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The solving integro-differential equations of fractional order with the ultraspherical functions

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Abstract In this paper, an integration method is presented based on using ultraspherical polynomials for solving a class of linear fractional integro-differential equations of Volterra types. This method is based on a new investigation of ultrasphreical integration to approximate the highest order derivative in the equations and generat approximations to the lower order derivatives through integration of the higher-order derivatives. Numerical example illustrate the efficiency and accuracy of the method.

Keywords. Fractional integro-differential equation, Ultraspherical functions, Caputo derivative.

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1. Introduction

The use of fractional differential equations for mathematical modeling of real world physical problems has been widely developed in recent years, e.g. the earth quake modeling, the fluid dynamic traffic modeling, measurement of viscoelastic material properties, and etc. Derivatives of non-integer order can be defined in different ways, e.g, Riemann-Liouville, Grunwald-Letnikow, Caputo and generalized functions approaches [13]. In this paper we focus on Caputo's definition which turns out to be more useful in real life applications, since it allows traditional initial and boundary conditions to be included in the formulation of the problem. In recent years, more attempts have been made to find analytical and numerical solutions for the fractional problems. These attempts include introducing finite difference methods [6, 10], collocation-shooting method [2], Adomian decomposition method [4], operational matrix methods [14] and etc.
The aim of this work is to present a new formulation of spectral integration matrix depends on using ultraspherical polynomials. By using ultraspherical integration matrix we approximate the highest order derivative in the given equation and generat approximations to the lower order derivatives through fractional integrations of the

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higher-order fractional derivatives. Then the problem reduces to an algebraic system which can be solved by using a standard numerical method.
Consider a linear Volterra fractional integro-differential equation in the from:

$$ {}^{c}D_0^{\alpha} y(x) = F(x)y(x) + \int_0^x K(x,t)y(t)dt, \qquad n-1 < \alpha \leq n, \quad n \in \mathbf{N} \quad (1.1) $$

with the initial conditions:

$$ y^{(k)}(0) = b_k \quad k = 0, \ldots, n-1 \quad (1.2) $$

where $F : [0,1] \longrightarrow \mathbb{R}$, $K : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ are given continuous functions and b_k are given constants. The rest of the paper is organized as follows:
Basic concepts of fractional integral and derivatives are discussed in section 2. In section 3, we introduce ultraspherical polynomials and some of their properties. In section 4, formulation of the problem is discussed in terms of ultraspherical polynomials . In section 5, illustrative examples are given.

2. Preliminaries

In this section, we recall some basic concepts of fractional calculus which are used throughout the paper.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_{\mu}, \mu \geq -1$ is defined as [4]

$$ I_a^{\alpha} f(\alpha) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(x)}{(x-t)^{1-\alpha}} dt, $$

where $n-1 < \alpha \leq n, n \in$N and $a \in \mathbb{R}$.

Definition 2.2. Let $f \in C_{-1}^{n}, n \in$N$\cup\{0\}$. Then the Caputo fractional derivative of order α is defined as [4]

$$ {}^{c}D_0^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-(n-1)}} dt $$

where $n-1 < \alpha \leq n, n \in$N.

Proposition 2.3. *Let $f \in C_{-1}^{n}$. Then the following properties hold* [6, 15]

(1) $I_a^{\alpha} x^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}, \quad \alpha > 0, \quad \beta > -1, \quad x > 0$
(2) $I_a^{\alpha}({}^{c}D_0^{\alpha} f(x)) = f(x) - \sum_{k=0}^{n-1} f^{k}(0^{+}) \dfrac{x^k}{k!}, \quad x > 0$
(3) ${}^{c}D_0^{\alpha} I^{\alpha} f(x) = f(x), \quad x > 0, \quad n-1 < \alpha \leq n.$
(4) ${}^{c}D_0^{\alpha} C = 0, \quad C$ *is constant.*
(5) *If* $\beta < [\alpha]$, *then* ${}^{c}D_0^{\alpha} x^{\beta} = 0, \quad x > 0.$
(6) *If* $\beta > [\alpha]$, *then* ${}^{c}D_0^{\alpha} x^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \quad x > 0.$

3. Ultraspherical polynomials and some properties

The ultraspherical (Gegenbauer) polynomials with the real parameter $\left(\lambda > -\frac{1}{2}, \lambda \neq 0\right)$, are a sequence of polynomials $\{C_j^{(\lambda)}(x)\}_{j=0}^{\infty}$ defined on $[-1,1]$, such that,

$$\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}C_j^{(\lambda)}(x)C_k^{(\lambda)}(x)dx = \begin{cases} 0 & j \neq k \\ \psi_j^{(\lambda)} & j = k \end{cases} \tag{3.1}$$

where

$$\psi_j^{(\lambda)} = 2^{1-2\lambda}\pi\frac{\Gamma(j+2\lambda)}{(j+\lambda)\{\Gamma(\lambda)\}^2\Gamma(j+1)}, \quad \lambda \neq 0 \tag{3.2}$$

is the normalization constant [15]. The shifted ultraspherical polynomials are defined on $[0,1]$ by

$$\tilde{C}_n^{(\lambda)}(x) = C_n^{(\lambda)}(2x-1).$$

All results of ultraspherical polynomials can be easily obtained for their shifted. The orthogonality relation for $\tilde{C}_n^{(\lambda)}(x)$ with respect to the weight function $(x-x^2)^{\lambda-1/2}$ is given by

$$\int_{0}^{1}(x-x^2)^{\lambda-\frac{1}{2}}\tilde{C}_j^{(\lambda)}(x)\tilde{C}_k^{(\lambda)}(x)dx = \begin{cases} 0 & j \neq k \\ 4^{-\lambda}\psi_j^{(\lambda)} & j = k, \end{cases}$$

where $\psi_j^{(\lambda)}$ is given in (3.2)(see [4])

Theorem 3.1. *The integral of ultraspherical polynomials is expressed in terms of ultraspherical polynomials as follows* [9]*:*

$$I(x_i) = \int_{-1}^{x_i} C_j^{(\lambda)}(x)dx = \sum_{r=0}^{\left[\frac{1}{2}j\right]}\frac{1}{j-2r+1}G_r^{(j)}(\lambda)(x_i^{j-2r+1} - (-1)^{j-2r+1}) \tag{3.3}$$

In the remaining parts of this paper, we assume that $f(x)$ is a smooth continuous function and

$$S = \left\{x_i \middle| \quad x_i = \frac{i}{N}; \quad i = 0, \dots, N\right\}.$$

Theorem 3.2. *Let* $\frac{\varphi(x)}{(t-x)^{1-\alpha}}$ *$(t \in S, t \neq x)$ be approximated on S by ultraspherical polynomials. Then* [5],

$$\frac{\varphi(x)}{(t-x)^{1-\alpha}} \simeq \sum_{j=0}^{N} a_j\tilde{C}_j^{(\lambda)}(x), \quad x \in [0,1] \tag{3.4}$$

where

$$a_j = \sum_{k=0}^{N}\frac{2\theta_k}{N}\left(\psi_j^{(\lambda)}\right)^{-1}(x_k - x_k^2)^{\lambda-1/2}\tilde{C}_j^{(\lambda)}(x_k)\frac{\varphi(x_k)}{(t-x_k)^{1-\alpha}}. \tag{3.5}$$

with

$$\theta_0 = \theta_N = \frac{1}{2}, \quad \theta_k = 1, for \quad k = 1, 2, \ldots, N-1.$$

Theorem 3.3. *Let* $\frac{\phi(x)}{(t-x)^{1-\alpha}}$ *$(t \in S, t \neq x)$ be approximated by ultraspherical polynomial. Then there exists a matrix $Q = [q_{ij}], i, j = 0 \ldots N$, satisfying* [5]

$$\int_0^{x_i} \frac{\phi(x)}{(x_i - x)^{1-\alpha}} dx \simeq \sum_{\substack{k=0 \\ i \neq k}}^{N} \tilde{q}_{ik}(\lambda) \frac{\phi(x_k)}{(x_i - x_k)^{1-\alpha}}, \tag{3.6}$$

where

$$\tilde{q}_{ik}(\lambda) = \sum_{j=0}^{N} \sum_{r=0}^{[\frac{1}{2}j]} \frac{4^{\lambda} 2\theta_k \tilde{G}_r^j(\lambda)(\psi_j^{(\lambda)})^{-1}}{N(j-2r+1)} (x_k - x_k^2)^{\lambda - \frac{1}{2}} \tilde{C}_j^{(\lambda)}(x_k)\big((2x_i - 1)^{j-2r+1} - (-1)^{j-2r+1}\big), \tag{3.7}$$

for $x_i, x_k \in S$, *with* $\theta_0 = \theta_N = \frac{1}{2}, \quad \theta_k = 1 \quad$ *for* $\quad k = 1, 2, \ldots, N-1.$

4. Description of the method

In this section we present the ultraspherical spectral integration method for solving the problems (1.1) – (1.2). For this purpose we give ultraspherical integration matrix for the highest order fractional derivative in the problem (1.1), i.e. ,

$$\varphi(x) = {}^{c}D_0^{\alpha} y(x), \quad n - 1 < \alpha \leq n. \tag{4.1}$$

An application of the integral operator I^{α} to both sides of (5.1) and using the initial conditions (1.2) and part 3 of proposition, yield (for $a = 0$)

$$y(x) = \sum_{k=0}^{n-1} b_k \frac{x^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}} dt.$$

Thus for $x \in S$ and use of the Theorem 4, we get

$$y(x_i) \simeq \sum_{k=0}^{n-1} b_k \frac{x_i^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{\substack{k=0 \\ i \neq k}}^{N} \frac{\varphi(x_k)\tilde{q}_{ik}(\lambda)}{(x_i - x_k)^{1-\alpha}}. \tag{4.2}$$

Substituting from (4.1) and (4.2) into (1.1), for $0 \leq i \leq N$ it can be written as:

$$\varphi(x_i) \simeq F(x_i) \left(\sum_{k=0}^{n-1} b_k \frac{x_i^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{\substack{k=0 \\ i \neq k}}^{N} \frac{\varphi(x_k)\tilde{q}_{ik}(\lambda)}{(x_i - x_k)^{1-\alpha}} \right) + \sum_{j=0}^{N} K(x_i, x_j) y(x_j) \tilde{q}_{ij}(\lambda),$$

or

$$\varphi(x_i) - F(x_i) \left(\sum_{k=0}^{n-1} b_k \frac{x_i^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{\substack{k=0 \\ i \neq k}}^{N} \frac{\varphi(x_k)\tilde{q}_{ik}(\lambda)}{(x_i - x_k)^{1-\alpha}} \right) - \sum_{j=0}^{N} K(x_i, x_j) y(x_j) \tilde{q}_{ij}(\lambda) \simeq 0,$$

For a given value of λ, this is a linear system of equations for the unknown values $\varphi(x_0), \varphi(x_1), ..., \varphi(x_N)$ and it can be solved by a standard method.

Theorem 4.1. *Let* $f(x) = \dfrac{\varphi(x)}{(t-x)^{1-\alpha}}, (t \neq x, t \in S)$ *in* (4.9). *Then, there exists* $\xi \in [0,1]$ *such that* [5, 12]

$$\int_0^{x_i} \frac{\varphi(x)}{(x_i - x)^{1-\alpha}} dx = \sum_{\substack{k=0 \\ i \neq k}}^{N} \frac{\varphi(x_k)\tilde{q}_{ik}(\lambda)}{(x_i - x_k)^{1-\alpha}} + E_N^{(\lambda)}(x_i, \xi), \tag{4.3}$$

where $x_i, x_k \in S,\ 0 \leq i \leq N,$

$$E_N^{(\lambda)}(x_i, \xi) = \frac{f^{(N+1)}(\xi)}{(N+1)! K_{N+1}^{[\lambda]}} \int_0^{x_i} \tilde{C}_{N+1}^{[\lambda]}(x) dx - \frac{2}{3N^2} \sum_{j=0}^{N} [\psi_j^{(\lambda)}]^{-1} H_j^{(2)}(\xi) \int_0^{x_i} \tilde{C}_j^{(\lambda)}(x) dx \tag{4.4}$$

5. Numerical examples

In this section we give computational results of some examples, to support our theoretical results.

Example 5.1. Consider the linear fractional integro-differential equation

$${}^cD_0^{(\frac{3}{4})} y(x) + \frac{x^2 e^x}{5} y(x) = \int_0^x e^x t y(t) dt + \frac{6x^{2.25}}{\Gamma(3.25)} \tag{5.1}$$

with the initial condition $y(0) = 0$ and the exact solution $y(x) = x^3$.
Let $\phi(x) = {}^cD_0^{\frac{3}{4}} y$. Using ultraspherical polynomials 1, 2 takes the form,

$$\phi_i + \frac{x_i^2 e^{x_i}}{5\Gamma(\frac{3}{4})} \left[\sum_{\substack{k=0 \\ i \neq k}}^{N} \frac{\phi(x_k)}{(x_i - x_k)^{1/4}} \tilde{q}_{ik}(\lambda) \right] - \frac{6x_i^{2.25}}{\Gamma(3.25)}$$

$$- \frac{1}{\Gamma(3/4)} \sum_{k=0}^{N} e^{x_i} x_k \left[\sum_{\substack{k=0 \\ i \neq k}}^{N} \frac{\phi_j \tilde{q}_{kj}(\lambda)}{(x_k - x_j)^{1/4}} \right] \tilde{q}_{ik}(\lambda) \simeq 0$$

where $\tilde{q}_{ik}(\lambda)$ is defined by 3.7 for $x_i \in S, i = 0, 1, ..., N$.
For a given value of λ, this is a linear system of equations for the unknown values $\varphi(x_0), \varphi(x_1), ..., \varphi(x_N)$ and it can be solved by a standard method. The numerical results reported in Table 1 for $N = 10, N = 20,$, $x_i = \frac{i}{N}$ and $\lambda = 0.75$,.

Table 1. Absolute error of Examples 5.1

N	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
10	0	0	3×10^{-4}	1×10^{-4}	2.4×10^{-3}	4.6×10^{-3}	8×10^{-3}	1.27×10^{-2}	1.9×10^{-2}	2.7×10^{-2}
20	0	0	0	1×10^{-4}	3×10^{-4}	6×10^{-4}	1.7×10^{-3}	1×10^{-3}	1.6×10^{-3}	2.4×10^{-3}

Example 5.2. Consider

$$ {}^{c}D_0^{(\frac{8}{3})}y(x) + e^x y(x) = \int_0^x 6ty(t)dt - x^6 + x^4e^x + \frac{24x^{\frac{4}{3}}}{\Gamma\left(\frac{7}{3}\right)} \tag{5.2} $$

with the initial conditions
$y(0) = y'(0) = y''(0) = 0$
and the exact solution $y(x) = x^4$. We set

$$ {}^{c}D_0^{\frac{8}{3}}y(x) = \phi(x), $$

and use fractional integration and the initial conditions to get

$$ y(x) = \frac{1}{\Gamma\left(\frac{8}{3}\right)} \int_0^x \frac{\phi(t)}{(x-t)^{-\frac{5}{3}}}dt $$

by the Theorem (3.3) and substituting in (5.2) we get the system:

$$ \phi_i - \frac{1}{\Gamma\left(\frac{8}{3}\right)} \sum_{k=0}^{N} \frac{\phi_k \tilde{q}_{ik}}{(x_i - x_k)^{-5/3}} - \frac{1}{\Gamma\left(\frac{8}{3}\right)} \sum_{k=0}^{N} \tilde{q}_{ik}(\lambda)e^{x_k} \left(\int_0^{x_k} \frac{\phi(s)}{(x_k - s)^{-5/3}}ds\right) \simeq B_i $$

$$ \phi_i - \frac{1}{\Gamma\left(\frac{8}{3}\right)} \sum_{k=0}^{N} \frac{\phi_k \tilde{q}_{ik}(\lambda)}{(x_i - x_k)^{-5/3}} - \frac{1}{\Gamma\left(\frac{8}{3}\right)} \sum_{k=0}^{N} \tilde{q}_{ik}(\lambda)e^{x_k} \left(\sum_{j=0}^{N} \frac{\phi_i \tilde{q}_{kj}(\lambda)}{(x_k - x_j)^{-5/3}}\right) - B_i \simeq 0, $$

where

$$ B_i = -x_i^6 + x_i^4 e^{x_i} + \frac{24x_i^{\frac{4}{3}}}{\Gamma\left(\frac{7}{3}\right)}. $$

and $\tilde{q}_{ik}(\lambda)$ is defined by 3.7 for $x_i \in S, i = 0, 1, ..., N$. For a given value of λ, a linear system is solved for the unknown values $\varphi(x_0), \varphi(x_1), ..., \varphi(x_N)$. The numerical results reported in Table 2 for $\lambda = 0.55$, and $N = 10, N = 20$,.

TABLE 2. Absolute error of Examples 5.2

N	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
10	0	0	1×10^{-4}	1.5×10^{-4}	1.6×10^{-3}	3.9×10^{-3}	8.1×10^{-3}	1.5×10^{-2}	2.56×10^{-2}	2.57×10^{-2}
20	0	0	0	0	1×10^{-4}	2×10^{-4}	5×10^{-4}	9×10^{-4}	1.6×10^{-3}	2.6×10^{-3}

CONCLUSION

A numerical method base on the orthogonal ulteraspherical polynomials designed for solving fractional order integro-differential equations of Volterra types.

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