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A new method for constructing exact solutions for a time-fractional differential equation

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Abstract In the present paper the process of finding new solutions from previous solutions of a given fractional differential equation (FDE) is considered. For this issue, first we should construct an exact solution by using the symmetry operators of the equation. Then, the commutator brackets of the obtained operators give new solutions from old ones via a systematic method.

Keywords. Lie point symmetry, Fractional calculus, Fractional differential equation, Exact solution.

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1. Introduction

In the last decade FPDEs have attracted considerable interest. This kind of equation plays an important role in various fields of sciences, for example engineering, electrochemistry, biology, economics, modeling, electronics, dynamics, and many other sciences [10, 12, 13]. Lie symmetries method have many efficient applications in physics and mathematics. As an important application of symmetry operators is the reduction procedure. This is possible from a similarity variable obtaining from the symmetry and Erdelyi-Kober. In this paper time FDE

$$D_t^{\alpha}u = xu_{xx} + f(x)u_x, \tag{1.1}$$

is considere, where $D_t^{\alpha}u$ is the fractional derivative of order α , $0 < \alpha \leq 1$ and $f(x)$ is an arbitrary function.

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Firstly, we present the complete algebra of Lie point symmetries for Eq. (1.1). With the aid of calculated symmetries, Eq. (1.1) is reduced and a list of exact solutions are found [1, 2, 8, 9]. The main goal of the paper is to build new solutions from the old ones by using the obtained exact solutions.

Some researches applied Lie group method for FDEs in the sense of the Caputo derivative and derived similarity solutions [4, 11]. In this work, we give group classification of Eq. (1.1), based on Riemann-Lioville derivative [5].

The organization of the paper sets in 6 sections; In section 2, we give some notations and preliminaries of equation with fractional order. The infinitesimal transformations and the determining equations of Lie symmetries are introduced in section 3. Section 4 is devoted to reduction precess for obtaining exact solutions of Eq. (1.1) in three separated cases. New exact solutions with the aid of obtained solutions, are presented in section 5. Finally, section 6 is dedicated to obtained results.

2. Fractional Calculus

There is no unique definition for fractional derivatives, such as modified Riemann-Liouville derivative, Grunwald-Letnikov derivative, Caputos, Riesz, Miller and Ross fractional derivative . Here we consider the most common definition named in Riemann and Liouvill derivative. In the sequel that, based on what is required in this work we give some basic definitions and properties of the fractional calculus theory [6]. Let us define

$$D_t^\alpha f(t) = \begin{cases} \dfrac{d^n f}{dt^n} & \alpha = n \\ \dfrac{d^n}{dt^n} I^{n-\alpha} f(t) & 0 \leq n-1 < \alpha < n, \end{cases} \tag{2.1}$$

where $n \in \mathbb{N}$, $I^\beta f(t)$ is the Riemann-Liouville fractional integral of order β , namely $I^\beta f(t) = \dfrac{1}{\Gamma(\mu)} \displaystyle\int_0^t (t-s)^{(\beta-1)} f(s) ds, \qquad \beta > 0.$

By definition, we have $I^0 f(t) = f(t)$ and it satisfies the stability property $I_t^{\nu_1} I_t^{\nu_2} f(t) = I_t^{\nu_1+\nu_2} f(t)$ and $\Gamma(\nu) = \displaystyle\int_0^\infty x^{\nu-1} e^{-x} dx, \quad \nu \in R^+$, is the standard gamma function.

Definition 2.1. The Riemann-Liouville fractional partial derivative is defined by,

$$\partial_t^\alpha u(t,x) = \begin{cases} \dfrac{\partial^n u}{\partial t^n} & \alpha = n \\ \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\partial^n}{\partial t^n} \displaystyle\int_0^t (t-s)^{n-\alpha-1} u(s,x) ds & 0 \leq n-1 < \alpha < n, \end{cases}$$

where ∂_t^n is the usual derivative of integer order n.

Some useful formulas and properties of Riemann-Liouville derivative have been summerized in [7].

The laplace transform of Riemann-Liouville fractional derivative of order $\alpha > 0$ is [14]

$$L\{D^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{k=0}^{\infty} s^{k} \{D_t^{\alpha-k-1} f(t)\}_{t=0}. \tag{2.2}$$

where $L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)dt$. A two-parameter function of Mittag-Leffler type was defined by the series expansion [15]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad \alpha, \beta \in \mathbb{C}, \qquad Re(\alpha) > 0, \qquad Re(\beta) > 0.$$

Some of the relationships are as follows:

$$D^{\nu}[t^{\beta-1} E_{\alpha,\beta}(at^{\alpha})] = t^{\beta-\nu-1} E_{\alpha,\beta-\nu}(at^{\alpha}), \qquad \nu > 0, \qquad \alpha > 0, \qquad a \in R.$$
$$L\{t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm at^{\alpha})\} = \frac{k! s^{\alpha-\beta}}{(s^{\alpha} \mp a)^{k+1}}, \qquad Re(s) > |a|^{\frac{1}{\alpha}}.$$

Definition 2.2. The Erdelyi-Kober fractional differential operator $P_{\beta}^{\tau,\alpha}$ of order α is defined as [6]

$$\left(P_{\beta}^{\tau,\alpha} g\right) := \prod_{j=0}^{n-1} \left(\tau + j - \frac{1}{\beta}\theta \frac{d}{d\theta}\right) \left(K_{\beta}^{\tau+\alpha, n-\alpha} g\right)(\theta),$$
$$n = \begin{cases} [\alpha]+1 & \alpha \notin \mathbb{N} \\ \alpha & \alpha \in \mathbb{N}, \end{cases}$$

where

$$\left(K_{\beta}^{\tau,\alpha} g\right) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_1^{\infty} (u-1)^{\alpha-1} u^{-(\tau+\alpha)} g(\theta u^{\frac{1}{\beta}}) du & \alpha > 0 \\ g(\theta) & \alpha = 0, \end{cases} \tag{2.3}$$

is the Erdelyi-Kober fractional integral operator.
Also we have

$$\begin{aligned} \frac{\partial^n}{\partial t^n} \left[t^{\chi} (K_{\beta}^{\tau, n-\alpha} g)\right](\theta) &= t^{\chi-n} \prod_{j=0}^{n-1} \left(\chi - n + 1 + j + a\theta \frac{d}{d\theta}\right) \left(K_{\beta}^{\chi-n+\alpha+1, n-\alpha} g\right)(\theta) \\ &= t^{\chi-n} \left(P_{\beta}^{\chi-n+1,\alpha}(\theta)\right), \end{aligned} \tag{2.4}$$

if

$$t\frac{\partial}{\partial t} g(\theta) = a\theta \frac{d}{d\theta} g(\theta), \qquad \theta = xt^{a}.$$

3. Symmetry analysis of FDEs.

Lie symmetries with the prolongation formula for FDEs have been given by Gazizov [3]. According to this method, Eq.FDE (1.1) is invariant under a one parameter continuous transformations with parameter ε [16, 17, 18],

$$\begin{aligned}
&t^* = t + \varepsilon\xi^0(t,x,u) + O(\varepsilon^2), \quad x^* = x + \varepsilon\xi^1(t,x,u) + O(\varepsilon^2),\\
&u^* = u + \varepsilon\eta(t,x,u) + O(\varepsilon^2), \quad \frac{\partial^\alpha u^*}{\partial t^*} = \frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon\zeta^{\alpha,t}(t,x,u) + O(\varepsilon^2),\\
&\frac{\partial u^*}{\partial x^*} = \frac{\partial u}{\partial x} + \varepsilon\zeta^{x}(t,x,u) + O(\varepsilon^2), \quad \frac{\partial^2 u^*}{\partial x^{*2}} = \frac{\partial^2 u}{\partial x^2} + \varepsilon\zeta^{xx}(t,x,u) + O(\varepsilon^2),
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
\zeta^x &= D_x\eta - u_t D_x\xi^0 - u_x D_x\xi^1, \qquad \zeta^{xx} = D_x\zeta^x - u_{xt}D_x\xi^0 - u_{xx}D_x\xi^1,\\
\zeta^{\alpha,t} &= \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\xi^0))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi^1) D_t^{\alpha-n}(u_x)\\
&+ \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\xi^0)\right] D_t^{\alpha-n}(u) + \mu,
\end{aligned} \tag{3.2}$$

and

$$\mu = \sum_{n=2}^{\infty}\sum_{m=1}^{n}\sum_{k=1}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}(-u)^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}}{\partial t^{n-m}\partial u^k},$$

also $\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1-n)}$.
Here D_x denotes total derivative operator defined by:

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + \dots.$$

If

$$X = \xi^0\frac{\partial}{\partial t} + \xi^1\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}, \tag{3.3}$$

be a symmetry operator for the Eq. (1.1), which is known in the literature as an infinitesimal operator or generator of the group G, we must have

$$\xi^0 = \frac{dt^*}{d\varepsilon}|_{\varepsilon=0} \quad \xi^1 = \frac{dx^*}{d\varepsilon}|_{\varepsilon=0} \quad \eta = \frac{du^*}{d\varepsilon}|_{\varepsilon=0}.$$

According to the infinitesimal invariance criterion, Eq. (1.1) admits transformation group (3) if the prolonged vector field $\mathrm{Pr}^{(\alpha,2)}X$ annihilates (1.1) on its solution, namely,

$$\mathrm{Pr}^{(\alpha,2)}X\,(\Delta)\,|_{\nabla=0} = 0, \qquad \Delta = D_t^\alpha u - xu_{xx} - f(x)u_x.$$

The operator $\mathrm{Pr}^{(\alpha,2)}X$ takes the form:

$$\mathrm{Pr}^{(\alpha,2)}X = X + \zeta^{\alpha,t}\partial_{\partial_t^\alpha u} + \zeta^x\partial_{u_x} + \zeta^{xx}\partial_{u_{xx}},$$

where ζ^x and ζ^{xx} are defined in (3.2). Now, we will investigate the invariance properties of the time fractional Eq. (1.1). The invariance criterion takes the form

$$\zeta^{\alpha,t} - (u_{xx} + f'(x)u_x)\xi^1 - f(x)\zeta^x - x\zeta^{xx} = 0. \tag{3.4}$$

Solving (3.4) along with (3.2), we derive the following characteristic system:

$$\begin{aligned}
&\xi^1_t = \xi^0_u = \xi^1_u = \xi^0_x = \eta_{uu} = 0,\\
&x\eta_u - \alpha x\xi^0_t - \xi^1 - x(\eta_u - 2\xi^1_x) = 0,\\
&-\alpha f(x)\xi^0_t - f'(x)\xi^1 + f(x)\xi^1_x - x(2\eta_{xu} - \xi^1_{xx}) = 0,\\
&\partial_t^\alpha\eta - u\partial_t^\alpha\eta_u - f(x)\eta_x - x\eta_{xx} = 0,\\
&\binom{\alpha}{n}\partial_t^n\eta_u - \binom{\alpha}{n+1}D_t^{n+1}\xi^0 = 0.
\end{aligned}$$

With classification of solution of this system, we obtain solution of Eq. (1.1), for arbitrary $f(x)$ and $\alpha \in (0,1]$, such as

$$\xi^0 = c_1t + c_2, \qquad \xi^1 = \alpha c_1 x + c_3\sqrt{x}, \qquad \eta = F_1(x)u + F_2(t,x),$$

where c_1, c_2 and c_3 are arbitrary constants.

For three cases we obtain different symmetries for Eq. (1.1).

- For $\beta = f(x) = \frac{1}{2}$, Eq. (1.1) translates to:

  $$D_t^\alpha u = xu_{xx} + \frac{1}{2}u_x. \tag{3.5}$$

  The Lie symmetries of Eq. (3.5) are found as follows:

  $$\begin{aligned}
  &X_1 = \alpha x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}, \qquad X_2 = \sqrt{x}\frac{\partial}{\partial x}, \qquad X_3 = \frac{\partial}{\partial t},\\
  &X_4 = u\frac{\partial}{\partial u}, \qquad X_{F_2} = F_2(t,x)\frac{\partial}{\partial u}.
  \end{aligned} \tag{3.6}$$

- For $\beta = f(x) = \frac{3}{2}$, we have

  $$D_t^\alpha u = xu_{xx} + \frac{3}{2}u_x, \tag{3.7}$$

  with Lie symmetries,

  $$\begin{aligned}
  &X_1 = \alpha x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}, \qquad X_3 = \frac{\partial}{\partial t}, \qquad X_4 = u\frac{\partial}{\partial u},\\
  &X_5 = \sqrt{x}\frac{\partial}{\partial x} - \frac{1}{2\sqrt{x}}u\frac{\partial}{\partial u}, \qquad X_{F_2} = F_2(t,x)\frac{\partial}{\partial u}.
  \end{aligned} \tag{3.8}$$

- For $\beta = f(x) = \frac{1+3\sqrt{x}}{2(1+\sqrt{x})}$, the Eq. (1.1) turns to

  $$D_t^\alpha u = xu_{xx} + \frac{1+3\sqrt{x}}{2(1+\sqrt{x})}u_x, \tag{3.9}$$

with following symmetries:

$$X_3 = \frac{\partial}{\partial t}, \qquad X_4 = u\frac{\partial}{\partial u}, \qquad X_6 = \alpha x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{\alpha}{2(1+\sqrt{x})}u\frac{\partial}{\partial u},$$
$$X_7 = \sqrt{x}\frac{\partial}{\partial x} - \frac{1}{2(1+\sqrt{x})}u\frac{\partial}{\partial u}, \qquad X_{F_2} = F_2(t,x)\frac{\partial}{\partial u}. \tag{3.10}$$

4. Exact Solutions or Reduction equations by using Lie method

In this section we give some exact solutions for Eq. (3.5).

- At first, we consider the symmetry $X_1 = \alpha x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}$, the corresponding characteristic equation is of the form:

$$\frac{dx}{\alpha x} = \frac{dt}{t} = \frac{du}{0}. \tag{4.1}$$

Integration of (4.1) provides the following similarity function

$$u = g(t,x) = g(\theta) = g(xt^{-\alpha}).$$

Let $n-\alpha < \alpha < n$, $n = 1,2,3,\ldots$.

According to the Riemann-Liouville fractional derivative, once we get:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial}{\partial t^n}\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\left(g(xs^{\frac{-\alpha}{2}})ds\right), \tag{4.2}$$

Let $\nu = \frac{t}{s}$, we have $ds = -\frac{t}{\nu^2}d\nu$. So the above statement can be expressed as:

$$\begin{aligned}
\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial}{\partial t^n}\frac{1}{\Gamma(n-\alpha)}\int_0^t t^{n-\alpha}\nu^{-(n-\alpha+1)}(\nu-1)^{n-\alpha-1}g(\theta\nu^\alpha)d\nu \\
&= \frac{\partial}{\partial t^n}\left[t^{n-\alpha}\left(K^{1,n-\alpha}_{\frac{2}{\alpha}}g\right)(\theta)\right] \\
&= t^{-\alpha}\prod_{j=0}^{n-1}\left(1-\alpha+j+\theta\frac{d}{d\theta}\right)\left(K^{1,n-\alpha}_{\frac{1}{\alpha}}g\right)(\theta) \\
&= t^{-\alpha}\left(P^{1-\alpha,\alpha}_{\frac{1}{\alpha}}g\right)(\theta).
\end{aligned} \tag{4.3}$$

By substituting the solution $u = g(xt^{\frac{-\alpha}{2}})$ into FPDE (3.5), one can get:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = xu_{xx} + \frac{1}{2}u_x = t^{-\alpha}(\theta g_{\xi\xi} + \frac{1}{2}g_\xi). \tag{4.4}$$

Thus, the time fractional equation (3.5) can be reduced into an FODE:

$$\left(P^{1-\alpha,\alpha}_{\frac{1}{\alpha}}g\right)(\theta) = \theta g'' + \frac{1}{2}g'. \tag{4.5}$$

- For $X_2 = \sqrt{x}\frac{\partial}{\partial x}$, with substituting $u = g(t)$ in Eq. (3.5) we have $\frac{\partial^\alpha g}{\partial t^\alpha} = 0$. So with the aid of Laplace transformation (2.2) we obtain $g(t) = \frac{t^{\alpha-1}}{(\alpha-1)!}$.
- For $X_3 = \frac{\partial}{\partial t}$, by using $u = g(x)$ and placing it in Eq. (3.5), we have $xg'' + \frac{1}{2}g' = 0$. So $g(x) = c_1 + c_2\sqrt{x}$ is a solution of Eq. (3.5).

Other solutions under symmetries $X_1 + X_4$, $X_2 + X_4$ and $X_3 + X_4$ are summerized in Table (1), where $A = 2\sqrt{xt^{-\alpha}e^{-t}E_{1,1-\alpha}(t)}$.

TABLE 1. Results For FDE (3.5)

Symmetries	Exact Solutions	Reduced equations
X_1	$u = g(xt^{-\alpha}) = g(\theta)$	$\left(P^{1-\alpha,\alpha}_{\frac{1}{\alpha}} g\right)(\theta) = \theta g'' + \frac{1}{2}g'.$
X_2	$u = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$	
X_3	$u = c_1 + c_2\sqrt{x}$	
$X_1 + X_4$	$u = tg(xt^{-\alpha}) = tg(\theta)$	$\left(P^{2-\alpha,\alpha}_{\frac{1}{\alpha}} g\right)(\theta) = \theta g'' + \frac{1}{2}g'.$
$X_2 + X_4$	$u = e^{2\sqrt{x}} t^{\alpha-1} E_{\alpha,\alpha}(t^\alpha)$	
$X_3 + X_4$	$u = e^t\left[c_1 \sinh(A) + c_2 \cosh(A)\right]$	

The exact solutions of Eq. (3.7) and (3.9) are given in Tables (2) and (3) respectively. Graph of solutions are shown in Figure 1, 2, 3 and 4 too.

TABLE 2. Results For FDE (3.7)

Symmetries	Exact Solutions	Reduced equations
X_1	$u = g(xt^{-\alpha}) = g(\theta)$	$\left(P^{1-\alpha,\alpha}_{\frac{1}{\alpha}} g\right)(\theta) = \theta g'' + \frac{3}{2}g'.$
X_3	$u = c_1 + \frac{c_2}{\sqrt{x}}$	
X_5	$u = \frac{1}{\sqrt{x}}\frac{t^{\alpha-1}}{\Gamma(\alpha)}$	
$X_3 + X_4$	$u = \frac{e^t}{\sqrt{x}}\left[c_1 \sinh(A) + c_2 \cosh(A)\right]$	

TABLE 3. Results For FDE (3.9)

Symmetries	Exact Solutions	Reduced equations
X_3	$u = c_1 + \frac{c_2}{1+\sqrt{x}}$	
X_6	$u = \frac{\sqrt{x}}{1+\sqrt{x}} g(xt^{-\alpha}) = \frac{\sqrt{x}}{1+\sqrt{x}} g(\theta)$	$\left(P^{1-\alpha,\alpha}_{\frac{1}{\alpha}} g\right)(\theta) = \theta g'' + \frac{3}{2}g'.$
X_7	$u = \frac{t^\alpha}{(1+\sqrt{x})\Gamma(\alpha)}$	
$X_3 + X_4$	$u = \frac{e^t}{1+\sqrt{x}}\left[c_1 \cosh(2\sqrt{ax}) + c_2 \frac{\sqrt{x}}{2}\frac{\sinh(2\sqrt{ax})}{\sqrt{ax}}\right]$	

5. New solutions obtaining from old ones

In the previous section we found some exact solutions by using symmetry operators. In this part we may construct further solutions by the property that symmetries map solutions to solutions. To construct new solutions, one uses the property that the Lie bracket of X_i with X_{F_2} gives another member of the class of symmetries of the form of X_{F_2}. This provides a route to the generation of new similarity solutions associated to X_i. The structure of the new solutions constructed from Lie brackets of the symmetries are coming in Table 4 with $B = \sqrt{x\frac{\Gamma(-\alpha,t)}{\Gamma(\alpha)}}$.

TABLE 4. New solutions

	Lie Brackets	New Solutions
Eq. (3.5)	$[X_2, X_{F_2}] = \left(\sqrt{x}\frac{\partial f_{\text{old}}}{\partial x}\right)\frac{\partial}{\partial u}$	$\sqrt{x}\frac{\partial f_{\text{old}}}{\partial x} = 0$
	$[X_3, X_{F_2}] = \left(\frac{\partial f_{\text{old}}}{\partial t}\right)\frac{\partial}{\partial u}$	$\frac{\partial f_{\text{old}}}{\partial t} = 0$
	$[X_2 + X_4, X_{F_2}] = \left(\sqrt{x}\frac{\partial f_{\text{old}}}{\partial x} - f_{\text{old}}\right)\frac{\partial}{\partial u}$	$\sqrt{x}\frac{\partial f_{\text{old}}}{\partial x} - f_{\text{old}} = 0$
	$[X_3 + X_4, X_{F_2}] = \left(\frac{\partial f_{\text{old}}}{\partial t} - f_{\text{old}}\right)\frac{\partial}{\partial u}$	$e^t\left[2\sinh(2\sqrt{B}) + 2\cosh(2\sqrt{B})\right] \times \left[\frac{xe^{-t}}{t^{1+\alpha}\Gamma(-\alpha)} + 2B\right]\left[\frac{xe^{-t}}{2t^{1+\alpha}\Gamma(-\alpha)\sqrt{B}}\right] - f_{\text{old}} = 0$
Eq. (3.7)	$[X_3, X_{F_2}] = \left(\frac{\partial f_{\text{old}}}{\partial t}\right)\frac{\partial}{\partial u}$	$\frac{\partial f_{\text{old}}}{\partial t} = 0$
	$[X_5, X_{F_2}] = \left(\sqrt{x}\frac{\partial f_{\text{old}}}{\partial x} + \frac{1}{2\sqrt{x}}f_{\text{old}}\right)\frac{\partial}{\partial u}$	$\sqrt{x}\frac{\partial f_{\text{old}}}{\partial x} + \frac{1}{2\sqrt{x}}f_{\text{old}} = 0$
	$[X_3 + X_4, X_{F_2}] = \left(\frac{\partial f_{\text{old}}}{\partial t} - f_{\text{old}}\right)\frac{\partial}{\partial u}$	$\frac{e^t}{\sqrt{x}}\left[2\sinh(2\sqrt{B}) + 2\cosh(2\sqrt{B})\right] \times \left[\frac{xe^{-t}}{t^{1+\alpha}\Gamma(-\alpha)} + 2B\right]\left[\frac{xe^{-t}}{2t^{1+\alpha}\Gamma(-\alpha)\sqrt{B}}\right] - f_{\text{old}} = 0$
Eq. (3.9)	$[X_3, X_{F_2}] = \left(\frac{\partial f_{\text{old}}}{\partial t}\right)\frac{\partial}{\partial u}$	$\frac{\partial f_{\text{old}}}{\partial t} = 0$
	$[X_7, X_{F_2}] = \left(\sqrt{x}\frac{\partial f_{\text{old}}}{\partial x} + \frac{f_{\text{old}}}{2(1+\sqrt{x})}\right)\frac{\partial}{\partial u}$	$\sqrt{x}\frac{\partial f_{\text{old}}}{\partial x} + \frac{f_{\text{old}}}{2(1+\sqrt{x})} = 0$
	$[X_3 + X_4, X_{F_2}] = \left(\frac{\partial f_{\text{old}}}{\partial t} - f_{\text{old}}\right)\frac{\partial}{\partial u}$	$\frac{\partial f_{\text{old}}}{\partial t} - f_{\text{old}} = 0$

6. CONCLUSION

In the present paper, we studied the time-fractional Eq. (1.1) in the sense of Lie point symmetry analysis. The complete classifications of the fractional Eq. (1.1) are presented and all of the geometric vector field of this equation are obtained. We have shown that, by using of Lie analysis, Eq. (1.1) can be transformed into FODE, and the reduced equation is an FODE with Erdelyi-Kober operator. Some exact solutions are found and consequently by using obtained solutions we present another solutions for Eq. (1.1).

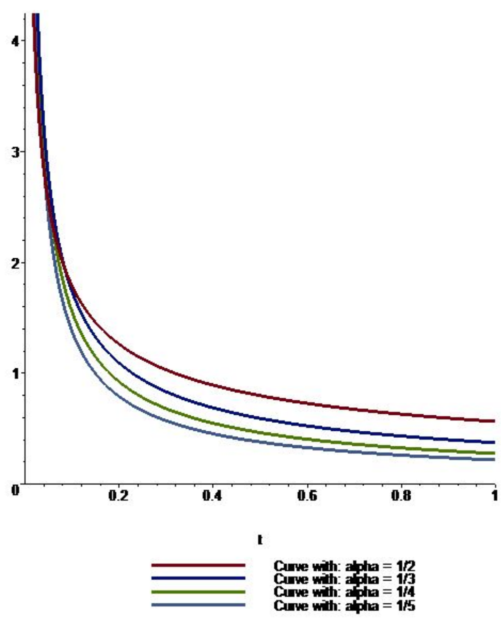

FIGURE 1. $u = t^{\alpha-1}/\Gamma(\alpha)$

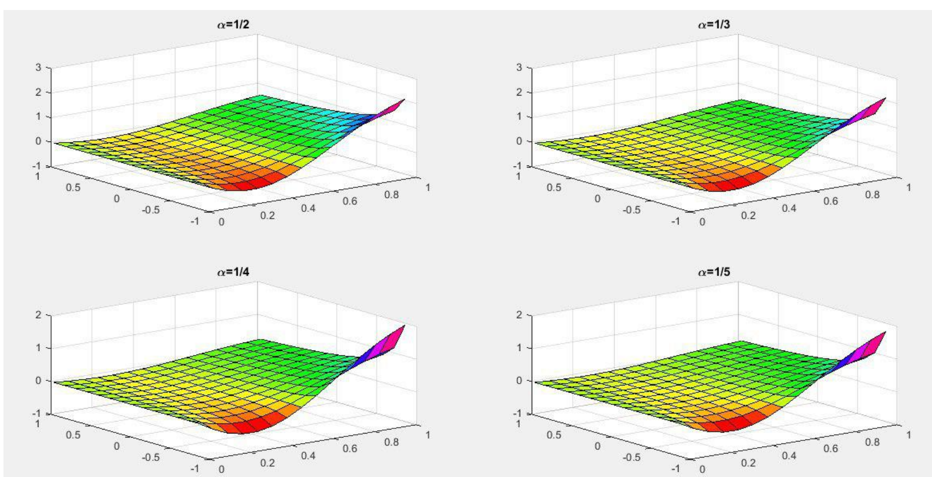

FIGURE 2. $u = \frac{1}{\sqrt{x}} \frac{t^{\alpha-1}}{\Gamma(\alpha)}$

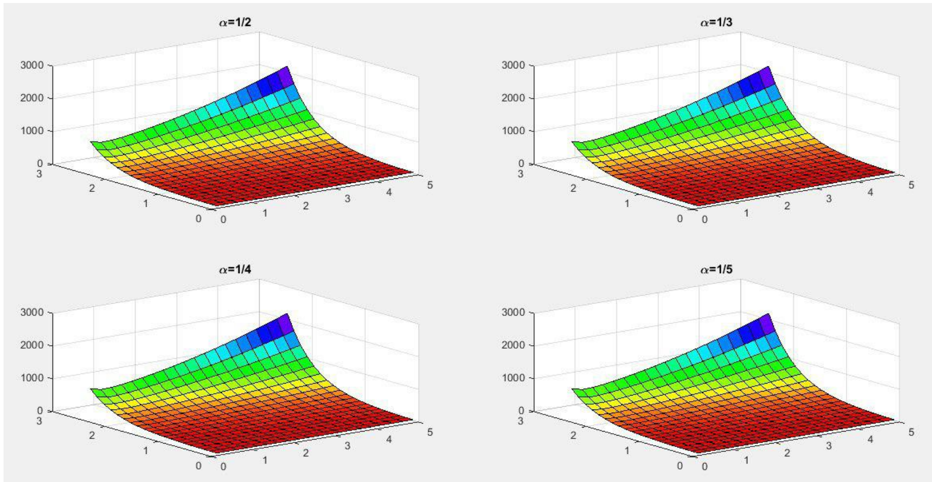

FIGURE 3. $u = \frac{e^t}{\sqrt{x}}\left[\sinh(A) + \cosh(A)\right]$

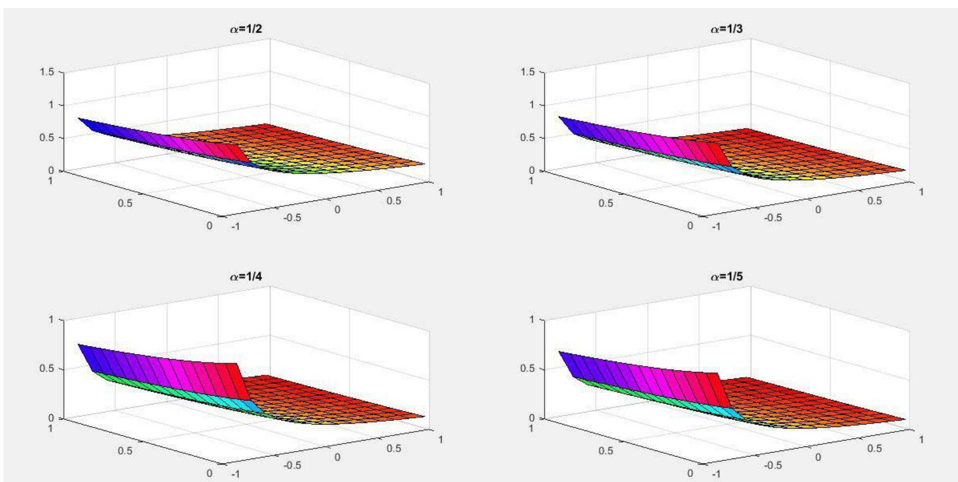

FIGURE 4. $u = \frac{t^{\alpha}}{(1+\sqrt{x})\Gamma(\alpha)}$

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