Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 8, No. 1, 2020, pp. 173-193
DOI:10.22034/cmde.2019.9621

A Study on Functional Fractional Integro-Differential Equations of Hammerstein type

Leila Saeedi
Department of Mathematics,
Shahed University, Tehran, Iran.
E-mail: l.saeedi62@gmail.com, L.saeedi@shahed.ac.ir

Abolfazl Tari*
Department of Mathematics,
Shahed University, Tehran, Iran.
E-mail: tari@shahed.ac.ir, at4932@gmail.com

Esmail Babolian
Department of Computer Science,
Kharazmi University, Tehran, Iran.
E-mail: Babolian@khu.ac.ir

Abstract In this paper, functional Hammerstein integro-differential equations of fractional order is studied. Here the existence and uniqueness of the solution is proved. A numerical method to approximate the solution of problem is also presented which is based on an improvement of the successive approximations method. Error estimation of the method is analyzed and error bound is obtained. The convergence and stability of the method are proved. At the end, application of the method is revealed by presenting some examples.

Keywords. Functional Hammerstein integrao-differential equations, Fractional order, Successive approximations, Spline interpolation, Trapezoidal quadrature rule.

2010 Mathematics Subject Classification. 65R20, 97N40.

1. Introduction

Fractional integro-differential equations arise in modeling many processes in applied sciences such as physics, chemistry, economy, electromagnetic, biology, engineering. Mathematical formulas of many phenomena such as oscillation of earthquake, fluid-dynamic traffic, statistical mechanics, astronomy, control theory and other areas of application contain integro-differential equations of fractional order [5, 6, 7, 8, 15, 16, 19, 31, 32].

On the other hand, solution of most of these equations can not be obtained analytically, so approximate methods should be used to solve them. For this type of equations, various methods have been used. For example, the Adomian decomposition method [14, 27, 28, 33], variational iteration method and homotopy perturbation method [2, 13, 23, 29, 37], Taylor expansion [17], wavelets [26, 34, 38, 39, 41],

Received: 9 October 2018 ; Accepted: 22 December 2018.
* corresponding author.

operational Tau method [21, 40], fractional differential transform method [4, 30], sinc-collocation method [3, 12], Laplace transform method [20, 22] and least squares method [25]. But, numerical solution of functional fractional integro-differential equations of Hammerstein type, have been studied in a few references.
In this paper, we consider functional fractional Hammerstein integro-differential equations (FFHIDEs) as

$$D^{\alpha}y(t) - \int_a^b K(t,s)F(s,y(s),y(\theta(s)))ds = g_1(t), \quad t \in [a,b] \tag{1.1}$$

with initial condition

$$y(a) = y_0, \tag{1.2}$$

where D^{α} denotes the Caputo fractional operator of order α, $0 < \alpha < 1$, $a, b \in \mathbb{R}$, $a < b$, $\theta : [a.b] \to [a,b]$, $\forall t \in [a,b]$, θ, $g_1 \in \mathbb{C}^1[a,b]$, and $K \in \mathbb{C}^1([a,b] \times [a,b])$.
Here, we combine the successive approximations method with the trapezoidal quadrature and natural cubic spline interpolation to solve the mentioned equations. Proving the convergence and numerical stability of the method only requires the Lipschitz properties.

2. Preliminary Results

2.1. **Preliminaries.** Assuming that $F \in \mathbb{C}^1([a,b] \times \mathbb{R} \times \mathbb{R})$, $K \in \mathbb{C}^1([a,b] \times [a,b])$, θ and $g \in \mathbb{C}^1[a,b]$ and $\theta : [a,b] \to [a,b]$. Consider the following conditions:

(i) there exist $\lambda,\ \mu \geq 0$ such that

$$|F(s,x_1,z_1) - F(s,x_2,z_2)| \leq \lambda |x_1 - x_2| + \mu |z_1 - z_2|, \tag{2.1}$$

for all $s \in [a,b]$, $(x_1,z_1),(x_2,z_2) \in \mathbb{R} \times \mathbb{R}$,

(ii)

$$2Q(b-a)^q(\lambda + \mu) < \Gamma(\alpha + 1) \tag{2.2}$$

where $Q = max\,\{|K(t,s)|, |\frac{\partial K(t,s)}{\partial t}| : (t,s) \in [a,b] \times [a,b]\}$ and
$(b-a)^q = max\,\{(b-a)^{\alpha+1}, (b-a)^{\alpha+2}\}$.

Let $F_0 : [a,b] \to \mathbb{R}$, $F_0(s) = F(s, g(s), g(\theta(s)))$, then F_0 is continuous on the compact set $[a,b]$ because F, g and θ are continuous and so there exists $M \geq 0$, such that $\mid F_0(s) \mid \leq M$ for all $s \in [a,b]$.

2.2. **Basic definitions of fractional calculus.** We recall the following definitions from [11]:

Definition 2.1. Let $\alpha \in \mathbb{R}_+$. The operator J_0^{α}, defined on the space $L_1[a,b]$ by

$$J_a^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x)dx, \qquad J_a^0 f(t) = f(t). \tag{2.3}$$

for $a \leq t \leq b$, is called the Riemann-Liouville fractional integral operator of order $\alpha > 0$.

Definition 2.2. The fractional derivative of f in the Caputo sense is defined by

$$D_a^\alpha f(t) = J_a^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{n-\alpha-1} \frac{d^n f(x)}{dx^n} dx, \qquad (2.4)$$
$$n-1 < \alpha \leq n, \; n \in \mathbb{N}, \; a \leq t \leq b, \; x > 0.$$

3. Existence and uniqueness of the solution

In this section, the existence and uniqueness of the solution of problem (1.1)-(1.2) are investigated.

Lemma 3.1. *Problem (1.1)-(1.2) is equivalent to the integral equation of the Hammerstein type*

$$y(t) = g(t) + \int_a^t H(t,s) F(s, y(s), y(\theta(s))) ds \qquad (3.1)$$

where

$$g(t) = y(a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} g_1(\tau) d\tau,$$
$$H(t,s) = \frac{1}{\Gamma(\alpha)} \int_a^b (t-\tau)^{\alpha-1} K(\tau, s) d\tau.$$

In other words, every solution of the integral equation (3.1) is a solution of problem (1.1)-(1.2), and vice versa.

Proof. Using the fractional integral operator on both sides of the equation (1.1) and by using of (1.2), we have

$$y(t) = y(a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} g_1(\tau) d\tau$$
$$+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \int_a^b K(\tau, s) F(s, y(s), y(\theta(s))) ds d\tau. \qquad (3.2)$$

By changing the order of integration in (3.2), equation (3.1) is obtained. □

Remark 3.2. If the functions g_1 and K are continuous on their domains, then g and H are also continuous.

Now we prove the existence and uniqueness of the solution by inspiration of [24].

Theorem 3.3. *If g and H are continuous functions on $[a, b]$ and $[a, b] \times [a, b]$ respectively, then the equation (3.1) has a unique continuous solution.*

Proof. We define the sequence $\{y_n(t)\}_{n=1}^{\infty}$ as follows:

$$y_n(t) = g(t) + \int_a^t H(t,s) F(s, y_{n-1}(s), y_{n-1}(\theta(s))) ds, \quad n = 1, 2, ..., \qquad (3.3)$$

where $y_0(t) = g(t)$. We introduce the sequence $\psi_n(t)$ as follows:

$$\psi_n(t) = y_n(t) - y_{n-1}(t), \quad n = 1, 2, ..., \qquad (3.4)$$

where $\psi_0(t) = g(t)$ and so we have

$$y_n(t) = \sum_{i=0}^{n} \psi_i(t). \tag{3.5}$$

By setting $H_1 = \max_{t,s\in[a,b]} |H(t,s)|$ and due to the $\theta : [a,b] \to [a,b]$, we have

$$\begin{aligned}
| \psi_n(t) | =| y_n(t) - y_{n-1}(t) | &\leq H_1 \int_a^t \lambda \mid y_{n-1}(s) - y_{n-2}(s) \mid ds \\
&+ H_1 \int_a^t \mu \mid y_{n-1}(\theta(s)) - y_{n-2}(\theta(s)) \mid ds \\
&\leq H_1(\lambda+\mu) \int_a^t \max_{s\in[a,b]} \mid y_{n-1}(s) - y_{n-2}(s) \mid ds \\
&= H_1(\lambda+\mu) \int_a^t \max_{s\in[a,b]} \mid \psi_{n-1}(s) \mid ds, \quad n = 1, 2, \ldots.
\end{aligned}$$

Now, by using induction, we prove the following inequality:

$$|\psi_n(t)| \leq \frac{G(H_1(\lambda+\mu)(t-a))^n}{n!}, \quad n = 0, 1, 2, ..., \tag{3.6}$$

where $G = \max_{t\in[a,b]} |g(t)|$. Obviously, the inequality holds for $n = 0$. Suppose that the inequality holds for $n-1$, we have:

$$\begin{aligned}
|\psi_n(t)| &\leq H_1(\lambda+\mu) \int_a^t \mid \psi_{n-1}(s) \mid ds \\
&\leq H_1(\lambda+\mu) \int_a^t \frac{G(H_1(\lambda+\mu)(s-a))^{n-1}}{(n-1)!} ds \\
&= \frac{G(H_1(\lambda+\mu))^n}{(n-1)!} \int_a^t (s-a)^{n-1} ds \\
&= \frac{G(H_1(\lambda+\mu)(t-a))^n}{n!}, \; n = 1, 2, \ldots.
\end{aligned}$$

Therefore, the sequence $y_n(t)$ in (3.5) is uniformly convergent and can be written as:

$$y(t) = \sum_{i=0}^{\infty} \psi_i(t). \tag{3.7}$$

Now, we show that the continuous function $y(t)$ in (3.7) satisfies the equation (3.1). For this purpose, we put

$$y(t) = y_n(t) + \Delta_n(t), \tag{3.8}$$

by using (3.3), we have

$$\begin{aligned}
y(t) - \Delta_n(t) = g(t) + \int_a^t & H(t,s).F(s, y(s) - \Delta_{n-1}(s), y(\theta(s)) \\
&- \Delta_{n-1}(\theta(s)))ds,
\end{aligned} \tag{3.9}$$

so that

$$y(t)-g(t) - \int_a^t H(t,s).F(s,y(s),y(\theta(s)))ds = \Delta_n(t) + \int_a^t H(t,s)$$
$$.\left[F(s,y(s)-\Delta_{n-1}(s),y(\theta(s))-\Delta_{n-1}(\theta(s))) - F(s,y(s),y(\theta(s)))\right]ds.$$

By applying (2.1), we have

$$\begin{aligned}
&\mid y(t)-g(t)-\int_a^t H(t,s).F(s,y(s),y(\theta(s)))ds \mid \\
&\leq \mid \Delta_n(t) \mid + H_1\left[\lambda \int_a^t \mid \Delta_{n-1}(s) \mid ds + \mu \int_a^t \mid \Delta_{n-1}(\theta(s)) \mid\right] \\
&\leq \mid \Delta_n(t) \mid + H_1(\lambda+\mu) \int_a^t \max_{s\in[a,b]} \mid \Delta_{n-1}(s) \mid ds \\
&\leq \mid \Delta_n(t) \mid + H_1(t-a)(\lambda+\mu) \parallel \Delta_{n-1} \parallel,
\end{aligned}$$

where $\parallel \Delta_{n-1} \parallel = \max_{a\leq s\leq b} \mid \Delta_{n-1}(s) \mid$. On the other hand $\lim_{n\to\infty} \mid \Delta_{n-1} \mid = 0$, Therefore, for sufficiently large values of n, the right-hand side of the inequality can be made as small as desired, and this concludes that the function $y(t)$ defined in the (3.7) satisfies the equation (3.1), and therefore is a solution of (3.1).
Now, to show uniqueness, we assume that $x(t)$ is another continuous solution of the equation (3.1), then

$$\mid x(t)-y(t) \mid \leq H_1 \int_a^t \lambda \mid x(s)-y(s) \mid ds + H_1 \int_a^t \mu \mid x(\theta(s)) - y(\theta(s)) \mid ds,$$

and by putting $B = \max_{t\in[a,b]} |x(t)-y(t)|$, we have

$$\mid x(t)-y(t) \mid \leq H_1(\lambda+\mu) \int_a^t \max_{s\in[a,b]} \mid x(s)-y(s) \mid ds \leq H_1 B(t-a)(\lambda+\mu).$$

Repeating the above process, leads to

$$\mid x(t)-y(t) \mid \leq \frac{B\left[H_1(t-a)(\lambda+\mu)\right]^n}{n!},$$

which yields $x(t)=y(t)$ when $n\to\infty$. □

4. Description of the method

To describe the method, by using integration by parts to Eq. (3.2), we obtain

$$\begin{aligned}
y(t) = g(t) &+ \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} \int_a^b K(a,s)F(s,y(s),y(\theta(s)))ds \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_a^t \int_a^b (t-\tau)^\alpha \frac{\partial K(\tau,s)}{\partial\tau} F(s,y(s),y(\theta(s)))dsd\tau \qquad (4.1)
\end{aligned}$$

The sequence of successive approximations for (4.1) is defined as

$$y_0(t) = g(t),$$

$$\begin{aligned} y_m(t) = g(t) &+ \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} \int_a^b K(a,s) F(s, y_{m-1}(s), y_{m-1}(\theta(s))) ds \\ &+ \frac{1}{\Gamma(\alpha+1)} \int_a^t \int_a^b (t-\tau)^\alpha \frac{\partial K(\tau,s)}{\partial \tau} F(s, y_{m-1}(s), y_{m-1}(\theta(s))) ds d\tau, \\ m \in \mathbb{N}, \end{aligned} \tag{4.2}$$

which can be considered as approximations of the exact solution of the equation (3.2). We need to compute the integrals in (4.2) by quadrature rules. For this purpose, consider the uniform partition of $[a, b]$:

$$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b \tag{4.3}$$

with $t_i = a + ih$, $i = 0, \ldots, n$, where $h = \frac{b-a}{n}$ and assume that f is a given Liepschitz function, i.e.

$$| f(s) - f(s') | \leq L \mid s - s' \mid, \tag{4.4}$$

where $L > 0$. Then the trapezoidal quadrature rule with an error estimate is [10]:

$$\int_a^b f(s) ds = \frac{h}{2} \cdot \sum_{j=1}^{n} \Big[f(t_{j-1}) + f(t_j) \Big] + e_n(f), \tag{4.5}$$

where

$$|e_n(f)| \leq \frac{(b-a)^2 L}{4n}, \tag{4.6}$$

and for bivariate function f with Lipschitz properties

$$| f(t', s) - f(t, s) | \leq L_1 \mid t - t' \mid, \ \ | f(t, s) - f(t, s') | \leq L_2 \mid s - s' \mid \tag{4.7}$$

where $L_1, L_2 > 0$, is

$$\begin{aligned} \int_a^b \int_a^b f(t,s) dt ds = \frac{h^2}{4} \cdot \sum_{k=1}^{i} \sum_{j=1}^{n} \big[& f(t_k, t_{j-1}) + f(t_k, t_j) + f(t_{k-1}, t_{j-1}) \\ & + f(t_{k-1}, t_j) \big] + r_n(f) \end{aligned} \tag{4.8}$$

where

$$|r_n(f)| \leq \frac{(b-a)^2}{4n} \left(L_2 + L_1 (b-a) \right). \tag{4.9}$$

By setting $t = t_i$ and using the trapezoidal rule (4.5) and (4.8) in (4.2), we have

$$y_0(t_i) = g(t_i), \ \ i = \overline{0, n} \tag{4.10}$$

$$\begin{aligned} y_1(t_i) = g(t_i) &+ \frac{(t_i - a)^\alpha}{\Gamma(\alpha+1)} \bigg(\frac{h}{2} \sum_{j=1}^{n} \big[K(a, t_{j-1}) F(t_{j-1}, g(t_{j-1}), g(\theta(t_{j-1}))) \\ &+ K(a, t_j) \cdot F(t_j, g(t_j), g(\theta(t_j))) \big] \bigg) \\ &+ \frac{1}{\Gamma(\alpha+1)} \bigg(\frac{h^2}{4} \sum_{k=1}^{i} \sum_{j=1}^{n} \big[(t_i - t_k)^\alpha \frac{\partial K(t_k, t_j)}{\partial \tau} F(t_j, g(t_j), g(\theta(t_j))) \end{aligned}$$

$$
\begin{aligned}
&+ (t_i - t_k)^\alpha \frac{\partial K(t_k, t_{j-1})}{\partial \tau} F(t_{j-1}, g(t_{j-1}), g(\theta(t_{j-1}))) \\
&+ (t_i - t_{k-1})^\alpha \cdot \frac{\partial K(t_{k-1}, t_j)}{\partial \tau} F(t_j, g(t_j), g(\theta(t_j))) \\
&+ (t_i - t_{k-1})^\alpha \frac{\partial K(t_{k-1}, t_{j-1})}{\partial \tau} F(t_{j-1}, g(t_{j-1}), g(\theta(t_{j-1})))\big]\Big) \\
&+ \underbrace{\frac{(t_i - a)^\alpha}{\Gamma(\alpha+1)} e_{1,i} + \frac{1}{\Gamma(\alpha+1)} r_{1,i}}_{R_{1,i}} = \overline{y_1(t_i)} + R_{1,i}, \;\; i = \overline{0,n}
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
y_2(t_i) =& g(t_i) + \frac{(t_i - a)^\alpha}{\Gamma(\alpha+1)} \Big(\frac{h}{2} \sum_{j=1}^{n} \big[K(a, t_{j-1}) F(t_{j-1}, \overline{y_1(t_{j-1})} \\
&+ R_{1,j-1}, y_1(\theta(t_{j-1}))) + K(a, t_j) F(t_j, \overline{y_1(t_j)} + R_{1,j}, y_1(\theta(t_j)))\big] \Big) \\
&+ \frac{1}{\Gamma(\alpha+1)} \Big(\frac{h^2}{4} \sum_{k=1}^{i} \sum_{j=1}^{n} \big[(t_i - t_k)^\alpha \cdot \frac{\partial K(t_k, t_j)}{\partial \tau} F(t_j, \overline{y_1(t_j)} + R_{1,j}, y_1(\theta(t_j))) \\
&+ (t_i - t_k)^\alpha \frac{\partial K(t_k, t_{j-1})}{\partial \tau} \cdot F(t_{j-1}, \overline{y_1(t_{j-1})} + R_{1,j-1}, y_1(\theta(t_{j-1}))) \\
&+ (t_i - t_{k-1})^\alpha \frac{\partial K(t_{k-1}, t_j)}{\partial \tau} \cdot F(t_j, \overline{y_1(t_j)} + R_{1,j}, y_1(\theta(t_j))) \\
&+ (t_i - t_{k-1})^\alpha \frac{\partial K(t_{k-1}, t_{j-1})}{\partial \tau} \cdot F(t_{j-1}, \overline{y_1(t_{j-1})} + R_{1,j-1}, y_1(\theta(t_{j-1})))\big] \Big) \\
&+ \underbrace{\frac{(t_i - a)^\alpha}{\Gamma(\alpha+1)} e_{2,i} + \frac{1}{\Gamma(\alpha+1)} r_{2,i}}_{R_{2,i}}
\end{aligned} \tag{4.12}
$$

Replacing y_1 by s_1 (where s_1 is the natural cubic spline interpolation of y_1) which is introduced below in (4.16), we obtain

$$
\begin{aligned}
y_2(t_i) = g(t_i) &+ \frac{(t_i - a)^\alpha}{\Gamma(\alpha+1)} \Big(\frac{h}{2} \sum_{j=1}^{n} \big[K(a, t_{j-1}) F(t_{j-1}, \overline{y_1(t_{j-1})}, s_1(\theta(t_{j-1}))) \\
&+ K(a, t_j) \cdot F(t_j, \overline{y_1(t_j)}, s_1(\theta(t_j)))\big] \Big) \\
&+ \frac{1}{\Gamma(\alpha+1)} \Big(\frac{h^2}{4} \sum_{k=1}^{i} \sum_{j=1}^{n} \big[(t_i - t_k)^\alpha \frac{\partial K(t_k, t_j)}{\partial \tau} \cdot F(t_j, \overline{y_1(t_j)}, s_1(\theta(t_j))) \\
&+ (t_i - t_k)^\alpha \frac{\partial K(t_k, t_{j-1})}{\partial \tau} F(t_{j-1}, \overline{y_1(t_{j-1})}, s_1(\theta(t_{j-1}))) \\
&+ (t_i - t_{k-1})^\alpha \frac{\partial K(t_{k-1}, t_j)}{\partial \tau} F(t_j, \overline{y_1(t_j)}, s_1(\theta(t_j)))
\end{aligned}
$$

$$+ (t_i - t_{k-1})^{\alpha} \frac{\partial K(t_{k-1}, t_{j-1})}{\partial \tau} \cdot F(t_{j-1}, \overline{y_1(t_{j-1})}, s_1(\theta(t_{j-1})))\big]\Big) + \overline{R}_{2,i}$$

$$= \overline{y_2(t_i)} + \overline{R}_{2,i}, \;\; i = \overline{0,n} \tag{4.13}$$

and for $m \geq 3$, we have

$$\begin{aligned}
y_m(t_i) &= g(t_i) + \frac{(t_i - a)^{\alpha}}{\Gamma(\alpha+1)} \Big(\frac{h}{2} \sum_{j=1}^{n} \big[K(a, t_{j-1}) \\
&\cdot F(t_{j-1}, \overline{y_{m-1}(t_{j-1})} + \overline{R}_{m-1,j-1}, y_{m-1}(\theta(t_{j-1}))) \\
&+ K(a, t_j) F(t_j, \overline{y_{m-1}(t_j)} + \overline{R}_{m-1,j}, y_{m-1}(\theta(t_j)))\big]\Big) + \frac{1}{\Gamma(\alpha+1)} \\
&\cdot \Big(\frac{h^2}{4} \sum_{k=1}^{i} \sum_{j=1}^{n} \big[(t_i - t_k)^{\alpha} \frac{\partial K(t_k, t_j)}{\partial \tau} F(t_j, \overline{y_{m-1}(t_j)} + \overline{R}_{m-1,j}, y_{m-1}(\theta(t_j))) \\
&+ (t_i - t_k)^{\alpha} \frac{\partial K(t_k, t_{j-1})}{\partial \tau} F(t_{j-1}, \overline{y_{m-1}(t_{j-1})} + \overline{R}_{m-1,j-1}, y_{m-1}(\theta(t_{j-1}))) \\
&+ (t_i - t_{k-1})^{\alpha} \frac{\partial K(t_{k-1}, t_j)}{\partial \tau} F(t_j, \overline{y_{m-1}(t_j)} + \overline{R}_{m-1,j}, y_{m-1}(\theta(t_j))) \\
&+ (t_i - t_{k-1})^{\alpha} \cdot \frac{\partial K(t_{k-1}, t_{j-1})}{\partial \tau} F(t_{j-1}, \overline{y_{m-1}(t_{j-1})} \\
&+ \overline{R}_{m-1,j-1}, y_{m-1}(\theta(t_{j-1})))\big]\Big) + \underbrace{\frac{(t_i - a)^{\alpha}}{\Gamma(\alpha+1)} e_{m,i} + \frac{1}{\Gamma(\alpha+1)} r_{m,i}}_{R_{m,i}}
\end{aligned} \tag{4.14}$$

and replacing $y_{m-1}(t_i)$ with $s_{m-1}(t_i)$, for $m \geq 3$ and $i = \overline{0,n}$, yields

$$\begin{aligned}
y_m(t_i) &= g(t_i) + \frac{(t_i - a)^{\alpha}}{\Gamma(\alpha+1)} \Big(\frac{h}{2} \sum_{j=1}^{n} \big[K(a, t_j) F(t_j, \overline{y_{m-1}(t_j)}, s_{m-1}(\theta(t_j))) \\
&+ K(a, t_{j-1}) \cdot F(t_{j-1}, \overline{y_{m-1}(t_{j-1})}, s_{m-1}(\theta(t_{j-1})))\big]\Big) \\
&+ \frac{1}{\Gamma(\alpha+1)} \Big(\frac{h^2}{4} \sum_{k=1}^{i} \sum_{j=1}^{n} \big[(t_i - t_k)^{\alpha} \frac{\partial K(t_k, t_j)}{\partial \tau} F(t_j, \overline{y_{m-1}(t_j)}, s_{m-1}(\theta(t_j))) \\
&+ (t_i - t_k)^{\alpha} \frac{\partial K(t_k, t_{j-1})}{\partial \tau} F(t_{j-1}, \overline{y_{m-1}(t_{j-1})}, s_{m-1}(\theta(t_{j-1}))) \\
&+ (t_i - t_{k-1})^{\alpha} \frac{\partial K(t_{k-1}, t_j)}{\partial \tau} F(t_j, \overline{y_{m-1}(t_j)}, s_{m-1}(\theta(t_j))) \\
&+ (t_i - t_{k-1})^{\alpha} \frac{\partial K(t_{k-1}, t_{j-1})}{\partial \tau} F(t_{j-1}, \overline{y_{m-1}(t_{j-1})}, s_{m-1}(\theta(t_{j-1})))\big]\Big) \\
&+ \overline{R_{m,i}} = \overline{y_m(t_i)} + \overline{R}_{m,i}, \;\; i = \overline{0,n}
\end{aligned} \tag{4.15}$$

where $s_{m-1}(t_i)$ is cubic spline interpolation for $\overline{y_{m-1}(t_i)}$, where its restriction on interval $[t_{i-1}, t_i], i = \overline{1,n}$ is as follows [1, 18]:

$$s_{m-1}^{(i)}(t) = \left[\frac{(t-t_{i-1})^2}{2} - \frac{(t-t_{i-1})^3}{6h} - \frac{h(t-t_{i-1})}{3}\right] \cdot M_{m-1}^{(i-1)} + \left[\frac{(t-t_{i-1})^3}{6h}\right.$$
$$\left. - \frac{h(t-t_{i-1})}{6}\right] \cdot M_{m-1}^{(i)} + \frac{t-t_{i-1}}{h} \cdot \overline{y_{m-1}}(t_i) + \frac{t_i - t}{h} \cdot \overline{y_{m-1}}(t_{i-1}), \tag{4.16}$$

with $M_{m-1}^{(0)} = M_{m-1}^{(n)} = 0$. And to compute $M_{m-1}^{(i)},\ i = \overline{1, n-1}$ we use the following algorithm:

$$a_i = 2,\ b_i = c_i = \frac{1}{2},$$
$$d_i = \frac{3}{h^2} \cdot [\overline{y_{m-1}}(t_{i+1}) - 2\overline{y_{m-1}}(t_i) + \overline{y_{m-1}}(t_{i-1})],\ i = \overline{1, n-1}$$
$$\alpha_1 = \frac{c_1}{a_1}, \quad \omega_i = a_i - \alpha_{i-1} \cdot b_i, \quad \alpha_i = \frac{c_i}{\omega_i},\ i = \overline{2, n-2},$$
$$\omega_{n-1} = a_{n-1} - \alpha_{n-2} \cdot b_{n-1} \quad z_1 = \frac{d_1}{2}, \quad z_i = \frac{d_i - b_i \cdot z_{i-1}}{\omega_i},\ i = \overline{2, n-1}$$

and by using of the backward recurrence, we have

$$M_{m-1}^{(n-1)} = z_{n-1}, \qquad M_{m-1}^{(i)} = z_i - \alpha_i \cdot M_{m-1}^{(i+1)}, \quad i = \overline{n-2, 1}.$$

Lemma 4.1. [9] *If $f : [a, b] \to \mathbb{R}$ is a uniformly continuous function and $s \in \mathbb{C}^2[a, b]$ is the cubic spline of interpolation generated with natural boundary conditions $s''(a) = s''(b) = 0$, such that $s(t_i) = f(t_i) = f_i,\ i = \overline{0, n}$, then the following error estimation holds:*

$$\max_{t \in [a,b]} \mid s(t) - f(t) \mid \leq \frac{7}{4}\omega(f, h), \tag{4.17}$$

where $\omega(f, h) = sup\,\{\mid f(t) - f(t') \mid : t, t' \in [a, b], \mid t - t' \mid \leq h\}$ is the uniform modulus of continuity.

5. Error bound and convergence analysis

In this section, we analyze error bound and convergence of the method presented in the previous section.

Theorem 5.1. *Suppose that the y^* be the exact solution of the equation (3.2), and consider the sequence of successive approximations (4.2). Then the following error estimation holds*

$$|y^*(t) - y_m(t)| \leq \frac{\left[\frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}\right]^m}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}} \cdot \frac{2(b-a)^q QM}{\Gamma(\alpha+1)},\ \forall t \in [a, b]. \tag{5.1}$$

Proof. By using of (4.2), we have

$$|y_1(t) - y_0(t)| \leq \frac{|(t-a)^\alpha|}{\Gamma(\alpha+1)} \int_a^b |K(a, s)| \cdot |F(s, y_0(s), y_0(\theta(s))|ds$$

$$+\frac{1}{\Gamma(\alpha+1)}\int_a^t\int_a^b|(t-\tau)^\alpha|\cdot|\frac{\partial K(\tau,s)}{\partial\tau}|\cdot|F(s,y_0(s),y_0(\theta(s))|dsd\tau$$
$$\leq\frac{2(b-a)^q}{\Gamma(\alpha+1)}QM, \tag{5.2}$$

and

$$|y_2(t)-y_1(t)|\leq\frac{|(t-a)^\alpha|}{\Gamma(\alpha+1)}\int_a^b|K(a,s)|\cdot|F(s,y_1(s),y_1(\theta(s)))$$
$$-F(s,y_0(s),y_0(\theta(s)))|ds+\frac{1}{\Gamma(\alpha+1)}\int_a^t\int_a^b|(t-\tau)^\alpha|$$
$$\cdot|\frac{\partial K(\tau,s)}{\partial\tau}|\cdot|F(s,y_1(s),y_1(\theta(s)))-F(s,y_0(s),y_0(\theta(s)))|dsd\tau$$
$$\leq\frac{2(b-a)^qQ(\lambda+\mu)}{\Gamma(\alpha+1)}\cdot\frac{2(b-a)^q}{\Gamma(\alpha+1)}QM,$$

by repeating the above process, we obtain

$$|y_m(t)-y_{m-1}(t)|\leq\left[\frac{2(b-a)^qQ(\lambda+\mu)}{\Gamma(\alpha+1)}\right]^{m-1}\cdot\frac{2(b-a)^q}{\Gamma(\alpha+1)}QM. \tag{5.3}$$

Now assume $n>m$, we have

$$|y_n(t)-y_m(t)|\leq|y_n(t)-y_{n-1}(t)|+|y_{n-1}(t)-y_{n-2}(t)|$$
$$+\cdots+|y_{m+1}(t)-y_m(t)|$$

and by using of (5.3), we obtain

$$|y_n(t)-y_m(t)|\leq\left[\frac{2(b-a)^qQ(\lambda+\mu)}{\Gamma(\alpha+1)}\right]^{n-1}\cdot\frac{2(b-a)^q}{\Gamma(\alpha+1)}QM+\frac{2(b-a)^q}{\Gamma(\alpha+1)}QM$$
$$\cdot\left[\frac{2(b-a)^qQ(\lambda+\mu)}{\Gamma(\alpha+1)}\right]^{n-2}+\cdots+\left[\frac{2(b-a)^qQ(\lambda+\mu)}{\Gamma(\alpha+1)}\right]^{m}\cdot\frac{2(b-a)^q}{\Gamma(\alpha+1)}QM$$
$$=\left(1+\frac{2(b-a)^qQ(\lambda+\mu)}{\Gamma(\alpha+1)}+\ldots+\left[\frac{2(b-a)^qQ(\lambda+\mu)}{\Gamma(\alpha+1)}\right]^{n-1-m}\right)$$
$$\cdot\left[\frac{2(b-a)^qQ(\lambda+\mu)}{\Gamma(\alpha+1)}\right]^{m}\cdot\frac{2(b-a)^q}{\Gamma(\alpha+1)}QM$$
$$=\frac{1-\left[\frac{2(b-a)^qQ(\lambda+\mu)}{\Gamma(\alpha+1)}\right]^{n-m}}{1-\frac{2(b-a)^qQ(\lambda+\mu)}{\Gamma(\alpha+1)}}\cdot\left[\frac{2(b-a)^qQ(\lambda+\mu)}{\Gamma(\alpha+1)}\right]^{m}\cdot\frac{2(b-a)^q}{\Gamma(\alpha+1)}QM,$$

therefore

$$|y_n(t)-y_m(t)|\leq\frac{1-\left[\frac{2(b-a)^qQ(\lambda+\mu)}{\Gamma(\alpha+1)}\right]^{n-m}}{1-\frac{2(b-a)^qQ(\lambda+\mu)}{\Gamma(\alpha+1)}}\cdot\left[\frac{2(b-a)^qQ(\lambda+\mu)}{\Gamma(\alpha+1)}\right]^{m}$$
$$\cdot\frac{2(b-a)^q}{\Gamma(\alpha+1)}QM.$$

Now if $n \to \infty$ and by using (2.2), we obtain

$$|y^*(t) - y_m(t)| \leq \frac{\left[\frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}\right]^m}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}} \cdot \frac{2(b-a)^q}{\Gamma(\alpha+1)} QM,$$

which is the desired error bound. □

Now, we give the following theorem for convergence analysis:

Theorem 5.2. *Under conditions (2.1) and (2.2), the sequence $(\overline{y_m(t_i)})_{m\in\mathbb{N}_0}$ approximates the solution $y^*(t_i)$ at the points $t_i = a + ih$, $i = \overline{0, n}$ with the following error:*

$$\begin{aligned} | y^*(t_i) - \overline{y_m(t_i)} | \leq & \frac{\frac{1}{4n\Gamma(\alpha+1)}\left[(b-a)^{\alpha+2}L + (b-a)^3 L_1 + (b-a)^2 L_2\right]}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}} \\ & + \frac{\frac{2(b-a)^q Q\mu}{\Gamma(\alpha+1)} \cdot \omega(V, h)}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}} + \frac{\left[\frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}\right]^m}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}} \cdot \frac{2(b-a)^q}{\Gamma(\alpha+1)} QM, \ m \in \mathbb{N}_0, \quad (5.4) \end{aligned}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and V_{m-1} is defined in (5.6).

Proof. By using the algorithm of the method, we have

$$\begin{aligned} | y^*(t_i) - \overline{y_m(t_i)} | &\leq | y^*(t_i) - y_m(t_i) | + | y_m(t_i) - \overline{y_m(t_i)} | \\ &= | y^*(t_i) - y_m(t_i) | + | \overline{R_{m,i}} |, \ \forall m \in \mathbb{N}_0, \ i = \overline{0, n}, \quad (5.5) \end{aligned}$$

From (5.1), we have

$$|y^*(t) - y_m(t)| \leq \frac{\left[\frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}\right]^m}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}} \cdot \frac{2(b-a)^q}{\Gamma(\alpha+1)} QM.$$

Therefore, it is enough, we find a bound for $| \overline{R_{m,i}} |$. Since $y_m(t_i) \neq \overline{y_m(t_i)}$, $\forall m \in \mathbb{N}_0$, $i = \overline{0, n}$, so s_m which interpolates the values $\overline{y_m(t_i)}$, $i = \overline{0, n}$, does not interpolate the values of $y_m(t_i)$. Thus, for every m, the function $V_m : [a, b] \to \mathbb{R}$, $m \in \mathbb{N}_0$ by its restriction on the subintervals $[t_{i-1}, t_i]$, $i = \overline{1, n}$ is defined as

$$\begin{aligned} V_m(t) =& y_m(t) + \left[\overline{y_m(t_i)} - y_m(t_i)\right] . \frac{t - t_{i-1}}{h} \\ &+ \left[\overline{y_m(t_{i-1})} - y_m(t_{i-1})\right] . \frac{t_i - t}{h}. \quad (5.6) \end{aligned}$$

We see that $V_m(t_i) = \overline{y_m(t_i)}$, $\forall i = \overline{0, n}$, that is, V_m interpolate the values of $\overline{y_m(t_i)}$, $i = \overline{0, n}$, and this function is continuous. Therefore, s_m interpolates the function V_m for every $m \in \mathbb{N}_0$ and V_m is uniformly continuous on the compact interval $[a, b]$. So V_m satisfies in the conditions of Lemma 4.1 and by using (4.17) we obtain

$$| V_m(t) - s_m(t) | \leq \frac{7\omega(V_m, h)}{4}, \ \forall t \in [a, b], \ \forall m \in \mathbb{N}. \quad (5.7)$$

Now, by using the algorithm of the method, we obtain

$$
\begin{aligned}
&\mid \overline{R_{2,i}} \mid = \mid y_2(t_i) - \overline{y_2(t_i)} \mid \leq \mid R_{2,i} \mid + \frac{|(t_i-a)^\alpha|}{\Gamma(\alpha+1)} \cdot \frac{(b-a)}{2n} \sum_{j=1}^{n} \Big[\mid K(a,t_{j-1}) \mid \\
&. \mid F(t_{j-1}, \overline{y_1(t_{j-1})} + R_{1,j-1}, y_1(\theta(t_{j-1}))) - F(t_{j-1}, \overline{y_1(t_{j-1})}, s_1(\theta(t_{j-1}))) \mid \\
&+ \mid K(a,t_j) \mid . \mid F(t_j, \overline{y_1(t_j)} + R_{1,j}, y_1(\theta(t_j))) - F(t_j, \overline{y_1(t_j)}, s_1(\theta(t_j))) \mid \Big] \\
&+ \frac{1}{\Gamma(\alpha+1)} \cdot \frac{(b-a)^2}{4n^2} \sum_{k=1}^{i} \sum_{j=1}^{n} |(t_i-t_k)^\alpha| \cdot |\frac{\partial K(t_k,t_j)}{\partial \tau}| \cdot \mid F(t_j, \overline{y_1(t_j)} \\
&+ R_{1,j}, y_1(\theta(t_j))) - F(t_j, \overline{y_1(t_j)}, s_1(\theta(t_j))) \mid + |(t_i-t_k)^\alpha| \cdot |\frac{\partial K(t_k,t_{j-1})}{\partial \tau}| \\
&\cdot \mid F(t_{j-1}, \overline{y_1(t_{j-1})} + R_{1,j-1}, y_1(\theta(t_{j-1}))) - F(t_{j-1}, \overline{y_1(t_{j-1})}, s_1(\theta(t_{j-1}))) \mid \\
&+ |(t_i-t_{k-1})^\alpha| \cdot |\frac{\partial K(t_{k-1},t_j)}{\partial \tau}| \cdot \mid F(t_j, \overline{y_1(t_j)} + R_{1,j}, y_1(\theta(t_j))) \\
&- F(t_j, \overline{y_1(t_j)}, s_1(\theta(t_j))) \mid + |(t_i-t_{k-1})^\alpha| \cdot |\frac{\partial K(t_{k-1},t_{j-1})}{\partial \tau}| \\
&\cdot \mid F(t_{j-1}, \overline{y_1(t_{j-1})} + R_{1,j-1}, y_1(\theta(t_{j-1}))) \\
&- F(t_{j-1}, \overline{y_1(t_{j-1})}, s_1(\theta(t_{j-1}))) \mid
\end{aligned} \tag{5.8}
$$

and for $m \geq 3$ we obtain

$$
\begin{aligned}
&\mid \overline{R_{m,i}} \mid = \mid y_m(t_i) - \overline{y_m(t_i)} \mid \leq \mid R_{m,i} \mid + \frac{|(t_i-a)^\alpha|}{\Gamma(\alpha+1)} \cdot \frac{(b-a)}{2n} \sum_{j=1}^{n} \Big[\mid K(a,t_{j-1}) \mid \\
&. \mid F(t_{j-1}, \overline{y_{m-1}(t_{j-1})} + \overline{R}_{m-1,j-1}, y_{m-1}(\theta(t_{j-1}))) \\
&- F(t_{j-1}, \overline{y_{m-1}(t_{j-1})}, s_{m-1}(\theta(t_{j-1}))) \mid + \mid K(a,t_j) \mid \\
&. \mid F(t_j, \overline{y_{m-1}(t_j)} + \overline{R}_{m-1,j}, y_{m-1}(\theta(t_j))) - F(t_j, \overline{y_{m-1}(t_j)}, s_{m-1}(\theta(t_j))) \mid \Big] \\
&+ \frac{1}{\Gamma(\alpha+1)} \cdot \frac{(b-a)^2}{4n^2} \sum_{k=1}^{i} \sum_{j=1}^{n} |(t_i-t_k)^\alpha| \cdot |\frac{\partial K(t_k,t_j)}{\partial \tau}| \\
&\cdot \mid F(t_j, \overline{y_{m-1}(t_j)} + \overline{R}_{m-1,j}, y_{m-1}(\theta(t_j))) - F(t_j, \overline{y_{m-1}(t_j)}, s_{m-1}(\theta(t_j))) \mid \\
&+ |(t_i-t_k)^\alpha| \cdot |\frac{\partial K(t_k,t_{j-1})}{\partial \tau}| \\
&\cdot \mid F(t_{j-1}, \overline{y_{m-1}(t_{j-1})} + \overline{R}_{m-1,j-1}, y_{m-1}(\theta(t_{j-1}))) \\
&- F(t_{j-1}, \overline{y_{m-1}(t_{j-1})}, s_{m-1}(\theta(t_{j-1}))) \mid + |(t_i-t_{k-1})^\alpha| \cdot |\frac{\partial K(t_{k-1},t_j)}{\partial \tau}| \\
&\cdot \mid F(t_j, \overline{y_{m-1}(t_j)} + \overline{R}_{m-1,j}, y_{m-1}(\theta(t_j))) - F(t_j, \overline{y_{m-1}(t_j)}, s_{m-1}(\theta(t_j))) \mid \\
&+ |(t_i-t_{k-1})^\alpha| \cdot |\frac{\partial K(t_{k-1},t_{j-1})}{\partial \tau}| \cdot \mid F(t_{j-1}, \overline{y_{m-1}(t_{j-1})}
\end{aligned}
$$

$$+\overline{R}_{m-1,j-1}, y_{m-1}(\theta(t_{j-1}))) - F(t_{j-1}, \overline{y_{m-1}(t_{j-1})}, s_{m-1}(\theta(t_{j-1}))) \mid .$$

Here we need to obtain an estimate for $\mid y_{m-1}(t) - s_{m-1}(t) \mid$. For this purpose we have

$$\begin{aligned}
&\mid y_{m-1}(t) - s_{m-1}(t) \mid \leq \mid y_{m-1}(t) - V_{m-1}(t) \mid + \mid V_{m-1}(t) - s_{m-1}(t) \mid \\
&\quad \leq \mid \frac{t-t_{i-1}}{h} \mid . \mid \overline{R_{m-1,i}} \mid + \mid \frac{t_i - t}{h} \mid . \mid \overline{R_{m-1,i-1}} \mid + \mid V_{m-1}(t) - s_{m-1}(t) \mid \\
&\quad \leq max(\overline{R_{m-1,i-1}}, \overline{R_{m-1,i}}) + \frac{7}{4}\omega(V_{m-1}, h), \ \forall t \in [t_{i-1}, t_i], \ \forall i = \overline{0, n}.
\end{aligned}$$

Now we can find a bound for $\mid \overline{R_{m,i}} \mid$. From (5.8) and (5.9) $\forall i = \overline{0, n}$ we conclude

$$\begin{aligned}
&\mid \overline{R_{2,i}} \mid \leq \mid R_{2,i} \mid + \frac{|(t_i - a)^\alpha|}{\Gamma(\alpha+1)} \cdot \frac{(b-a)}{2n} \sum_{j=1}^{n} \Big[Q(\lambda \mid R_{1,j-1} \mid \\
&\quad + \mu \mid y_1(\theta(t_{j-1})) - s_1(\theta(t_{j-1})) \mid) \\
&\quad + Q\left(\lambda \mid R_{1,j} \mid + \mu \mid y_1(\theta(t_j)) - s_1(\theta(t_j)) \mid\right) \Big] + \frac{1}{\Gamma(\alpha+1)} \\
&\quad \cdot \frac{(b-a)^2}{4n^2} \sum_{k=1}^{i} \sum_{j=1}^{n} \Big[|(t_i - t_k)^\alpha| \cdot Q\left(\lambda \mid R_{1,j} \mid + \mu \mid y_1(\theta(t_j)) - s_1(\theta(t_j)) \mid\right) \\
&\quad + |(t_i - t_k)^\alpha| \cdot Q\left(\lambda \mid R_{1,j-1} \mid + \mu \mid y_1(\theta(t_{j-1})) - s_1(\theta(t_{j-1})) \mid\right) \\
&\quad + |(t_i - t_{k-1})^\alpha| \cdot Q\left(\lambda \mid R_{1,j} \mid + \mu \mid y_1(\theta(t_j)) - s_1(\theta(t_j)) \mid\right) \\
&\quad + |(t_i - t_{k-1})^\alpha| \cdot Q\left(\lambda \mid R_{1,j-1} \mid + \mu \mid y_1(\theta(t_{j-1})) - s_1(\theta(t_{j-1})) \mid\right) \Big]
\end{aligned} \tag{5.9}$$

therefore

$$\begin{aligned}
&\mid \overline{R_{2,i}} \mid \leq \frac{1}{4n\Gamma(\alpha+1)} \left[(b-a)^{\alpha+2} L + (b-a)^3 L_1 + (b-a)^2 L_2 \right] + \frac{1}{4n\Gamma(\alpha+1)} \\
&\quad \cdot \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)} \cdot \left[(b-a)^{\alpha+2} L + (b-a)^3 L_1 + (b-a)^2 L_2 \right] \\
&\quad + \frac{2(b-a)^q Q\mu}{\Gamma(\alpha+1)} \cdot \frac{7}{4}\omega(V, h).
\end{aligned} \tag{5.10}$$

With the induction for $m \geq 3$ and for every $i = \overline{0, n}$, also using (2.2), we have

$$\begin{aligned}
&\mid \overline{R_{m,i}} \mid \leq \left[1 + \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)} + \left(\frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)} \right)^2 \right. \\
&\quad \left. + \ldots + \left(\frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)} \right)^{m-1} \right] \cdot \frac{1}{4n\Gamma(\alpha+1)} \\
&\quad \cdot \left[(b-a)^{\alpha+2} L + (b-a)^3 L_1 + (b-a)^2 L_2 \right] + \frac{2(b-a)^q Q\mu}{\Gamma(\alpha+1)} \cdot \frac{7}{4}\omega(V, h)
\end{aligned}$$

$$\cdot \left[1 + \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)} + \left(\frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}\right)^2 + \ldots + \left(\frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}\right)^{m-2}\right] = \frac{\frac{2(b-a)^q Q\mu}{\Gamma(\alpha+1)} \cdot \frac{7}{4}\omega(V,h)}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}}$$

$$+ \frac{\frac{1}{4n\Gamma(\alpha+1)}\left[(b-a)^{\alpha+2}L + (b-a)^3 L_1 + (b-a)^2 L_2\right]}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}} \tag{5.11}$$

Now, by replacing (5.11) and the bound obtained in (5.3), in (5.5), the inequality (5.4) is obtained. □

Corollary 5.3. *Under the conditions of the theorem 5.2, when $n \to \infty$ and $m \to \infty$, we have*

$$| y^*(t_i) - \overline{y_m(t_i)} | \to 0, \ \forall i = \overline{0, n}$$

which concludes the proposed method is convergent.

Proof. Since $\lim\limits_{h\to 0} \omega(V_{m-1}, h) = 0$ and λ, μ, M, Q and $(b-a)$ are fixed values, so

$$\lim_{h\to 0} \frac{\frac{1}{4n\Gamma(\alpha+1)}\left[(b-a)^{\alpha+2}L + (b-a)^3 L_1 + (b-a)^2 L_2\right]}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}} = 0,$$

$$\lim_{h\to 0} \frac{\frac{2(b-a)^q Q\mu}{\Gamma(\alpha+1)} \cdot \frac{7}{4}\omega(V,h)}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}} = 0,$$

also, using (2.2), we have

$$\lim_{m\to\infty} \frac{\left[\frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}\right]^m}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}} \cdot \frac{2(b-a)^q}{\Gamma(\alpha+1)} QM = 0.$$

□

6. Stability analysis

To prove the stability of the method, we consider a small perturbation in the first step $y_0 = g$, so we consider the equation (3.2) as follows

$$x(t) = h(t) + \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} \int_a^b K(a,s).F(s, x(s), x(\theta(s)))ds$$
$$+ \frac{1}{\Gamma(\alpha+1)} \int_a^t \int_a^b (t-\tau)^\alpha \frac{\partial K(\tau,s)}{\partial \tau}.F(s, x(s), x(\theta(s)))dsd\tau, \ t \in [a,b], \tag{6.1}$$

where $h \in \mathbb{C}^1[a,b]$ and $| g(t) - h(t) | < \varepsilon$ for small $\varepsilon > 0$. Also, assume that $M' \geq 0$ is such that

$$M' = max\{| F(s, h(s), h(\theta(s))) | : s \in [a,b]\}.$$

Using the proposed method for the equation (6.1), we get the sequence of successive approximations on the points $t_i = a + ih$, $i = \overline{0, n}$:

$$x_0(t_i) = h(t_i), \ i = \overline{0, n},$$
$$x_m(t_i) = h(t_i) + \frac{(t_i - a)^\alpha}{\Gamma(\alpha+1)} \int_a^b K(a,s).F(s, x(s), x(\theta(s)))ds$$
$$+ \frac{1}{\Gamma(\alpha+1)} \int_a^{t_i} \int_a^b (t_i - \tau)^\alpha \frac{\partial K(\tau, s)}{\partial \tau}.F(s, x(s), x(\theta(s)))dsd\tau, \ m \in \mathbb{N},$$

with computations such as (4.15), we have

$$x_0(t_i) = h(t_i), \ i = \overline{0, n},$$
$$x_m(t_i) = \overline{x_m(t_i)} + \overline{R'_{m,i}}, \ i = \overline{0, n}, \ m \in \mathbb{N}.$$

Therefore

$$| \ x_0(t) - y_0(t) \ | < \varepsilon, \ \forall t \in [a, b].$$

Definition 6.1. [9] The presented method is stable if there are $p \in \mathbb{N}_0$, a sequence of continuous functions $\mu_m : [0, b-a] \to [0, \infty]$, $m \in \mathbb{N}_0$ with the property $\lim\limits_{h \to 0} \mu_m(h) = 0, \ \forall m \in \mathbb{N}_0$ and the constants $K_1, \ K_2, \ K_3 > 0$ independent of h so that

$$| \ \overline{x_m(t_i)} - \overline{y_m(t_i)} \ | \leq K_1 \varepsilon + K_2.h^p + K_3.\mu_m(h), \ \forall i = \overline{0, n}, \ m \in \mathbb{N}_0.$$

Theorem 6.2. *Under the conditions of the theorem 5.2, the presented method is stable.*

Proof. We have

$$| \ \overline{x_m(t_i)} - \overline{y_m(t_i)} \ | \leq | \ \overline{x_m(t_i)} - x_m(t_i) \ | + | \ x_m(t_i) - y_m(t_i) \ |$$
$$+ | \ y_m(t_i) - \overline{y_m(t_i)} \ | \leq | \ \overline{R_{m,i}} \ | + | \ x_m(t_i) - y_m(t_i) \ | + | \ \overline{R'_{m,i}} \ |, \tag{6.2}$$

and based on the relation (5.11):

$$| \ \overline{R_{m,i}} \ | \leq \frac{\frac{1}{4n\Gamma(\alpha+1)} \left[(b-a)^{\alpha+2} L + (b-a)^3 L_1 + (b-a)^2 L_2 \right]}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}}$$
$$+ \frac{\frac{2(b-a)^q Q\mu}{\Gamma(\alpha+1)} \cdot \frac{7}{4}\omega(V, h)}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}}, \tag{6.3}$$

$$| \ \overline{R'_{m,i}} \ | \leq \frac{\frac{1}{4n\Gamma(\alpha+1)} \left[(b-a)^{\alpha+2} L' + (b-a)^3 L'_1 + (b-a)^2 L'_2 \right]}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}}$$
$$+ \frac{\frac{2(b-a)^q Q\mu}{\Gamma(\alpha+1)} \cdot \frac{7}{4}\omega(V, h)}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}}, \tag{6.4}$$

where $L', L_1', L_2' \geq 0$ are Lipschitz constants similar to (4.4) and (4.7).
Since $\mid x_0(t) - y_0(t) \mid \leq \varepsilon,\ \forall t \in [a,b]$, so

$$\begin{aligned} \mid x_1(t) - y_1(t) \mid \leq & \mid x_0(t) - y_0(t) \mid \frac{|(t-a)^\alpha|}{\Gamma(\alpha+1)} \int_a^b |K(a,s)|.|F(s, x_0(s), x_0(\theta(s))) \\ & - F(s, y_0(s), y_0(\theta(s)))|ds + \frac{1}{\Gamma(\alpha+1)} \int_a^t \int_a^b \mid (t-\tau)^\alpha \mid . \mid \frac{\partial K(\tau,s)}{\partial \tau} \mid \\ & . \mid F(s, x_0(s), x_0(\theta(s))) - F(s, y_0(s), y_0(\theta(s))) \mid ds d\tau \\ & \leq \left[1 + \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}\right] \cdot \varepsilon, \end{aligned}$$

and for $m \geq 2$, we have

$$\begin{aligned} \mid x_m(t) - y_m(t) \mid \leq & \mid x_0(t) - y_0(t) \mid \frac{|(t-a)^\alpha|}{\Gamma(\alpha+1)} \int_a^b |K(a,s)| \\ & \cdot |F(s, x_{m-1}(s), x_{m-1}(\theta(s))) - F(s, y_{m-1}(s), y_{m-1}(\theta(s)))|ds \\ & + \frac{1}{\Gamma(\alpha+1)} \int_a^t \int_a^b \mid (t-\tau)^\alpha \mid . \mid \frac{\partial K(\tau,s)}{\partial \tau} \mid \\ & . \mid F(s, x_{m-1}(s), x_{m-1}(\theta(s))) - F(s, y_{m-1}(s), y_{m-1}(\theta(s))) \mid ds d\tau \\ & \leq \Big[1 + \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)} + \Big(\frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}\Big)^2 + \dots \\ & + \Big(\frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}\Big)^m\Big] \cdot \varepsilon \leq \frac{\varepsilon}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}}. \end{aligned} \tag{6.5}$$

And by replacing (6.3), (6.4) and (6.5) into (6.2), we obtain

$$\begin{aligned} \mid \overline{x_m(t_i)} - \overline{y_m(t_i)} \mid \leq & \frac{\frac{(b-a)^q Q\mu}{\Gamma(\alpha+1)} \cdot 7\omega(V,h)}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}} + \frac{\varepsilon}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}} \\ & + \left[(b-a)^{\alpha+2}(L+L') + (b-a)^3(L_1+L_1') + (b-a)^2(L_2+L_2')\right] \\ & \cdot \frac{\frac{1}{4n\Gamma(\alpha+1)}}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}} \leq K_1\varepsilon + K_2.h + K_3.\mu_m(h), \end{aligned}$$

where

$$K_1 = \frac{1}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}},\ K_3 = \frac{\frac{(b-a)^q Q\mu}{\Gamma(\alpha+1)} \cdot 7\omega(V,h)}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}},$$

$$K_2 = \frac{\frac{1}{4n\Gamma(\alpha+1)}\left[(b-a)^{\alpha+2}(L+L') + (b-a)^3(L_1+L_1') + (b-a)^2(L_2+L_2')\right]}{1 - \frac{2(b-a)^q Q(\lambda+\mu)}{\Gamma(\alpha+1)}},$$

with $p = 1$ and $\mu_m(h) = \omega(V_{m-1}, h)$. Therefore by definition 6.1, the presented method is stable. □

7. Application of the method

In this section, we apply the presented method to obtain the approximate solution of functional fractional Hammerstein integro-differential equations.

Example 7.1. Consider the following fractional Hammerstein integro-differential equation of [35, 36, 41]

$$D^{\frac{1}{2}}y(t) = \frac{1}{\Gamma(\frac{1}{2})}\left(\frac{8}{3}t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right) - \frac{t}{1260} + \int_0^1 ts[y(s)]^4 ds, \tag{7.1}$$

with the initial condition $y(0) = 0$ and exact solution is $y(t) = t^2 - t$. Using the fractional integral operator on both sides of the equation (7.1) and by using of initial condition, we obtain

$$y(t) = t^2 - t - \frac{t^{\frac{3}{2}}}{945\sqrt{\pi}} + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t \int_0^1 (t-\tau)^{\frac{-1}{2}} \tau s[y(s)]^4 ds d\tau. \tag{7.2}$$

We apply the presented method on the (7.2) and the numerical results of the presented method are given in TABLE 1. We calculate its polynomial interpolation $p(t)$ for $(t_i, \overline{y_m(t_i)})$, $i = 1, 2, \ldots, n$ and $\| y(t) - p(t) \|_2$ for different values of n, which are shown in TABLE 2. It is obviously seen that the numerical results of our method are more accurate than of the results obtained from referred methods.

Table 1. Numerical results of Example 7.1

t	*exact solution*	$n=8$	$n=16$	$n=24$	$n=32$	$n=48$
$t=0.125$	-0.10937500	-0.10938159	-0.10937749	-0.10937640	-0.10937592	-0.10937551
$t=0.250$	-0.18750000	-0.18750705	-0.18750261	-0.18750145	-0.18750095	-0.18750052
$t=0.375$	-0.23437500	-0.23438225	-0.23437766	-0.23437647	-0.23437596	-0.23437553
$t=0.500$	-0.25000000	-0.25000736	-0.25000269	-0.25000149	-0.25000097	-0.25000053
$t=0.625$	-0.23437500	-0.23438243	-0.23437771	-0.23437649	-0.23437597	-0.23437554
$t=0.750$	-0.18750000	-0.18750748	-0.18750272	-0.18750150	-0.18750097	-0.18750054
$t=0.875$	-0.10937500	-0.10938251	-0.10937772	-0.10937651	-0.10937598	-0.10937554
$t=1.000$	0.00000000	$-0.7534e-5$	$-.2733e-5$	$-0.1515e-5$	$-0.9815e-6$	$-0.5504e-6$

Table 2. Absolute errors of Example 7.1

	$\| e_8 \|_2$	$\| e_{12} \|_2$	$\| e_{16} \|_2$	$\| e_{24} \|_2$	$\| e_{32} \|_2$	$\| e_{48} \|_2$
Presented method	$7.2850e-6$	$4.0478e-6$	$2.6559e-6$	$1.4716e-6$	$9.6072e-7$	$5.3383e-7$
Ref. [35]	$-$	$1.0438e-4$	$-$	$2.6403e-5$	$-$	$-$
Ref. [41]	$6.0313e-5$	$-$	$1.4484e-6$	$-$	$2.3374e-7$	$5.3445e-6$
Ref. [36]	$-$	$7.7110e-4$	$-$	$2.0755e-5$	$-$	$5.3445e-6$

Example 7.2. As the second example, consider the following fractional Hammerstein integro-differential equation of Ref. [36]

$$D^{\frac{5}{6}}y(t) - \int_0^1 te^s[y(s)]^4 ds = \frac{3}{\Gamma(\frac{1}{6})}\left(2t^{\frac{1}{6}} - \frac{431}{91}t^{\frac{13}{6}} + t(248e - 674)\right), \tag{7.3}$$

with the initial condition $y(0) = 0$ and exact solution $y(t) = t - t^3$. Using the fractional integral operator on both sides of the equation (7.3) and using of initial condition, yields

$$y(t) = t - t^3 - \frac{1}{\Gamma(\frac{5}{6})}\left(\frac{8928}{55} t^{\frac{11}{6}} e - \frac{24264 t^{\frac{11}{6}}}{55}\right) + \frac{1}{\Gamma(\frac{5}{6})}\int_0^t \int_0^1 (t-\tau)^{\frac{-1}{6}} \tau e^s [y(s)]^4 ds d\tau. \tag{7.4}$$

$\| y(t) - p(t) \|_2$ of the presented method and Ref. [36] are given in TABLE 3.

TABLE 3. Numerical results of Example 7.2

$\| e_n \|_2$	*presented method*	*Ref.* [36]
$\| e_{12} \|_2$	$1.79091184e-4$	$2.0862e-3$
$\| e_{24} \|_2$	$9.29306806e-5$	$6.3440e-4$

Example 7.3. Consider the following functional fractional Hammerstein integro-differential equation

$$D^{\frac{1}{2}} y(t) - \int_0^1 (t-s).\big((s-1) + sy(\frac{s}{2})\big) ds = \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}} + \frac{7}{16} t - \frac{7}{60}, \tag{7.5}$$

with the initial condition $y(0) = 0$ and exact solution $y(t) = t^2$. Similar to the previous example, by using the fractional integral operator on both sides of the equation (7.5) and by using of initial condition, we obtain

$$y(t) = t^2 - \frac{1}{\sqrt{\pi}}\left(\frac{7t^{\frac{3}{2}}}{12} - \frac{7\sqrt{t}}{30}\right) + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t \int_0^1 (t-\tau)^{\frac{-1}{2}} (\tau - s).\big((s-1) + sy(\frac{s}{2})\big) ds d\tau. \tag{7.6}$$

The numerical results of the presented method are given in TABLE 4. Also, $\| y(t) - p(t) \|_2$ is reported in last line.

8. Conclusion

In this paper, we investigated on functional Hammerstein integro-differential equations of fractional order. Here we also presented an approximate method to solve these equations. We proved convergence and stability of the method, too. At the end, we gave some numerical examples, which show the accuracy of the method.

Acknowledgment

The authors would like to thank the anonymous referees for their valuable comments that helped the authors to improve the paper.

TABLE 4. Numerical results of Example 7.3

t	*exact solution*	$n = 8$	$n = 24$	$n = 48$
$t = 0.125$	0.01562500	0.01758606	0.01619027	0.01586463
$t = 0.502$	0.06250000	0.06410829	0.06302393	0.06273235
$t = 0.375$	0.14062500	0.14193456	0.14111765	0.14085170
$t = 0.500$	0.25000000	0.25107388	0.25047139	0.25022308
$t = 0.625$	0.39062500	0.39151664	0.39108344	0.39084622
$t = 0.750$	0.56250000	0.56325437	0.56295248	0.56272087
$t = 0.875$	0.76562500	0.76628063	0.76607752	0.76584684
$t = 1.000$	1.00000000	1.00059055	1.00045784	1.00022396
$\parallel e_n \parallel_2$		$1.1959e-3$	$4.9113e-4$	$2.2686e-4$

References

[1] J. H. Ahlberg, E. N. Nilson and J. L. Walsh, *The Theory of splines and their applications*, Academic Press, New York, London, 1967.

[2] F. Alhendi, W. Shammakh and H. Al-Badrani, *Numerical solutions for quadratic integro-differential equations of fractional orders*, Open Journal of Applied Sciences (OJAppS), *7* (2017), 157–170.

[3] S. Alkan and V. F. Hatipoglu, *Approximate solutions of Volterra-Fredholm integro-differential equations of fractional order*, Tbilisi Mathematical Journal, *10*(2) (2017), 1–13.

[4] A. Arikoglu and I. Ozkol, *Solution of fractional integro-differential equations by using fractional differential transform method*, Chaos, Solitons & Fractals, *40*(2) (2009), 521–529.

[5] T. M. Atanacković, S. Pilipović, B. Stanković and D. Zorica, , *Fractional calculus with applications in mechanics: vibrations and diffusion processes*, ISTE Ltd and John Wiley & Sons, Inc, Great Britain and the United States, 2014.

[6] T. M. Atanacković, S. Pilipović, B. Stanković and D. Zorica, *Fractional calculus with applications in mechanics: wave propagation, impact and variational principles*, ISTE Ltd and John Wiley & Sons, Inc, Great Britain and the United States, 2014.

[7] G. W. Bohannan, *Analog fractional order controller in temperature and motor control applications*, Journal of Vibration and Control, *14* (2008), 1487–1498.

[8] R. Baillie, *Long memory processes and fractional integration in econometrics*, Journal of Econometrics, *73* (1996), 5–59.

[9] A. M. Bica, M. Curila and S. Curila, *About a numerical method of successive interpolations for functional Hammerstein integral equations*, Journal of Computational and Applied Mathematics, 2*36* (2012), 2005–2024.

[10] P. Cerone and S. Dragomir, *Trapezoidal and midpoint-type rules from inequalities point of view*, G. A. Anastassiou (Ed.), Hand book of Analytic Computational Methods in Applied Mathematics, Chapman Hall, CRC Press, Boca Raton, London, New York, Washington, DC, 2000.

[11] K. Diethelm, *The analysis of fractional differential equations*, Springer, 2004.

[12] I. Emiroglu, *An approximation method for fractional integro-differential equations*, Open Physics, *13* (2015), 370–376.

[13] A. A. Elbeleze, A. Klman and B. M. Taib, *Approximate solution of integro-differential equation of fractional (arbitrary) order*, Journal of King Saud University - Science, *28* (2016), 61–68.

[14] A. A. Hamoud and K. P. Ghadle, *Modified Laplace decomposition method for fractional Volterra-Fredholm integro-differential equations*, Journal of Mathematical Modeling, *6*(1) (2018), 91–104.

[15] J. H. He, *Nonlinear oscillation with fractional derivative and its applications*, in: Proc. International conference on vibrating engineering, China: Dalian, (1998), 288–291.

[16] R. Hilfer, *Application of fractional calculus in physics*, World Scientific, Singapore, 2000.

[17] L. Huang, X. F. Li, Y. Zhao and X. Y. Duan, *Approximate solution of fractional integro-differential equations by Taylor expansion method*, Computers & Mathematics with Applications, *62*(3) (2011), 1127–1134.

[18] C. Iancu, *On the cubic spline of interpolation*, Seminar on functional analysis and numerical methods, Cluj-Napoca, *4* (1981), 52–71.

[19] C. Ionescu, A. Lopes, D. Copot, J. A. T. Machado and J. H. T. Bates , *The role of fractional calculus in modeling biological phenomena: A review*, Communications in Nonlinear Science and Numerical Simulation, *51* (2017), 141–159.

[20] H. K. Jassim, *The analytical solutions for Volterra integro-differential equations within local fractional operators by Yang-Laplace transform*, Sahand Communications in Mathematical Analysis (SCMA), *6*(1) (2017), 69–76.

[21] S. Karimi Vanani and A. Aminataei, *Operational Tau approximation for a general class of fractional integro-differential equations*, Journal of Computational and Applied Mathematics, *30*(3) (2011), 655–674.

[22] A. Kilicman and W. A. Ahmood, *Solving multi-dimensional fractional integro-differential equations with the initial and boundary conditions by using multi-dimensional Laplace Transform method*, Tbilisi Mathematical Journal, *10*(1) (2017), 105–115.

[23] M. Kurulay and A. Secer, *Variational iteration method for solving nonlinear fractional integro-differential equations*, International Journal of Computer Science & Emerging Technologies (IJCSET), *2*(1) (2011), 18–20.

[24] P. Linz, *Analytical numerical methods for Volterra equations*, SIAM, Studies in Applied Mathematics, 1985.

[25] A. M. S. Mahdy, *Numerical studies for solving fractional integro-differential equations*, Journal Of Ocean Engineering And Science (JOES), *3* (2018), 127–132.

[26] Z. Meng, L. Wang, H. Li and W. Zhang, *Legendre wavelets method for solving fractional integro-differential equations*, International Journal of Computer Mathematics, *92*(6) (2015), 1275–1291.

[27] R. C. Mittal and R. Nigam, *Solution of fractional integro-differential equations by Adomian decomposition method*, International Journal of Applied Mathematics and Mechanics, *4*(2) (2008), 87–94.

[28] S. Momani and M. Noor, *Numerical methods for fourth order fractional integro-differential equations*, Applied Mathematics and Computation, *182* (2006), 754–760.

[29] Y. Nawaz, *Variational iteration method and homotopy perturbation method for fourth order fractional integro-differential equations*, Computers and Mathematics with Applications, *61* (2011), 2330–2341.

[30] D. Nazari and S. Shahmorad, *Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions*, Journal of Computational and Applied Mathematics, 234(3) (2010), pp. 883-891.

[31] K. Oldham, *Fractional differential equations in electrochemistry*, Advances in Engineering Software, *41* (2010), 9–17.

[32] I. Podlubny, *Fractional differential equations, Mathematics in Science and Engineering*, Academic Press, San Diego, 1999.

[33] S. S. Ray, *Analytical solution for the space fractional diffusion equation by two-step Adomian decomposition method*, Communications in Nonlinear Science and Numerical Simulation, *14* (2009), 1295–1306.

[34] L. J. Rong and P. Chang, *Jacobi wavelet operational matrix of fractional integration for solving fractional integro-differential equation*, Journal of Physics: Conference Series, *693* (2016), 1–14.

[35] H. Saeedi, *Application of Haar wavelets in solving nonlinear fractional Fredholm integro-differential equations*, Journal of Mahani Mathematical Research Center (JMMRC), *2*(1) (2013), 15–28.

[36] H. Saeedi, M. Mohseni Moghadam, N. Mollahasani and G. N. Chuev, *A CAS wavelet method for solving nonlinear Fredholm integro-differential equations of fractional order*, Communications in Nonlinear Science and Numerical Simulation, *16* (2011), 1154–1163.

[37] H. Saeedi and F. Samimi, *Hes homotopy perturbation method for nonlinear Fredholm integro-differential equations of fractional order*, International Journal of Engineering Research and Applications (IJERA), *2*(5) (2010), 052–056.

[38] Y. Wang and L. Zhu, *Solving nonlinear Volterra integro-differential equations of fractional order by using Euler wavelet method*, Advances in Difference Equations, *27* (2017), 1–16.

[39] J. Wang, T. Z. Xu, Y. Q.Wei and J. Q. Xie, *Numerical simulation for coupled systems of nonlinear fractional order integro-differential equations via wavelets method*, Applied Mathematics and Computation, *324* (2018), 36–50.

[40] A. Youse, T. Mahdavi-Rad and S. G. Shaei, *A quadrature Tau method for solving fractional integro-differential equations in the Caputo sense*, International Journal of Mathematics and Computer Science, *15* (2015), 97–107.

[41] L. Zhu and Q. Fan, *Solving fractional nonlinear Fredholm integro-differential equations by the second kind Chebyshev wavelets*, Communications in Nonlinear Science and Numerical Simulation, *17* (2012), 2333–2341.