x(m)) \end{pmatrix}.$$

For example, if $m = 8$ and $\alpha = 0.9$, the Haar wavelet operational matrix of fractional integration is given by:

$$P^{0.9}_{8\times 8} = \begin{pmatrix} 0.0857 & 0.2305 & 0.3650 & 0.4941 & 0.6195 & 0.7421 & 0.8625 & 0.9811 \\ 0.0857 & 0.2305 & 0.3650 & 0.4941 & 0.4480 & 0.2812 & 0.1325 & -0.0071 \\ 0.0857 & 0.2305 & 0.1935 & 0.0331 & -0.0248 & -0.156 & -0.115 & -0.0091 \\ 0 & 0 & 0 & 0 & 0.0857 & 0.2305 & 0.1935 & 0.0331 \\ 0.0857 & 0.0590 & -0.0102 & -0.0054 & -.0037 & -0.0028 & -0.0022 & -0.0018 \\ 0 & 0 & 0.0857 & 0.0590 & -0.0102 & -0.0054 & -0.0037 & -0.0028 \\ 0 & 0 & 0 & 0 & 0.0857 & 0.0590 & -0.0102 & -0.0054 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0857 & 0.0590 \end{pmatrix}.$$

2.2. **Haar wavelet operational matrix of fractional integration for boundary value problems.** We derive another operational matrix of fractional integration to solve the fractional boundary value problems. Let $\eta > 0$, $g : [0, \eta] \to \mathbb{R}$ be a continuous function and $[0, \eta]$ be the Haar function support. Put

$$\begin{aligned} g(x) I^{\alpha}_0 h_1(\eta) &= g(x) \int_0^{\eta} (\eta - s)^{\alpha - 1} ds, \\ v^{\alpha,\eta,1} &= g(x) C_{\alpha,1} \end{aligned}$$

and

$$\begin{aligned} g(x) I^{\alpha}_0 h_i(\eta) &= g(x) \left\{ \int_{a(i)}^{b(i)} (\eta - s)^{\alpha-1} ds - \int_{b(i)}^{c(i)} (\eta - s)^{\alpha-1} ds \right\}, \\ v^{\alpha,\eta,i} &= g(x) C_{\alpha,i}, \end{aligned}$$

where $C_{\alpha,1} = \frac{\eta^{\alpha}}{\Gamma(\alpha+1)}$ and $C_{\alpha,i} = \frac{1}{\Gamma(\alpha+1)} \left[(\eta - a(i))^{\alpha} - 2(\eta - b(i))^{\alpha} + (\eta - c(i))^{\alpha} \right]$. By using the introduced collocation points, we obtain

$$V^{\alpha,\eta,g(x)}_{m\times m} = \begin{pmatrix} g(x(1)) I^{\alpha}_0 h_1(\eta) & g(x(2)) I^{\alpha}_0 h_1(\eta) & \cdots & g(x(m)) I^{\alpha}_0 h_1(\eta) \\ g(x(1)) I^{\alpha}_0 h_2(\eta) & g(x(2)) I^{\alpha}_0 h_2(\eta) & \cdots & g(x(m)) I^{\alpha}_0 h_2(\eta) \\ \vdots & \vdots & \ddots & \vdots \\ g(x(1)) I^{\alpha}_0 h_m(\eta) & g(x(2)) I^{\alpha}_0 h_m(\eta) & \cdots & g(x(m)) I^{\alpha}_0 h_m(\eta) \end{pmatrix}.$$

3. Method of solution

In this section, we describe the solving method of the time fractional modified Fisher's equation. The first step is to convert the fractional nonlinear partial differential equation to a linear partial differential equation by using the iteration technique. The second step is to solve the obtained discretized fractional partial differential equation by Haar wavelet operational matrix method.

Consider the problem:

$$\begin{aligned} &D^{\alpha}_t u(x,t) = D_{xx} u(x,t) + \lambda u(x,t)(1 - u^n(x,t)) + q(x,t), \\ &0 < \alpha \leq 1, \ \lambda \in \mathbb{R}, \ n \geq 1, \end{aligned} \tag{3.1}$$

with the initial and boundary conditions:

$$u(x,0) = g(x), \qquad u(0,t) = y_1(t), \qquad u(1,t) = y_2(t) \ \ (t \geq 0, \ 0 \leq x \leq 1).$$

By applying the iteration method of Picard to Eq. (3.1) (see [2]), we get the following problem:

$$\frac{\partial^\alpha u_{r+1}}{\partial t^\alpha} - \frac{\partial^2 u_{r+1}}{\partial x^2} = \lambda u_r(x,t)\left(1-u_r(x,t)\right) + q(x,t), (0<\alpha\le 1,\ r\ge 0) \tag{3.2}$$

with the initial and boundary conditions:

$$u_{r+1}(x,0)=g(x), u_{r+1}(0,t)=y_1(t), u_{r+1}(1,t)=y_2(t), (t\ge 0,\ 0\le x\le 1).$$

For applying the Haar wavelet method to Eq. (3.2), we suppose that

$$\frac{\partial^2 u_{r+1}}{\partial x^2} = \sum_{i=1}^{m}\sum_{j=1}^{m} C_{i,j}^{r+1} h_i(x)h_i(t) = H^T(x)C^{r+1}H(t). \tag{3.3}$$

By applying the fractional integral operator I_x^2 on the Eq. (3.3), we obtain:

$$u_{r+1}(x,t) = (P_x^2)^T C^{r+1}H(t) + p(t)x + q(t). \tag{3.4}$$

By using the boundary conditions and putting $x=0$ and $x=1$, we get $q(t)=y_1(t)$ for $x=0$ and $p(t)=y_2(t)-y_1(t)-(P_x^2(1))^T C^{r+1}H(t)$ for $x=1$. Thus, the Eq. (3.4) can be rewritten as:

$$\begin{aligned} u_{r+1}(x,t) &= (P_x^2)^T C^{r+1}H(t) - x\big\{(P_x^2(1))^T C^{r+1}H(t)\big\} \\ &\quad + x(y_2(t)-y_1(t)) + y_1(t). \end{aligned} \tag{3.5}$$

Now for simplicity, we denote the right side of Eq. (3.2) by $S(x,t)$, that is,

$$\begin{aligned} S(x,t) &= \lambda u_r(x,t)(1-u_r(x,t)) + q(x,t) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m} m_{i,j}h_i(x)h_j(t) = H^T(x).M.H(t), \end{aligned} \tag{3.6}$$

where $m_{i,j} = \big\langle h_i(x), \big\langle S(x,t), h_j(t)\big\rangle\big\rangle$. By substituting Eq. (3.6) and Eq. (3.3) in Eq. (3.2), we get:

$$\frac{\partial^\alpha u_{r+1}}{\partial t^\alpha} = H^T(x)C^{r+1}H(t) + H^T(x)MH(t). \tag{3.7}$$

By applying fractional integral operator I_t^α to Eq. (3.7) and using the initial condition, we obtain:

$$u_{r+1}(x,t) = H^T(x)C^{r+1}P_t^\alpha + H^T(x)MP_t^\alpha + g(x). \tag{3.8}$$

Now, from Eqs. (3.8) and (3.5) we get:

$$\begin{aligned} & K(x,t) + (P_x^2)^T C^{r+1}H(t) - x\Big((P_x^2(1))^T C^{r+1}H(t)\Big) \\ & -H^T(x)C^{r+1}P_t^\alpha - H^T(x)MP_t^\alpha = 0, \end{aligned} \tag{3.9}$$

where $K(x,t) = -g(x) + x\big(y_2(t)-y_1(t)\big) + y_1(t)$. By discrediting Eq. (3.9) and using collocation points, we obtain the matrix form:

$$\Big((P_x^2)^T - V^{2,1,g(x)}\Big)C^{r+1}H - H^T C^{r+1}P_t^\alpha - H^T MP_t^\alpha + K = 0, \tag{3.10}$$

where H is the $m \times m$ Haar matrix, $V^{2,1,g(x)} = g(x)I_1^2 H^T = x(P^2(1))^T$ is the $m \times m$ fractional integration matrix for boundary value problem, $P_x^\alpha = I_x^\alpha H^T$ and $P_t^\alpha = I_t^\alpha H$ are the $m \times m$ matrices of fractional integration of the Haar function. Also, $K = K(x(i), t(i))$ $(i = 1, 2, \dots, m)$ is the matrix determined by using the collocation points. Note that, the Eq. (3.10) can be written by:

$$\underbrace{(H^T)^{-1}\left((P^2)^T - V^{2,1,g(x)}\right)}_{A} C^{r+1} - C^{r+1} \underbrace{P_t^\alpha(H^{-1})}_{-B} = \underbrace{(H^T)^{-1}\left(H^T M P_t^\alpha - K\right)(H^{-1})}_{D}, \tag{3.11}$$

where it is the Sylvester equation(A C+C B=D). We solve the Eq. (3.11) for C^{r+1} which is $m \times m$ coefficients matrix. By substituting C^{r+1} in Eq. (3.8) or Eq. (3.5), we get solution $u_{r+1}(x,t)$ at the collocation points. By considering an initial approximation $u_0(x,t)$, we obtain a linear fractional partial differential equation in $u_1(x,t)$ by substituting $r = 0$ in Eq. (3.2). Again, we solve it by the above procedure. Similarly, for $r = 1$, we obtain $u_2(x,t)$ and so on.

4. Convergence

Theorem 4.1. *Consider the functions $u_m(x,t)$ obtained by the Haar wavelet with approximation $u(x,t)$. Then, $\|u(x,t) - u_m(x,t)\|_E \leq \frac{K}{\sqrt{3}m}$, where*

$$\|u(x,t)\|_E = \left(\int_0^1 \int_0^1 u^2(x,t)dxdt\right)^{1/2}.$$

Proof. Since $u_m(x,t) = \sum_{n=0}^{m-1} \sum_{l=0}^{m-1} u_{nl} h_n(x) h_l(t)$, we have

$$u(x,t) - u_m(x,y) = \sum_{n=m}^{\infty} \sum_{l=m}^{\infty} u_{nl} h_n(x) h_l(t) = \sum_{n=2^{p+1}}^{\infty} \sum_{l=2^{p+1}}^{\infty} u_{nl} h_n(x) h_l(t).$$

The orthogonality of the functions $h_i(x)$ on $[0,1)$ implies that $h_l(.) = 2^{\frac{j}{2}} h(2^j(.) - k)$. Hence,

$$\begin{aligned} \|u(x,t) - u_m(x,t)\|_E^2 &= \int_0^1 \int_0^1 (u(x,t) - u_m(x,t))^2 dxdt \\ &= 2^j \sum_{n=2^{p+1}}^{\infty} \sum_{l=2^{p+1}}^{\infty} \sum_{n'=2^{p+1}}^{\infty} \sum_{l'=2^{p+1}}^{\infty} u_{nl} u_{n'l'} \\ &\quad \left(\int_0^1 h_n(x) h_{n'}(x) dx\right)\left(\int_0^1 h_l(t) h_{l'}(t) dt\right) \\ &= 2^j \sum_{n=2^{p+1}}^{\infty} \sum_{l=2^{p+1}}^{\infty} u_{nl}^2 \end{aligned} \tag{4.1}$$

where $u_{nl} = \Big\langle h_n(x), \langle u(x,t), h_l(t)\rangle \Big\rangle$. By using Eq. (2.1), we get:

$$\Big\langle u(x,t), h_l(t) \Big\rangle = \int_0^1 u(x,t)h_l(t)dt = 2^{\frac{j}{2}}\Bigg(\int_{\frac{k-1}{2^j}}^{\frac{k-0.5}{2^j}} u(x,t)dt - \int_{\frac{k-0.5}{2^j}}^{\frac{k}{2^j}} u(x,t)dt \Bigg).$$

By using mean value theorem of integrals, there exist $\frac{k-1}{2^j} \le t_1 < \frac{k-0.5}{2^j}$ and $\frac{k-0.5}{2^j} \le t_2 < \frac{k}{2^j}$ such that

$$\begin{aligned} \Big\langle u(x,t), h_l(t) \Big\rangle &= 2^{\frac{j}{2}}\Big((\frac{k-0.5}{2^j} - \frac{k-1}{2^j})u(x,t_1) - (\frac{k}{2^j} - \frac{k-0.5}{2^j})u(x,t_2) \Big) \\ &= \frac{2^{\frac{j}{2}}}{2^{j+1}}\Big(u(x,t_1) - u(x,t_2) \Big) \end{aligned}$$

and

$$\begin{aligned} u_{nl} &= \Big\langle h_n(x), \frac{1}{2^{\frac{j}{2}+1}}(u(x,t_1) - u(x,t_2)) \Big\rangle \qquad (4.2) \\ &= \frac{1}{2^{j+1}} \int_0^1 h_n(x)(u(x,t_1) - u(x,t_2))dx \\ &= \frac{2^{\frac{j}{2}}}{2^{\frac{j}{2}+1}} \Big(\int_{\frac{k-1}{2^j}}^{\frac{k-0.5}{2^j}} u(x,t_1)dx - \int_{\frac{k-0.5}{2^j}}^{\frac{k}{2^j}} u(x,t_1)dx \\ &- \int_{\frac{k-1}{2^j}}^{\frac{k-0.5}{2^j}} u(x,t_2)dx + \int_{\frac{k-0.5}{2^j}}^{\frac{k}{2^j}} u(x,t_2)dx \Big). \end{aligned}$$

By using mean value theorem of integrals, choose $\frac{k-1}{2^j} \le x_1, x_2 < \frac{k-0.5}{2^j}$ and $\frac{k-0.5}{2^j} \le x_3, x_4 < \frac{k}{2^j}$ such that

$$\begin{aligned} u_{nl} &= \frac{1}{2}\Big\{ (\frac{k-0.5}{2^j} - \frac{k-1}{2^j})u(x_1,t_1) - (\frac{k-}{j} - \frac{k-0.5}{2^j})u(x_2,t_1) - \\ &(\frac{k-0.5}{2^j} - \frac{k-1}{2^j})u(x_3,t_2) + (\frac{k}{2^j} - \frac{k-0.5}{2^j})u(x_4,t_2) \Big\} \\ &= \frac{1}{2^{j+2}}\Big\{ (u(x_1,t_1) - u(x_2,t_1)) - (u(x_3,t_2) - u(x_4,t_2)) \Big\} \end{aligned}$$

and so $u_{nl}^2 = \frac{1}{2^{2j+4}}\Big\{ (u(x_1,t_1) - u(x_2,t_1)) - (u(x_3,t_2) - u(x_4,t_2)) \Big\}^2$. By using mean value theorem of derivatives, there exist $x_1 \le \xi_1 < x_2$ and $x_3 \le \xi_2 < x_4$ such that

$$\begin{aligned} u_{nl}^2 \le \frac{1}{2^{2j+4}} & \Big\{ (x_2 - x_1)^2 [\frac{\partial u(\xi_1,t_1)}{\partial x}]^2 + (x_4 - x_3)^2 [\frac{\partial u(\xi_1,t_1)}{\partial x}]^2 + \\ & 2(x_2 - x_1)(x_4 - x_3) \Big| \frac{\partial u(\xi_1,t_1)}{\partial x} \Big| \Big| \frac{\partial u(\xi_2,t_2)}{\partial x} \Big| \Big\}. \end{aligned}$$

Since $\frac{\partial u(x,t)}{\partial x}$ is bounded on $(0,1)\times(0,1)$, there exists $K>0$ such that $\left|\frac{\partial u(x,t)}{\partial x}\right| \le K$ for all $x,t \in (0,1)\times(0,1)$. This implies that $u_{nl}^2 \le \left(\frac{1}{2^{2j+4}}\right)\frac{4K^2}{2^{2j}} = \frac{4K^2}{2^{4j+4}}$ and so

$$\begin{aligned}
||u(x,t)-u_m(x,t)||_E^2 &= \sum_{j=p+1}^{\infty}\left(\sum_{n=2^j}^{2^{j+1}-1}\sum_{n=2^j}^{2^{j+1}-1} u_{nl}^2\right) \le \sum_{j=p+1}^{\infty}\left(\sum_{n=2^j}^{2^{j+1}-1}\sum_{n=2^j}^{2^{j+1}-1}\frac{4K^2}{2^{4j+4}}\right)\\
&= 4K^2\sum_{j=p+1}^{\infty}\left(\sum_{n=2^j}^{2^{j+1}-1}\sum_{n=2^j}^{2^{j+1}-1}\frac{1}{2^{4j+4}}\right)\\
&= \frac{K^2}{3}\frac{1}{4^{p+1}} = \frac{K^2}{3m^2}.
\end{aligned}$$

Thus, $\|u(x,t)-u_m(x,t)\|_E \le \frac{K}{\sqrt{3}m}$ and so $\|u(x,t)-u_m(x,t)\|_E \to 0$ when $m\to\infty$. By using a similar procedure, we can show that $\|u_{r+1}(x,t)-u_{r+1}^m(x,t)\|_E \le \frac{K}{\sqrt{3}m}$ which implies that error between the exact and approximate solution at the (r+1)th iteration is inversely proportional to the maximal level of resolution. This shows that $u_{r+1}^m(x,t)$ converges to $u_{r+1}(x,t)$ as $m\to\infty$. Since $u_{r+1}(x,t)$is obtained at (r+1)th iteration of Picard technique, we conclude that $u_{r+1}(x,t)$ converges to $u(x,t)$ as $r\to\infty$. Thus, $\lim_{m,r\to\infty} u_{r+1}^m(x,t) = u(x,t)$. □

5. Experiments and results

In this section, we use the Haar wavelet collocation iteration method (HWCIM) to solve the fractional Fisher's equations. We provide four examples to illustrate the method.

Example 1. *Consider the problem (1.1) for* $\lambda=1$, $n=6$, $q(x,t)=0$.

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u(x,t)\left(1-u^6(x,t)\right) \tag{5.1}$$

with the initial and boundary conditions:

$$\begin{aligned}
&u(x,0)=g(x)=\left(\frac{1}{1+e^{\frac{3x}{2}}}\right)^{\frac{1}{3}}, u(0,t)=y_1(t)=\left(\frac{1}{1+e^{\frac{-15t}{4}}}\right)^{\frac{1}{3}},\\
&u(1,t)=y_2(t)=\left(\frac{1}{1+e^{\frac{6-15t}{4}}}\right)^{\frac{1}{3}}.
\end{aligned}$$

It is known that the problem has the exact solution $u_{exact} = \left(\frac{1}{1+e^{\frac{-15t+6x}{4}}}\right)^{\frac{1}{3}}$ *for* $\alpha=1$.

We put $u_0(x,t)=\left(\frac{1}{1+e^{\frac{3x}{2}}}\right)^{\frac{1}{3}}$ and apply the Haar wavelet iteration technique. By applying the iteration technique of Picard, we obtain:

$$\frac{\partial^\alpha u_{r+1}}{\partial t^\alpha} - \frac{\partial^2 u_{r+1}}{\partial x^2} = u_r\left(1-(u_r)^6\right) \tag{5.2}$$

with the initial and boundary conditions: $u_{r+1}(x,0) = g(x)$, $u_{r+1}(0,t) = y_1(t)$ and $u_{r+1}(1,t) = y_2(t)$. We suppose that

$$\frac{\partial^2 u_{r+1}}{\partial x^2} = H^T(x)C^{r+1}H(t) \tag{5.3}$$

Now, by applying the fractional integral operator I_x^2 on the Eq. (5.3), we get:

$$u_{r+1}(x,t) = (P_x^2)^T C^{r+1}H(t) + p(x)t + q(t). \tag{5.4}$$

By using the boundary conditions and putting $x = 0$ and $x = 1$, we get $q(t) = y_1(t)$ for $x = 0$ and $p(t) = y_2(t) - y_1(t) - (P_x^2)^T C^{r+1}H(t)$ for $x = 1$. Thus, the Eq. (5.4) can be rewritten as:

$$\frac{\partial^2 u_{r+1}}{\partial x^2} = H^T(x)C^{r+1}H(t) - x(P_x^2)^T C^{r+1}H(t) + x(y_2(t) - y_1(t)) + y_1(t). \tag{5.5}$$

We put the right of Eq. (5.2) by $S(x,t)$, and we estimate by Haar wavelet, that is,

$$\begin{aligned} u_r(x,t)\left(1-(u_r(x,t))^6\right) &= S(x,t) \qquad (5.6)\\ &= \sum_{i=1}^{m}\sum_{j=1}^{m} m_{i,j}h_i(x)h_j(t) = H^T M H(t), \end{aligned}$$

where $m_{i,j} = \big\langle h_i(x), \big\langle S(x,t), h_j(t)\big\rangle\big\rangle$. We substitute Eqs. (5.6) and (5.3) in Eq. (5.2),

$$\frac{\partial^\alpha u_{r+1}}{\partial t^\alpha} = H^T(x)C_{r+1}H(t) + H^T(x)MH(t). \tag{5.7}$$

We apply fractional integral operator I_t^α to Eq. (5.7) and use the initial condition,

$$u_{r+1}(x,t) = H^T(x)C_{r+1}P_t^\alpha + H_T(x)MP_t^\alpha + g(x). \tag{5.8}$$

Now, from Eqs. (5.8) and (5.4) we get:

$$\begin{aligned} &K(x,t) + (P_x^2)^T C^{r+1}H(t) - x\Big((P_x^2(1))^T C^{r+1}H(t)\Big) \qquad (5.9)\\ &-H^T(x)C^{r+1}P_t^\alpha - H^T(x)MP_t^\alpha = 0, \end{aligned}$$

where $K(x,t) = -g(x) + x\big(y_2(t) - y_1(t)\big) + y_1(t)$. By discrediting Eq. (5.9) and using collocation points, we obtain the matrix form as follow:

$$\Big((P_x^2)^T - V^{2,1,g(x)}\Big)C^{r+1}H - H^T C^{r+1}P_t^\alpha - H^T M P_t^\alpha + K = 0, \tag{5.10}$$

where H is the $m \times m$ Haar matrix, $V^{2,1,g(x)} = g(x)I_1^2 H^T = x(P^2(1))^T$ is the $m \times m$ fractional integration matrix for boundary value problem, $P_x^\alpha = I_x^\alpha H^T$ and $P_t^\alpha = I_t^\alpha H$ are the $m \times m$ matrices of fractional integration of the Haar function. Also, $K = K(x(i), t(i))$ $(i = 1, 2, \ldots, m)$ is the matrix determined by using the collocation points. Note that, the Eq. (5.10) can be rewritten by:

$$\underbrace{(H^T)^{-1}\Big((P^2)^T - V^{2,1,g(x)}\Big)}_{A} C^{r+1} - C^{r+1}\underbrace{P_t^\alpha(H^{-1})}_{-B} = \underbrace{(H^T)^{-1}\Big(H^T M P_t^\alpha - K\Big)(H^{-1})}_{D}, \tag{5.11}$$

where is the Sylvester equation. By considering $r = 0$ and initial approximation $u_0(x,t)$ and solving the Sylvester equation by Matlab software, we obtain C^1. Then,

TABLE 1. Absolute error for Example 1, with $m = 64$ and $\alpha \to 1$

method	HWCIM	HWCIM	HWCIM	HWCIM	MVIM[16]	HPM [12]
(x, t)	$\alpha = 0.5$ $\lvert u_4 - u_{ex}\rvert$	$\alpha = 0.7$ $\lvert u_4 - u_{ex}\rvert$	$\alpha = 0.9$ $\lvert u_4 - u_{ex}\rvert$	$\alpha = 1$ $\lvert u_4 - u_{ex}\rvert$	$\alpha = 1$ $\lvert u_4 - u_{ex}\rvert$	$\alpha = 1$ $\lvert u_4 - u_{ex}\rvert$
$(\frac{15}{128}, \frac{15}{128})$	1.34×10^{-2}	9.51×10^{-3}	3.79×10^{-3}	4.06×10^{-6}	2.47×10^{-2}	4.09×10^{-2}
$(\frac{31}{128}, \frac{31}{128})$	1.40×10^{-2}	9.48×10^{-3}	3.84×10^{-3}	5.59×10^{-6}	4.92×10^{-2}	1.10×10^{-1}
$(\frac{47}{128}, \frac{47}{128})$	7.74×10^{-3}	4.52×10^{-3}	1.57×10^{-3}	4.02×10^{-5}	6.30×10^{-2}	2.29×10^{-1}
$(\frac{63}{128}, \frac{63}{128})$	3.08×10^{-4}	5.35×10^{-4}	6.23×10^{-4}	2.60×10^{-4}	5.93×10^{-2}	4.26×10^{-1}
$(\frac{79}{128}, \frac{79}{128})$	5.12×10^{-3}	4.34×10^{-3}	1.66×10^{-3}	7.36×10^{-4}	3.69×10^{-2}	7.27×10^{-1}
$(\frac{95}{128}, \frac{95}{128})$	7.31×10^{-3}	5.34×10^{-3}	1.61×10^{-3}	1.17×10^{-3}	6.75×10^{-3}	1.15×10^{0}
$(\frac{111}{128}, \frac{111}{128})$	5.87×10^{-3}	3.98×10^{-3}	9.98×10^{-4}	1.01×10^{-4}	4.08×10^{-2}	1.73×10^{0}
$(\frac{127}{128}, \frac{127}{128})$	4.95×10^{-4}	3.19×10^{-4}	6.87×10^{-5}	8.90×10^{-5}	7.83×10^{-2}	2.46×10^{0}

by substituting C^1 to Eq. (5.8) or Eq. (5.4) we get $u_1(x,t)$ at the collocation points. Similarly for $r = 1$, we obtain $u_2(x,t)$ and so on.

The numerical results for $m = 64$ are given in Figure 1 and Table 1. Note that, Table 1 shows that the solutions obtained by the present method for different value of α, convergence to the exact solution whenever α tends to 1. Also, the obtained numerical results have been compared with HPM [12], [22] and MVIM [16].

Example 2. *Consider the problem (1.1) for $\lambda = 1$, $n = 1$ and $q(x,t) = 0$,*

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u(x,t)(1 - u(x,t))$$

with the initial and boundary conditions

$$u(x,0) = \beta, u(0,t) = \frac{\beta e^t}{1 - \beta + \beta e^t}, u(1,t) = \frac{\beta e^t}{1 - \beta + \beta e^t},$$

Where β be a constant. It is known that the problem has the exact solution $u_{exact} = \frac{\beta e^t}{1-\beta+\beta e^t}$ for $\alpha = 1$.

Put $u_0(x,t) = \beta$ and apply the Haar wavelet iteration technique. The numerical results for $m = 64$ and $\beta = 2/3$ at 4 iterations are given in Figure 2, Figure 3 and Table 2. Note that, Table 2 shows that the solutions obtained by the present method for different values of α, convergence to the exact solution whenever α tends to 1. Also, the obtained numerical results have been compared with HPM [24] and MVIM [16] and Figure 3 shows that the approximate results are closer to the exact solution by increasing the resolution (m).

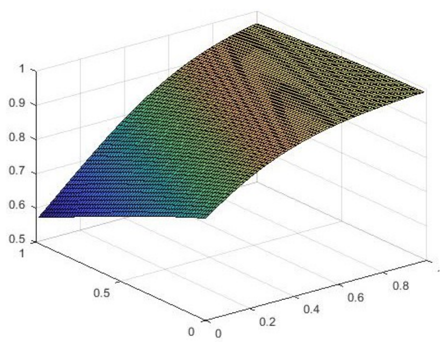
Example 3. *Consider the problem (1.1) for $\lambda = 6$, $n = 1$ and $q(x,t) = 0$*

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + 6u(x,t)(1 - u(x,t))$$

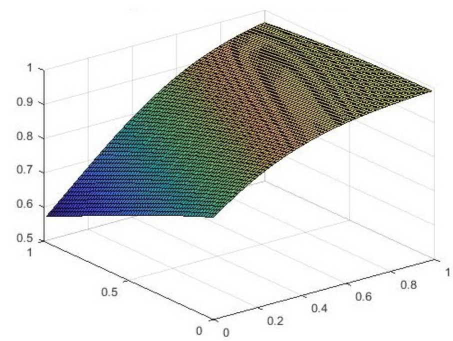
with the initial and boundary conditions:

$$u(x,0) = \frac{1}{(1 + e^x)^2}, u(0,t) = \frac{1}{(1 + e^{-5t})^2}, u(1,t) = \frac{1}{(1 + e^{1-5t})^2}.$$

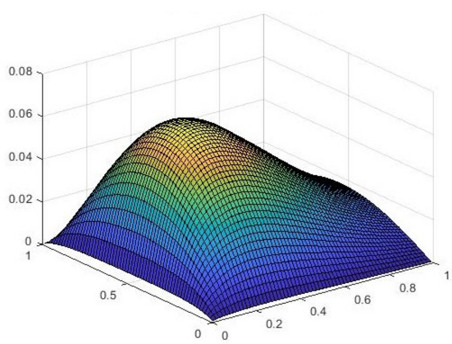
Figure 1. Comparison of the exact solution and the Haar wavelet collocation iteration solution for Example 1



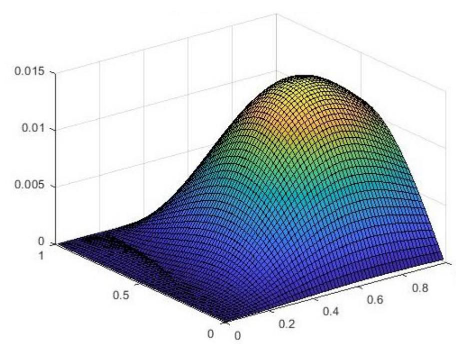
(a) exact solution for $\alpha = 1$



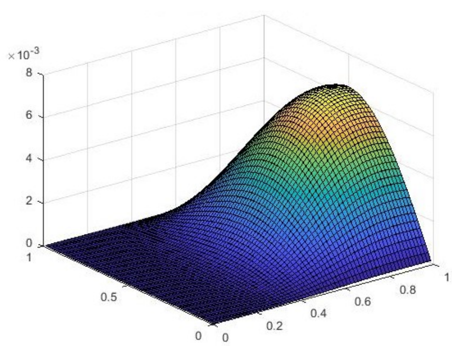
(b) approximate solution at the **4th** iteration with $m = 64$, $\alpha = 1$



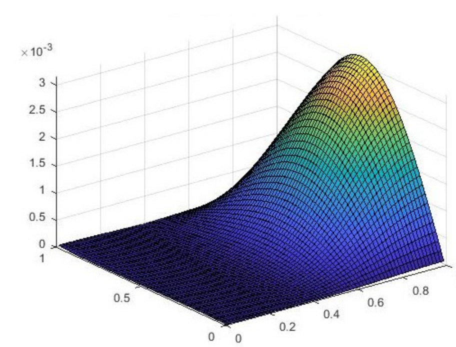
(c) absolute error for the **1st** iteration with $m = 64$, $\alpha = 1$



(d) absolute error for the **2nd** iteration with $m = 64$, $\alpha = 1$

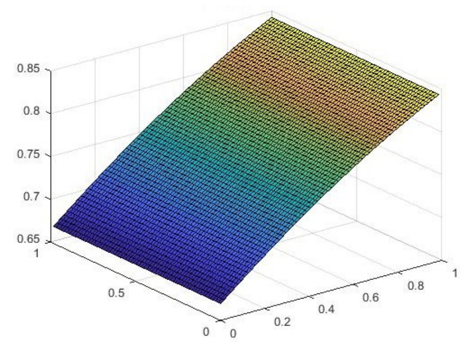


(e) absolute error for the **3rd** iteration with $m = 64$, $\alpha = 1$

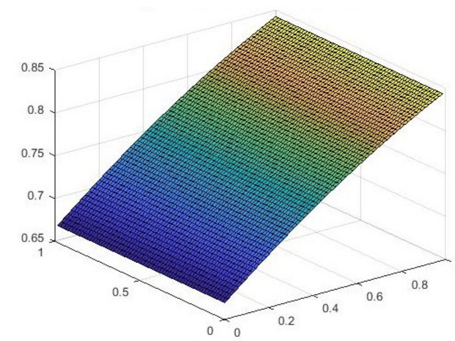


(f) absolute error for the **4th** iteration with $m = 64$, $\alpha = 1$

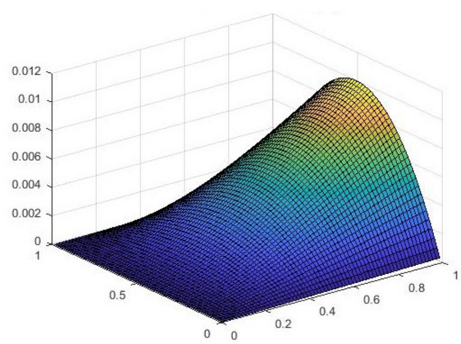
FIGURE 2. Comparison of the exact solution and the Haar wavelet collocation iteration solution for Example 2



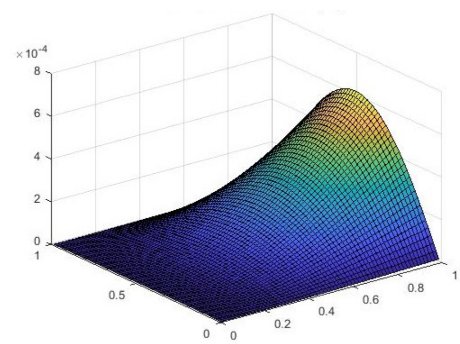
(a) exact solution for $\alpha = 1$



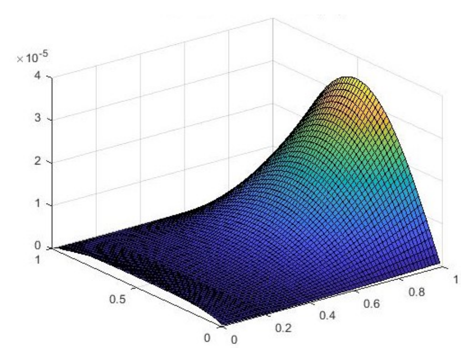
(b) approximate solution at the **4th** iteration with $m = 64$, $\alpha = 1$



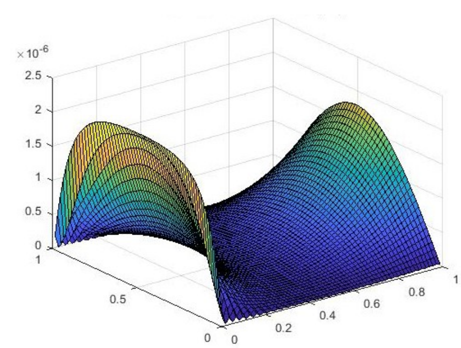
(c) absolute error for the **1st** iteration with $m = 64$, $\alpha = 1$



(d) absolute error for the **2nd** iteration with $m = 64$, $\alpha = 1$



(e) absolute error for the **3rd** iteration with $m = 64$, $\alpha = 1$



(f) absolute error for the **4th** iteration with $m = 64$, $\alpha = 1$

TABLE 2. Absolute error for Example 2, $m = 64$ and $\alpha \to 1$.

method (x,t)	HWCIM $\alpha=0.5$ $\lvert u_4-u_{ex}\rvert$	HWCIM $\alpha=0.7$ $\lvert u_4-u_{ex}\rvert$	HWCIM $\alpha=0.9$ $\lvert u_4-u_{ex}\rvert$	HWCIM $\alpha=1$ $\lvert u_4-u_{ex}\rvert$	HPM [24] $\alpha=1$ $\lvert u_4-u_{ex}\rvert$	MVIM [16] $\alpha=1$ $\lvert u_4-u_{ex}\rvert$
$(\frac{31}{128},\frac{31}{128})$	7.34×10^{-3}	4.91×10^{-3}	1.92×10^{-3}	2.59×10^{-7}	2.88×10^{-7}	2.17×10^{-3}
$(\frac{31}{128},\frac{63}{128})$	2.67×10^{-3}	$1,39\times10^{-3}$	3.73×10^{-4}	1.74×10^{-7}	6.26×10^{-6}	8.98×10^{-3}
$(\frac{31}{128},\frac{95}{128})$	8.88×10^{-4}	1.02×10^{-3}	5.60×10^{-4}	5.01×10^{-7}	2.19×10^{-5}	2.04×10^{-2}
$(\frac{31}{128},\frac{127}{128})$	3.61×10^{-4}	2.73×10^{-3}	1.14×10^{-3}	1.24×10^{-6}	9.10×10^{-6}	3.68×10^{-2}
$(\frac{63}{128},\frac{31}{128})$	9.95×10^{-3}	6.69×10^{-3}	2.65×10^{-3}	3.78×10^{-7}	2.88×10^{-7}	2.17×10^{-3}
$(\frac{63}{128},\frac{63}{128})$	3.66×10^{-3}	1.94×10^{-3}	5.38×10^{-4}	2.49×10^{-7}	6.26×10^{-6}	8.98×10^{-3}
$(\frac{63}{128},\frac{95}{128})$	1.15×10^{-3}	1.34×10^{-3}	7.38×10^{-4}	7.24×10^{-7}	2.19×10^{-5}	2.04×10^{-2}
$(\frac{63}{128},\frac{127}{128})$	4.84×10^{-3}	3.66×10^{-3}	1.53×10^{-3}	1.80×10^{-6}	9.10×10^{-6}	3.68×10^{-2}
$(\frac{95}{128},\frac{31}{128})$	7.65×10^{-3}	5.12×10^{-3}	2.01×10^{-3}	2.72×10^{-7}	2.88×10^{-7}	2.17×10^{-3}
$(\frac{95}{128},\frac{63}{128})$	2.78×10^{-3}	1.45×10^{-3}	3.91×10^{-4}	1.82×10^{-7}	6.26×10^{-6}	8.98×10^{-3}
$(\frac{95}{128},\frac{95}{128})$	9.21×10^{-4}	1.06×10^{-3}	5.82×10^{-4}	5.26×10^{-7}	2.19×10^{-5}	2.04×10^{-2}
$(\frac{95}{128},\frac{127}{128})$	3.75×10^{-3}	2.84×10^{-3}	1.19×10^{-3}	1.30×10^{-6}	9.10×10^{-6}	3.68×10^{-2}
$(\frac{127}{128},\frac{31}{128})$	3.13×10^{-4}	2.06×10^{-4}	7.85×10^{-5}	3.20×10^{-9}	2.88×10^{-7}	2.17×10^{-3}
$(\frac{127}{128},\frac{63}{128})$	1.10×10^{-4}	5.50×10^{-5}	1.32×10^{-5}	1.10×10^{-9}	6.26×10^{-6}	8.98×10^{-3}
$(\frac{127}{128},\frac{95}{128})$	4.24×10^{-5}	4.80×10^{-5}	2.57×10^{-5}	1.26×10^{-8}	2.19×10^{-5}	2.04×10^{-2}
$(\frac{127}{128},\frac{127}{128})$	1.59×10^{-4}	1.20×10^{-4}	5.00×10^{-5}	4.01×10^{-8}	9.10×10^{-7}	3.68×10^{-2}

It is known that the problem has the exact solution $u_{exact} = \frac{1}{(1+e^{x-5t})^2}$ *for* $\alpha = 1$.

We put $u_0(x,t) = \frac{1}{(1+e^{x})^2}$ and apply the Haar wavelet iteration technique. The numerical results included absolute error and approximate solutions for $m = 64$ at 4 iteration are given in Figure 4 which shows that, numerical solution is in very good coincide with exact solution by increasing iterations. The present method solution (HWCIM) has been compared with HPM [24] and MVIM [16] at Table (3), which shows Haar wavelet collocation iteration method is more accurate than the HPM and MVIM methods.

Example 4. *Consider the problem (1.1) (non-homogenous time fractional fisher equation) for* $\lambda = 1$, $n = 3$ *and*

$$q(x,t) = t\left(-2 - x(t+x)(1 - t^3x^3(t+x)^3) + \frac{x^2t^{-\alpha}}{\Gamma(2-\alpha)} + \frac{2xt^{1-\alpha}}{\Gamma(3-\alpha)}\right),$$

with the initial and boundary conditions $u(x,0) = 0$, $u(0,t) = 0$ *and* $u(1,t) = t^2 + t$, *where it is known that the problem has the exact solution* $u_{exact}(x,t) = xt^2 + tx^2$.

Put $u_0(x,t)$ and apply the Haar wavelet collocation iteration technique. The numerical results included absolute error and approximate solutions for $m = 64$ at 4 iterations are shown in Figure 5 and Table 4.

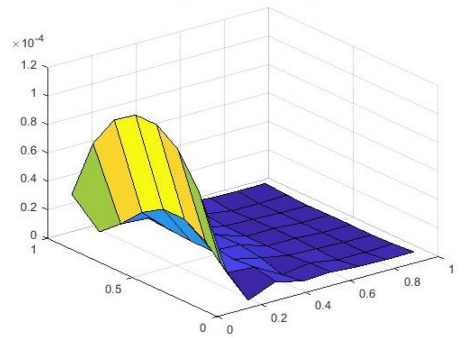
TABLE 3. Absolute error for Example 3, $m = 64$ and $\alpha \to 1$

method	HWCIM	HWCIM	HWCIM	HWCIM	HPM [24]	MVIM [16]
(x,t)	$\alpha=0.5$	$\alpha=0.7$	$\alpha=0.9$	$\alpha=1$	$\alpha=1$	$\alpha=1$
	$\|u_4-u_{ex}\|$	$\|u_4-u_{ex}\|$	$\|u_4-u_{ex}\|$	$\|u_4-u_{ex}\|$	$\|u_4-u_{ex}\|$	$\|u_4-u_{ex}\|$
$(\frac{31}{128},\frac{31}{128})$	0.60×10^{-2}	4.25×10^{-2}	1.72×10^{-2}	4.11×10^{-5}	2.07×10^{-2}	5.12×10^{-3}
$(\frac{31}{128},\frac{63}{128})$	3.61×10^{-3}	2.83×10^{-4}	6.06×10^{-4}	2.77×10^{-5}	2.11×10^{-1}	7.99×10^{-2}
$(\frac{31}{128},\frac{95}{128})$	2.44×10^{-2}	1.82×10^{-2}	7.36×10^{-3}	4.98×10^{-4}	5.63×10^{-1}	1.12×10^{-1}
$(\frac{31}{128},\frac{127}{128})$	3.15×10^{-2}	2.16×10^{-2}	7.87×10^{-3}	7.76×10^{-4}	8.85×10^{-1}	1.91×10^{-1}
$(\frac{63}{128},\frac{31}{128})$	8.68×10^{-2}	6.10×10^{-2}	2.49×10^{-2}	7.75×10^{-5}	2.30×10^{-1}	1.54×10^{-3}
$(\frac{63}{128},\frac{63}{128})$	9.38×10^{-3}	3.71×10^{-3}	5.64×10^{-4}	2.85×10^{-5}	2.96×10^{-1}	8.95×10^{-2}
$(\frac{63}{128},\frac{95}{128})$	3.17×10^{-2}	2.38×10^{-2}	9.72×10^{-3}	7.06×10^{-4}	9.97×10^{-1}	2.90×10^{-1}
$(\frac{63}{128},\frac{127}{128})$	4.28×10^{-2}	2.95×10^{-2}	1.08×10^{-2}	1.12×10^{-3}	2.07×10^{-1}	1.91×10^{-1}
$(\frac{95}{128},\frac{31}{128})$	6.89×10^{-2}	4.85×10^{-2}	1.95×10^{-2}	6.62×10^{-5}	2.01×10^{-1}	1.10×10^{-2}
$(\frac{95}{128},\frac{63}{128})$	1.08×10^{-2}	5.15×10^{-3}	1.20×10^{-3}	1.20×10^{-5}	3.16×10^{-1}	5.40×10^{-2}
$(\frac{95}{128},\frac{95}{128})$	2.43×10^{-2}	1.86×10^{-2}	7.67×10^{-3}	5.00×10^{-4}	1.21×10^{-1}	3.18×10^{-1}
$(\frac{95}{128},\frac{127}{128})$	3.43×10^{-2}	2.37×10^{-2}	8.75×10^{-3}	8.10×10^{-4}	2.70×10^{-1}	6.52×10^{-1}
$(\frac{127}{128},\frac{31}{128})$	2.85×10^{-3}	1.89×10^{-3}	2.80×10^{-4}	1.63×10^{-6}	1.39×10^{-2}	2.00×10^{-2}
$(\frac{127}{128},\frac{63}{128})$	2.58×10^{-3}	1.62×10^{-3}	5.59×10^{-4}	1.37×10^{-7}	2.76×10^{-1}	8.65×10^{-3}
$(\frac{127}{128},\frac{95}{128})$	5.04×10^{-4}	5.15×10^{-4}	1.47×10^{-3}	1.66×10^{-5}	1.20×10^{-1}	2.29×10^{-1}
$(\frac{127}{128},\frac{127}{128})$	1.88×10^{-3}	1.37×10^{-3}	4.72×10^{-4}	2.72×10^{-5}	2.98×10^{-1}	6.44×10^{-1}

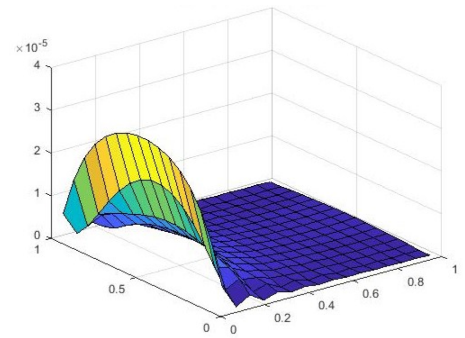
TABLE 4. Absolute error for Example 4, $m = 64$ and $\alpha \to 1$.

method	HWCIM	HWCIM	HWCIM	HWCIM
(x,t)	$\alpha=0.5$	$\alpha=0.7$	$\alpha=0.9$	$\alpha=1$
$(\frac{31}{128},\frac{31}{128})$	2.04×10^{-2}	1.37×10^{-2}	5.26×10^{-2}	2.80×10^{-6}
$(\frac{31}{128},\frac{63}{128})$	2.26×10^{-2}	1.45×10^{-2}	5.30×10^{-3}	4.59×10^{-6}
$(\frac{31}{128},\frac{95}{128})$	2.08×10^{-2}	1.28×10^{-2}	4.44×10^{-3}	7.45×10^{-6}
$(\frac{31}{128},\frac{127}{128})$	1.34×10^{-2}	8.17×10^{-3}	2.65×10^{-3}	8.84×10^{-5}
$(\frac{63}{128},\frac{31}{128})$	3.55×10^{-2}	2.37×10^{-2}	9.06×10^{-3}	4.03×10^{-6}
$(\frac{63}{128},\frac{63}{128})$	3.80×10^{-2}	2.43×10^{-2}	8.74×10^{-3}	6.37×10^{-6}
$(\frac{63}{128},\frac{95}{128})$	3.37×10^{-2}	2.06×10^{-2}	7.04×10^{-3}	8.31×10^{-6}
$(\frac{63}{128},\frac{127}{128})$	1.87×10^{-2}	1.12×10^{-2}	3.42×10^{-3}	3.89×10^{-4}
$(\frac{95}{128},\frac{31}{128})$	3.50×10^{-2}	2.33×10^{-2}	8.77×10^{-3}	2.89×10^{-6}
$(\frac{95}{128},\frac{63}{128})$	3.57×10^{-2}	2.25×10^{-2}	7.96×10^{-3}	4.12×10^{-6}
$(\frac{95}{128},\frac{95}{128})$	3.00×10^{-2}	1.81×10^{-2}	6.01×10^{-3}	3.33×10^{-6}
$(\frac{95}{128},\frac{127}{128})$	1.16×10^{-2}	6.40×10^{-3}	1.22×10^{-3}	1.19×10^{-3}
$(\frac{127}{128},\frac{31}{128})$	5.45×10^{-3}	3.56×10^{-3}	1.28×10^{-3}	3.93×10^{-8}
$(\frac{127}{128},\frac{63}{128})$	5.00×10^{-3}	3.03×10^{-3}	1.00×10^{-3}	1.18×10^{-7}
$(\frac{127}{128},\frac{95}{128})$	3.92×10^{-3}	2.21×10^{-3}	6.67×10^{-4}	2.38×10^{-7}
$(\frac{127}{128},\frac{127}{128})$	1.07×10^{-3}	1.99×10^{-4}	3.92×10^{-4}	5.52×10^{-4}

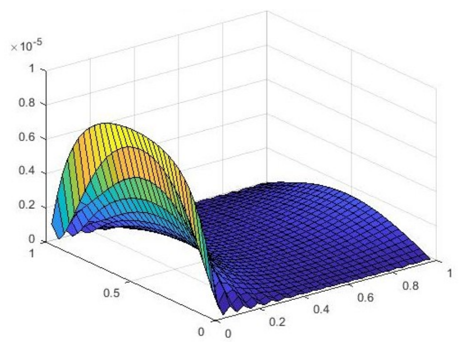
Figure 3. Absolute error of Haar wavelet collocation iteration solution with different values of M for Example 2



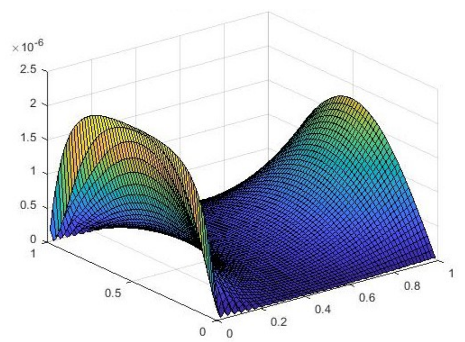
(a) absolute error for the **4th** iteration with $m = 8$, $\alpha = 1$



(b) absolute error for the **4th** iteration with $m = 16$, $\alpha = 1$



(c) absolute error for the **4th** iteration with $m = 32$, $\alpha = 1$

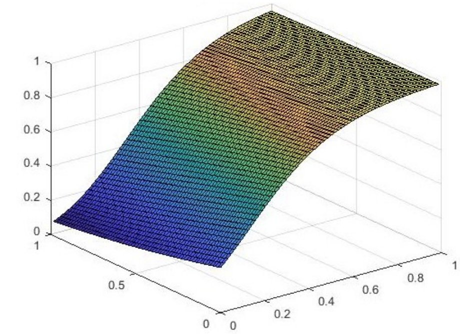


(d) absolute error for the **4th** iteration with $m = 64$, $\alpha = 1$

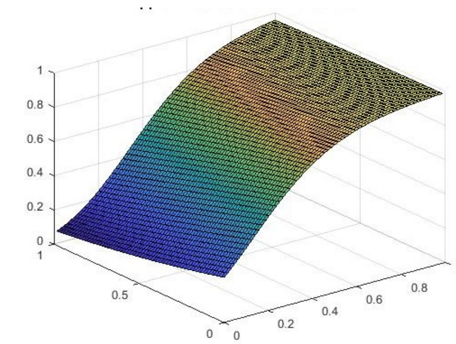
6. Conclusion

In this work, we successfully apply the combination of Haar wavelet operational matrices method and the iteration Picard technique to obtain the solution of fractional fisher's equation. By the use of iteration technique, we transform the nonlinear fractional partial differential equation to the linear equation and Sylvester equation. The obtained results have been compared with exact solutions, the HPM method, and the MVIM method, which shows that numerical solutions are in very good coincide with the exact solution by increasing iterations or level of resolution or both. These results have been cited in the tables in order to justify the accuracy and efficiency of the proposed method.

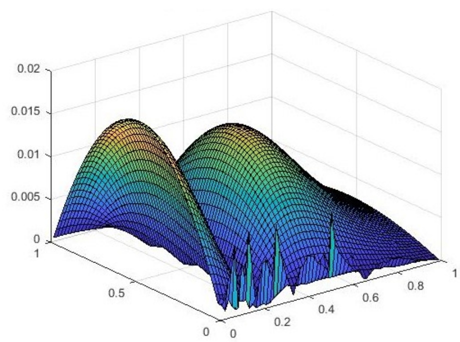
FIGURE 4. Comparison of the exact solution and the Haar wavelet collocation iteration solution for Example 3



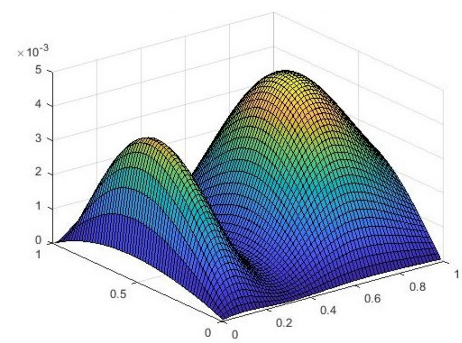
(a) exact solution for $\alpha = 1$



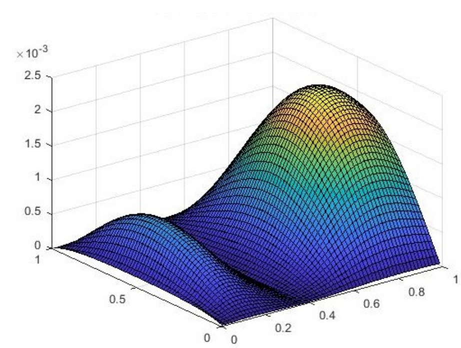
(b) approximate solution at the **4th** iteration with $m = 64$, $\alpha = 1$



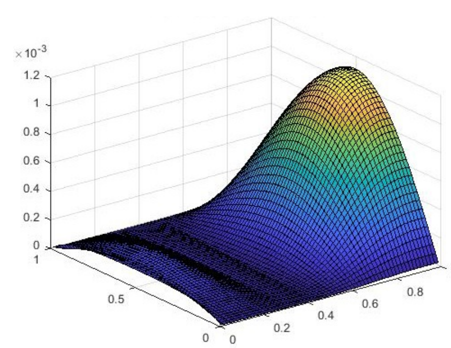
(c) absolute error for the **1st** iteration with $m = 64$, $\alpha = 1$



(d) absolute error for the **2nd** iteration with $m = 64$, $\alpha = 1$

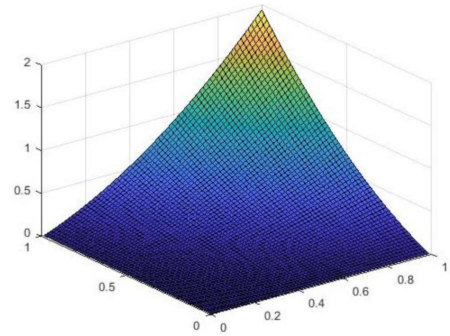


(e) absolute error for the **3rd** iteration with $m = 64$, $\alpha = 1$

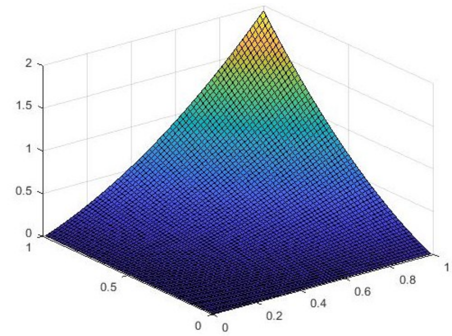


(f) absolute error for the **4th** iteration with $m = 64$, $\alpha = 1$

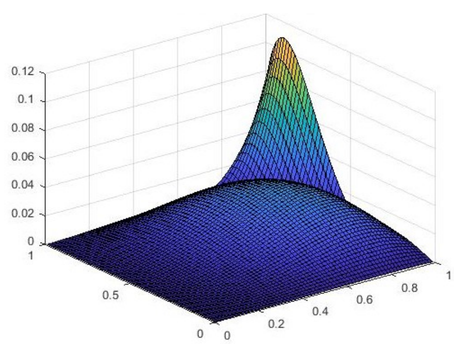
FIGURE 5. Comparison of the exact solution and the Haar wavelet collocation iteration solution for Example 4



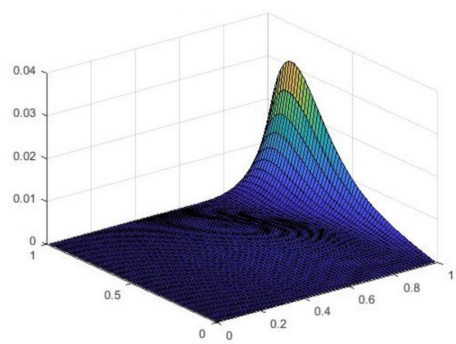
(a) exact solution for $m = 64$, $\alpha = 1$



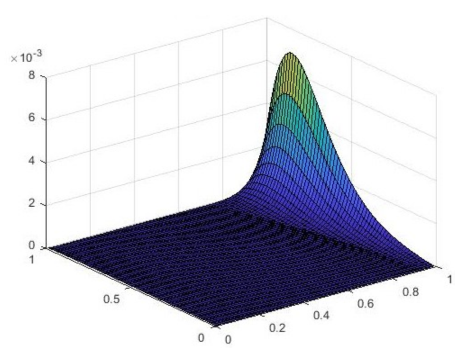
(b) approximate solution at the**4th** iteration with $m = 64$, $\alpha = 1$



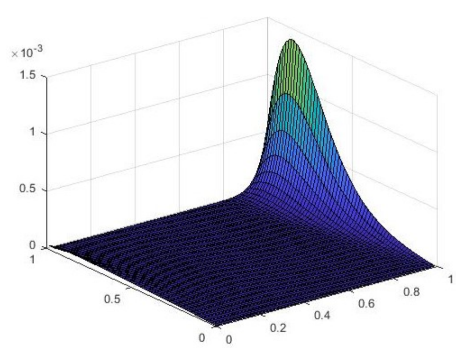
(c) absolute error for the **1st** iteration with $m = 64$, $\alpha = 1$



(d) absolute error for the **2nd** iteration with $m = 64$, $\alpha = 1$



(e) absolute error for the **3rd** iteration with $m = 64$, $\alpha = 1$



(f) absolute error for the **4th** iteration with $m = 64$, $\alpha = 1$

References

[1] N. Aghazadeh and S. Fathi, *Convergence of an approach for solving Fredholm functional integral equations*, Iranian Journal of Mathematical Sciences and Informatics, *11*(1) (2016), 35-46.

[2] R. E. Bellman and R. E. Kalaba, *Quasilinearization and Nonlinear Boundary-value Problems*, American Elsevier Publishing Company, 1965.

[3] C. Cattani, *Haar wavelet spline*, J. Interdiscip. Math, *4* (2001), 35–47.

[4] C. F. Chen and C. H. Hsiao, *Haar wavelet method for solving lumped and distributed-parameter systems*, IEE Proc. Pt.D, *144* (1997), 87–94.

[5] R. A. Fisher, *The wave of advance of advantageous genes.* Ann. Eugenics, *7*(4) (1937), 355–369.

[6] A. K. Gupta and S. Saha Ray, *On the solutions of fractional Burgers-Fisher and generalized Fisher's equations using two reliable methods*, Inter. J. Math. Math. Sci. (2014) Article ID 682910, 16 pages.

[7] G. Hariharan,K. Kannan, *Haar wavelet method for solving Cahn-Allen equation*, Appl. Math. Sci, *3* (2009), 2523–2533.

[8] G. Hariharan, K. Kannan, and K. R. Sharma, *Haar wavelet method for solving Fisher's equation*, Appl. Math. Comput, *211* (2009), 284–292.

[9] H. N. A. Ismail, K. R. Raslan, and A. A. Aziza, *Adomian decomposition method for Burger's-Huxley and Burger's-Fisher equations*, Appl. Math. Comput, *159* (2004), 291–301.

[10] A. Kolmogorov, N. Petrovsky, and S. Piscounov, *Etude de I equations de la diffusion avec croissance de la quantitate de matiere et son application a un probolome biologique* , Bull. Univ. Mosku, *1* (1937), 1–25.

[11] U. Lepik, *Solving fractional integral equations by the Haar wavelet method*, Appl. Math. Comput, *214* (2009), 468–478.

[12] M. Matinfar and M. Ghanbari, *Homotopy Perturbation Method for the Fisher's Equation and Its Generalized*, International Journal of Nonlinear Science, *4* (2009), 448–455

[13] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley, New York (1993).

[14] M. Mirzazadeh, *A novel approach for solving fractional Fisher equation using differential transform method*, Pramana J. Phys, *86* (2016), 957–963.

[15] M. Moghimi and F. Hejazi, *Variational iteration method for solving generalized Burger-Fisher and Burger equations*, Chaos, Solit. Fract, *33* (2007), 1756–1761.

[16] S. T. Mohyud-Din and M. A. Noor, *Modified variational iteration method for solving Fisher's equations*, J. Appl. Math. Comput, *31* (2009), 295–308.

[17] B. Sahoo, *A study on solution of differential equations using Haar wavelet collocation method*, MSc Thesis, National Institute of Technology, India (2012).

[18] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach, Yverdon (1993).

[19] S. C. Shiralashetti and A. B. Deshi, *An efficient Haar wavelet collocation method for the numerical solution of multi-term fractional differential equations*, Nonlinear Dynam, *83* (2016), 293–303.

[20] L. Wang, Y. Ma, and Z. Meng, *Haar wavelet method for solving fractional partial differential equations numerically*, Appl. Math. Comput, *227* (2014), 66–76.

[21] A. M. Wazwaz, *The tanh method for generalized forms of nonlinear heat conduction and Burger's-Fisher equations*, Appl. Math. Comput, *169* (2004), 321–338.

[22] A. M. Wazwaz and A. Gorguis, *An analytic study of Fisher's equation by using Adomian decomposition method*, Applied Mathematics and Computation, *154* (2004), 609–620

[23] B. J. West, M. Bologna, and P. Grigolini, *Physics of fractal operator*, Springer-Verlag, New York (2003).

[24] X. Zhang and J. Liu, *An analytic study on time-fractional Fisher equqtion using homotopy perturbation method*, Walailak J. Sci. Tech, *11* (2014), 975–985.