Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 8, No. 1, 2020, pp. 54-68
DOI:10.22034/cmde.2019.9498

Neumann method for solving conformable fractional Volterra integral equations

Mousa Ilie
Department of Mathematics, Rasht Branch,
Islamic Azad University, Rasht, Iran.
E-mail: ilie@iaurasht.ac.ir

Jafar Biazar*
Department of Applied Mathematics, Faculty of Mathematical Sciences,
University of Guilan, P.O. Box. 41335-1914, Guilan, Rasht, Iran.
E-mail: biazar@iaurasht.ac.ir
E-mail: biazar@guilan.ac.ir

Zainab Ayati
Department of Engineering Sciences, Faculty of Technology and Engineering,
East of Guilan, University of Guilan, P.C. 44891-63157, Rudsar-Vajargah, Iran.
E-mail: ayati.zainab@gmail.com

Abstract This paper deals with the solution of a class of Volterra integral equations in the sense of the conformable fractional derivative. For this goal, the well-organized Neumann method is developed and some theorems related to existence, uniqueness, and sufficient condition of convergence are presented. Some illustrative examples are provided to demonstrate the efficiency of the method in solving conformable fractional Volterra integral equations.

Keywords. Volterra integral equations; Conformable fractional derivative; Neumann method; Existence, uniqueness, and sufficient condition of convergence.

2010 Mathematics Subject Classification. 45D99-65R20-34A08.

1. Introduction

Fractional calculus is as old as usual calculus. During the past several years, many researchers have been trying to generalize the concept of the usual calculus to fractional calculus and some definitions for fractional derivative and integral, by Riemann, Liouville, Grenville, Caputo, and others are presented (see [29]). However, some of these definitions suffer disadvantages that caused their application confront difficulties such as: satisfying the derivative product rule, the derivative quotient rule, and the chain rule. In 2014, Khalil and his collaborators presented a new definition for fractional derivative and integral, called conformable fractional derivative and integral, that removes all of the drawbacks of aforementioned definitions (see [1, 27]). In what follows same basic definitions are stated referred to (see [27]).

Received: 22 February 2018 ; Accepted: 24 September 2018.
$*$ Corresponding author.

Definition 1.1. Consider a function $f : [0,\infty) \to \mathbb{R}$, then conformable fractional derivative of f of order α is defined by

$$T_\alpha(f)(x) = \lim_{\varepsilon\to 0} \frac{f(x+\varepsilon x^{1-\alpha}) - f(x)}{\varepsilon},$$

for all $x > 0$ and $\alpha \in (0,1]$. If f is α-differentiable in some open interval $(0,a)$, and $\lim_{x\to 0^+} T_\alpha(f)(x)$ exists, then one can define $T_\alpha(f)(0) = \lim_{x\to 0^+} T_\alpha(f)(x)$.

If the conformable derivative of f of order α exists, then we simply say that f is α-differentiable (see [1, 15–25, 27]).

Definition 1.2. Given a function $f : [a,\infty) \to \mathbb{R}, a \geq 0$, then the conformable fractional integral of f is defined by

$$I^a_\alpha(f)(x) = \int_a^x \frac{f(t)}{t^{1-\alpha}} dt,$$

where the integral is the usual Riemann improper integral and $\alpha \in (0,1)$ (see [1, 15–25, 27]).

For the sake of simplicity, let us consider $I^0_\alpha(f)(x) = I_\alpha(f)(x)$. One of the most useful facts, in this concept, is the following statement (see [1, 15–25, 27]).
Many phenomena in our real world are described by fractional differential equations (FDEs) and fractional integral equations (FIEs). In recent years, many effective methods have been proposed and exerted to solve fractional integral equations (see [2–6, 8–14, 26, 28–30]). In this study, the Neumann method is successfully developed to handle an important class of Volterra integral equations in the sense of conformable fractional derivative.
The organization of the paper is as follows: In Section 2, some fundamental concepts related to conformable fractional Volterra integral equations of the second kind are given. In Section 3, the Neumann method is presented to solve conformable fractional integral equation. In Section 4, some illustrative examples are provided to show the efficiency of the method. Finally, conclusion is given in Section 5.

2. Some fundamental concepts related to conformable fractional Volterra integral equation (CFVIEs) of the second kind

In this section, some fundamental concepts such as regular value, existence, uniqueness, sufficient condition of convergence, and conformable fractional Neumann series related to conformable fractional Volterra integral equations are presented.

2.1. **Regular value.** Consider the conformable fractional Volterra integral equations as the following form

$$x(s) = y(s) + \lambda I^a_\alpha \left(K(s,t)x(t)\right), \ \forall\alpha \in (0,1] \tag{2.1}$$

where y and K are known function, λ and a are constants and x is an unknown function(see [7, 25]). Applying conformable fractional integral definition on equation (2.1), results in

$$x(s) = y(s) + \lambda \int_a^s \frac{K(s,t)x(t)}{t^{1-\alpha}} dt. \tag{2.2}$$

By considering

$$K^{\alpha}(s,t) = \frac{K(s,t)}{t^{1-\alpha}}, \tag{2.3}$$

as conformable fractional Volterra kernel, and substituting (2.3) into (2.2), we obtain

$$x(s) = y(s) + \lambda \int_a^s K^{\alpha}(s,t)x(t)dt. \tag{2.4}$$

According to equation (2.4), the operator form of CFVIEs (2.1), can be denoted as follows

$$x = y + \lambda K^{\alpha} x, \qquad \forall \alpha \in (0,1], \tag{2.5}$$

or

$$L^{\alpha} x = (I - \lambda K^{\alpha})x = y. \tag{2.6}$$

Definition 2.1. Let's consider $\lambda = \lambda_0$, $\alpha = \alpha_0$, and $(L^{\alpha_0})^{-1}$ as an $\mathcal{L}^2$ operator, exists and satisfies

$$(L^{\alpha_0})^{-1} L^{\alpha_0} = L^{\alpha_0} (L^{\alpha_0})^{-1} = I, \tag{2.7}$$

then λ_0 is called a regular value of the conformable fractional operator K^{α_0} (see [7, 25]).

Theorem 2.2. *If for a given $\alpha = \alpha_0$ and $\lambda = \lambda_0$, the operator $(L^{\alpha_0})^{-1}$ exists, then it is unique (see [7, 25]).*

Proof. Suppose that $(L_1^{\alpha_0})^{-1}$ and $(L_2^{\alpha_0})^{-1}$ are two $\mathcal{L}^2$ operators that satisfy Eq. (2.7), and let

$$H = (L_1^{\alpha_0})^{-1} - (L_2^{\alpha_0})^{-1}.$$

Regarding Eq. (2.7), one has

$$\begin{aligned} &(L_1^{\alpha_0})^{-1} L^{\alpha_0} = L^{\alpha_0} (L_1^{\alpha_0})^{-1} = I, \\ &(L_2^{\alpha_0})^{-1} L^{\alpha_0} = L^{\alpha_0} (L_2^{\alpha_0})^{-1} = I, \end{aligned} \tag{2.8}$$

and subtracting these two relations results in

$$H L^{\alpha_0} = L^{\alpha_0} = 0. \tag{2.9}$$

Multiplying Eq. (2.9) by the conformable fractional operator $(L_1^{\alpha_0})^{-1}$ and regarding Eq. (2.8), we get $H = 0$. □

Theorem 2.3. *If λ is a regular value of the conformable fractional operator K^{α}, with inverse conformable fractional operatore $(L^{\alpha_0})^{-1}$, then for any $\mathcal{L}^2$ function y, Eq. (2.6) has an unique $\mathcal{L}^2$ solution, say, x, satisfying (see [7, 25])*

$$x = (L^{\alpha_0})^{-1} y. \tag{2.10}$$

Proof. By Substitution of equation (2.10) into equation (2.6), we have

$$L^{\alpha}(L^{\alpha})^{-1}y = y, \tag{2.11}$$

and since $L^{\alpha}(L^{\alpha})^{-1} = I$, thus the function x, defined by Eq. (2.10), is a solution of Eq. (2.6). To show the uniqueness, Lets x_1 and x_2 be two different solutions of (2.6), then

$$L^{\alpha}(x_1 - x_2) = 0,$$

hence

$$(L^{\alpha})^{-1}L^{\alpha}(x_1 - x_2) = 0.$$

So

$$x_1 = x_2,$$

which completes the proof. □

2.2. **Conformable fractional Neumann series.** If λ is a regular value of conformable fractional operator K^{α}, then the Eq. (2.6) has a unique solution

$$x = (L^{\alpha})^{-1}y = (I - \lambda K^{\alpha})^{-1}y.$$

So

$$(L^{\alpha})^{-1} = (I - \lambda K^{\alpha})^{-1} = I + \lambda K^{\alpha} + (\lambda K^{\alpha})^2 + (\lambda K^{\alpha})^3 + (\lambda K^{\alpha})^4 + \cdots,$$

$$(L^{\alpha})^{-1} = I + \sum_{n=1}^{\infty}(\lambda K^{\alpha})^n, \tag{2.12}$$

where Eq. (2.12) is called the conformable fractional Neumann series for the invers conformable fractional operator $(L^{\alpha})^{-1}$.
We set

$$\begin{aligned}
x_0 &= y,\\
x_1 &= y + \lambda K^{\alpha}x_0 = y + \lambda K^{\alpha}y,\\
x_2 &= y + \lambda K^{\alpha}x_1 = y + \lambda K^{\alpha}y + (\lambda K^{\alpha})^2 y,\\
&\vdots
\end{aligned}$$

so, the nth approximation to x_n, can be presented as below

$$x_n = y + \lambda K^{\alpha}x_{n-1} = y + \sum_{i=1}^{n}(\lambda K^{\alpha})^i y.$$

Therefore, if the sequence of functions x_n have a limit as $n \to \infty$, then

$$x = \lim_{n\to\infty} x_n = y + \sum_{i=1}^{\infty}(\lambda K^{\alpha})^i y, \tag{2.13}$$

where Eq. (2.13) is called the conformable fractional Neumann series for the solution x of CFVIEs (2.1) (see [7]).
The following theorem states the sufficient condition for convergence.

Theorem 2.4. *The conformable fractional Neumann series (2.12), for $(L^{\alpha})^{-1}$, $\alpha \in (0,1]$ converges strongly if $||\lambda K^{\alpha}|| < 1$ (see [7, 25]).*

Proof. Assume that α is given and considered as a constant throughout the proof. Define

$$S_n = \sum_{i=0}^{n} (\lambda K^{\alpha})^i, \tag{2.14}$$

and take $n > m$. Regarding equation (2.14), we have

$$\|S_n - S_m\| \le \sum_{i=m+1}^{n} \|\lambda K^{\alpha}\|^i = \frac{\|\lambda K^{\alpha}\| \left(\|\lambda K^{\alpha}\|^m - \|\lambda K^{\alpha}\|^n\right)}{1 - \|\lambda K^{\alpha}\|}. \tag{2.15}$$

Since $\|\lambda K^{\alpha}\| < 1$, thus

$$\lim_{n\to\infty} \|\lambda K^{\alpha}\|^n = 0, \tag{2.16}$$

by considering equations (2.15) and (2.16), we derive

$$\lim_{n,m\to\infty} \|S_n - S_m\| = 0. \tag{2.17}$$

So, the sequence $\{S_n\}$ is a Cauchy sequence, so the limit S_n exists.
Now, lets consider the residual R_n as the following form

$$R_n = I - (I - \lambda K^{\alpha}) S_n. \tag{2.18}$$

Setting equation (2.14) in equation (2.18), results in

$$R_n = (\lambda K^{\alpha})^{n+1},$$
$$\|R_n\| \le \|\lambda K^{\alpha}\|^{n+1}.$$

Since $\|\lambda K^{\alpha}\| < 1$, therefore

$$\lim_{n\to\infty} \|R_n\| = 0.$$

Then, conformable fractional operator $(L^{\alpha})^{-1}$, is a right inverse of L^{α}, a similar proof shows that it is also a left inverse of conformable fractional operatore L^{α}. □

Lemma 2.5. *If K^{α} is an $\mathcal{L}^2$ conformable fractional Volterra operator for a given α, and $b > a$, then*

$$\left|(K^{\alpha})^{n+1}(s,t)\right| \le \frac{\|K^{\alpha}\|_E^{n+1}}{[(n-1)!]^{\frac{1}{2}}} k_1^{\alpha}(s)\, k_2^{\alpha}(s),$$

where (see [7, 25])

$$k_1^{\alpha}(s) = \left[\int_a^s |K^{\alpha}(s,t)|^2 ds\right]^{\frac{1}{2}},$$

$$k_2^{\alpha}(s) = \left[\int_t^b |K^{\alpha}(s,t)|^2 ds\right]^{\frac{1}{2}}.$$

Proof. For $\alpha = 1$, refer to books [7, 31, 32]. □

Theorem 2.6. *If K^α is an $\mathcal{L}^2$ conformable fractional Volterra operator for a given α, the Neumann series (2.12), converges strongly for all λ to the inverse conformable fractional operator of K^α (see [17, 25]).*

Proof. According to Eq. (2.14), for $n > m$, we obtain

$$\|S_n - S_m\|_E \leq \sum_{i=m+1}^{n} \|\lambda K^\alpha\|_E^i. \tag{2.19}$$

But from lemma (2.5) and Euclidean norm, we get

$$\|\lambda K^\alpha\|_E^i \leq |\lambda|^i \frac{\|K^\alpha\|_i^E}{[(i-2)!]^{\frac{1}{2}}},$$

and hence, for allλ,

$$\lim_{i\to\infty} \|\lambda K^\alpha\|_E^i = 0. \tag{2.20}$$

By considering equations (2.19) and (2.20), we persuade the sequence $\{S_n\}$ is Cauchy, so the Neumann series (2.12), converges strongly for all λ to the inverse conformable fractional operator of K^α. □

3. The conformable fractional Neumann method (CFNM)

Consider the conformable fractional Volterra integral equations of second kind as follows

$$x(s) = y(s) + \lambda I_\alpha^a \left(K(s,t)x(t)\right),$$

where y, K are known functions and λ, a are constants and x an unknown function. We define

$$\begin{aligned}
x_0\left(s\right) &= y\left(s\right),\\
x_1\left(s\right) &= y\left(s\right) + \lambda I_\alpha^a\left(K\left(s,t\right)x_0(t)\right) = y\left(s\right) + \lambda I_\alpha^a\left(K\left(s,t\right)y(t)\right),\\
x_2\left(s\right) &= y\left(s\right) + \lambda I_\alpha^a\left(K\left(s,t\right)x_1(t)\right)\\
&= y\left(s\right) + \lambda I_\alpha^a\left(K\left(s,t\right)y(t)\right) + \lambda^2 I_\alpha^a\left(K\left(s,t\right)I_\alpha^a\left(K(t,t_1)y(t_1)\right)\right),\\
x_3\left(s\right) &= y\left(s\right) + \lambda I_\alpha^a\left(K\left(s,t\right)x_2(t)\right)\\
&= y\left(s\right) + \lambda I_\alpha^a\left(K\left(s,t\right)y(t)\right) + \lambda^2 I_\alpha^a\left(K\left(s,t\right)I_\alpha^a\left(K(t,t_1)y(t_1)\right)\right)\\
&+ \lambda^3 I_\alpha^a\left(K\left(s,t\right)I_\alpha^a\left(K(t,t_1)\left(I_\alpha^a\left(K(t_1,t_2)y(t_2)\right)\right)\right)\right),\\
&\vdots
\end{aligned}$$

moreover the n-th approximations x_n, to x, will be as

$$\begin{aligned} x_n(s) &= y(s) + \lambda I_\alpha^a \left(K(s,t)\, x_{n-1}(t)\right) \\ &= y(s) + \lambda I_\alpha^a \left(K(s,t)\, y(t)\right) \\ &+ \lambda^2 I_\alpha^a \left(K(s,t)\, I_\alpha^a \left(K(t,t_1) y(t_1)\right)\right) \\ &+ \cdots + \lambda^n I_\alpha^a \left(K(s,t)\, I_\alpha^a \left(K(t,t_1) \cdots \left(I_\alpha^a \left(K(t_{n-1},t_{n-2}) y(t_n)\right)\right)\right)\right). \end{aligned}$$

The solution of CFVIEs is

$$x(s) = \lim_{n\to\infty} x_n(s).$$

4. Examples

In this section, some illustrative examples are provided to demonstrate the efficiency of the method in solving conformable fractional Volterra integral equations of second kind.

Example 4.1. Consider the following conformable fractional Volterra integral equation (see [17, 25])

$$x(s) = 1 - I_\alpha \left(\left(\frac{1}{\alpha} s^\alpha - \frac{1}{\alpha} t^\alpha \right) x(t) \right), \tag{4.1}$$

where its exact solution is as follows

$$x(s) = \cos\left(\frac{1}{\alpha} s^\alpha\right).$$

According to the proposed conformable fractional Neumann method, we have

$$\begin{aligned} & x_0(s) = 1, \\ & x_1(s) = 1 - I_\alpha \left(\frac{1}{\alpha} s^\alpha - \frac{1}{\alpha} t^\alpha \right), \qquad (4.2) \\ & x_2(s) = 1 - I_\alpha \left(\frac{1}{\alpha} s^\alpha - \frac{1}{\alpha} t^\alpha \right) + I_\alpha \left(\left(\frac{1}{\alpha} s^\alpha - \frac{1}{\alpha} t^\alpha \right) I_\alpha \left(\frac{1}{\alpha} s^\alpha - \frac{1}{\alpha} t^\alpha \right) \right), \\ & x_3(s) = 1 - I_\alpha \left(\frac{1}{\alpha} s^\alpha - \frac{1}{\alpha} t^\alpha \right) + I_\alpha \left(\left(\frac{1}{\alpha} s^\alpha - \frac{1}{\alpha} t^\alpha \right) I_\alpha \left(\frac{1}{\alpha} s^\alpha - \frac{1}{\alpha} t^\alpha \right) \right) \\ & \qquad - I_\alpha \left(\left(\frac{1}{\alpha} s^\alpha - \frac{1}{\alpha} t^\alpha \right) I_\alpha \left(\left(\frac{1}{\alpha} s^\alpha - \frac{1}{\alpha} t^\alpha \right) I_\alpha \left(\frac{1}{\alpha} s^\alpha - \frac{1}{\alpha} t^\alpha \right) \right) \right), \\ & \qquad \vdots \end{aligned}$$

By solving this sequence of integral equations, the solution of equation (4.1), can be obtained as the following form

$$x(s) = 1 - \frac{1}{2}\left(\frac{1}{\alpha} s^\alpha\right)^2 + \frac{1}{24}\left(\frac{1}{\alpha} s^\alpha\right)^4 - \frac{1}{720}\left(\frac{1}{\alpha} s^\alpha\right)^6 + \cdots = \cos\left(\frac{1}{\alpha} s^\alpha\right).$$

This solution is the same as the exact solution.

Example 4.2. Consider the following CFVIE (see [17, 25])

$$x(s) = \exp\left(\frac{1}{\alpha}s^{\alpha}\right) + I_{\alpha}\left(\exp\left(\frac{1}{\alpha}s^{\alpha} - \frac{1}{\alpha}t^{\alpha}\right)x(t)\right), \tag{4.3}$$

whit the exact solution

$$x(s) = \exp\left(\frac{2}{\alpha}s^{\alpha}\right).$$

According to the CFNM approach, we have

$$\begin{aligned}
x_0(s) &= \exp\left(\frac{1}{\alpha}s^{\alpha}\right), \\
x_1(s) &= \exp\left(\frac{1}{\alpha}s^{\alpha}\right) + I_{\alpha}\left(\left(\frac{1}{\alpha}s^{\alpha} - \frac{1}{\alpha}t^{\alpha}\right)\exp\left(\frac{1}{\alpha}t^{\alpha}\right)\right), \qquad (4.4)\\
x_2(s) &= \exp\left(\frac{1}{\alpha}s^{\alpha}\right) + I_{\alpha}\left(\left(\frac{1}{\alpha}s^{\alpha} - \frac{1}{\alpha}t^{\alpha}\right)\exp\left(\frac{1}{\alpha}t^{\alpha}\right)\right) \\
&\quad + I_{\alpha}\left(\left(\frac{1}{\alpha}s^{\alpha} - \frac{1}{\alpha}t^{\alpha}\right)I_{\alpha}\left(\left(\frac{1}{\alpha}s^{\alpha} - \frac{1}{\alpha}t^{\alpha}\right)\exp\left(\frac{1}{\alpha}t_1^{\alpha}\right)\right)\right), \\
x_3(s) &= \exp\left(\frac{1}{\alpha}s^{\alpha}\right) + I_{\alpha}\left(\left(\frac{1}{\alpha}s^{\alpha} - \frac{1}{\alpha}t^{\alpha}\right)\exp\left(\frac{1}{\alpha}t^{\alpha}\right)\right) \\
&\quad + I_{\alpha}\left(\left(\frac{1}{\alpha}s^{\alpha} - \frac{1}{\alpha}t^{\alpha}\right)I_{\alpha}\left(\left(\frac{1}{\alpha}t^{\alpha} - \frac{1}{\alpha}t_1^{\alpha}\right)\exp\left(\frac{1}{\alpha}t_1^{\alpha}\right)\right)\right) \\
&\quad + I_{\alpha}\left(\left(\frac{1}{\alpha}s^{\alpha} - \frac{1}{\alpha}t^{\alpha}\right)I_{\alpha}\left(\left(\frac{1}{\alpha}t^{\alpha} - \frac{1}{\alpha}t_1^{\alpha}\right)I_{\alpha}\left(\left(\frac{1}{\alpha}t_1^{\alpha} - \frac{1}{\alpha}t_2^{\alpha}\right)\exp\left(\frac{1}{\alpha}t_2^{\alpha}\right)\right)\right)\right), \\
&\ \ \vdots
\end{aligned}$$

By solving this sequence of integral equations, n-th order approximation of Eq. (4.2) is

$$\begin{aligned}
x_n(s) = \exp\left(\frac{1}{\alpha}s^{\alpha}\right) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.5)\\
\left[1 + \left(\frac{1}{\alpha}s^{\alpha}\right) + \frac{1}{2}\left(\frac{1}{\alpha}s^{\alpha}\right)^2 + \frac{1}{6}\left(\frac{1}{\alpha}s^{\alpha}\right)^3 + \cdots + \frac{1}{n!}\left(\frac{1}{\alpha}s^{\alpha}\right)^n\right].
\end{aligned}$$

According to the expansion of the exponential function and equation (4.5), it is obvious that

$$x(s) = \lim_{n\to\infty} x_n(s) = \exp\left(\frac{2}{\alpha}s^{\alpha}\right).$$

This solution is the same as the exact solution.

Example 4.3. Consider the following Volterra conformable fractional integral equation (see [17, 25])

$$x(s) = 2 + s^2 + I_{\alpha}((s-t)x(t)), \tag{4.6}$$

where for $\alpha = 1$, the exact solution of Eq. (4.6) is as follows

$$x(s) = 4\cosh -2.$$

Applying to the proposed conformable fractional Neumann method, results in

$$
\begin{aligned}
&x_0\left(s\right) = 2 + s^2, \\
&x_1\left(s\right) = 2 + s^2 + I_\alpha\left(\left(s-t\right)\left(2+t^2\right)\right), \qquad\qquad (4.7)\\
&x_2\left(s\right) = 2 + s^2 + I_\alpha\left(\left(s-t\right)\left(2+t^2\right)\right) + I_\alpha\left(\left(s-t\right) I_\alpha\left(\left(t-t_1\right)\left(2+t_1^2\right)\right)\right), \\
&x_3\left(s\right) = 2 + s^2 + I_\alpha\left(\left(s-t\right)\left(2+t^2\right)\right) + I_\alpha\left(\left(s-t\right) I_\alpha\left(\left(t-t_1\right)\left(2+t_1^2\right)\right)\right) \\
&\qquad + I_\alpha\left(\left(s-t\right) I_\alpha\left(\left(t-t_1\right) I_\alpha\left(\left(t_1-t_2\right)\left(2+t_2^2\right)\right)\right)\right), \\
&\qquad \vdots
\end{aligned}
$$

The corresponding solutions of these sequences are as below

$$
\begin{aligned}
&x_0\left(s\right) = 2 + s^2, \\
&x_1(s) = \frac{1}{\alpha\left(\alpha+1\right)\left(\alpha+2\right)\left(\alpha+3\right)}\left(12\alpha + 22\alpha^2 + 12\alpha^3 + 2\alpha^4\right. \\
&\qquad + \left(12 + 10\alpha + 2\alpha^2\right) s^{\alpha+1} + \left(\alpha + \alpha^2\right) s^{\alpha+3} \\
&\qquad \left. + \left(6\alpha + 11\alpha^2 + 6\alpha^3 + \alpha^4\right) s^2\right) \\
&\qquad \vdots
\end{aligned}
$$

The seven-terms approximate solutions of Eq. (4.6), for $\alpha = 0.4, 0.6, 0.8, 1.0$, will be obtained, respectively as follows

$$
\begin{aligned}
x_7^{0.4}\left(s\right) = {} & 2 + s^2 + 0.1225490196 s^{\frac{17}{5}} + 3.571428571 s^{\frac{7}{5}} + 0.006718696249 s^{\frac{24}{5}} \\
& + 0.7086167799 s^{\frac{14}{5}} + 0.0002083962858 s^{\frac{31}{5}} + 0.05272446284 s^{\frac{21}{5}} \\
& + 0.000041546308954 s^{\frac{38}{5}} + 0.002046757097 s^{\frac{28}{5}} + 5.770320685\ 10^{-8} s^9 \\
& + 0.0000487323118 s^7 + 5.902537525\ 10^{-10} s^{\frac{52}{5}} + 7.839818501\ 10^{-7} s^{\frac{42}{5}},
\end{aligned}
$$

$$
\begin{aligned}
x_7^{0.6}\left(s\right) = {} & 2 + s^2 + 0.106837606 s^{\frac{18}{5}} + 2.083333334 s^{\frac{8}{5}} + 0.004891831816 s^{\frac{26}{5}} \\
& + 0.295928030 s^{\frac{16}{5}} + 0.0001240322468 s^{\frac{34}{5}} + 0.01622412446 s^{\frac{24}{5}} \\
& + 0.00004154630895 s^{\frac{38}{5}} + 0.002046757097 s^{\frac{28}{5}} + 5.770320685\ 10^{-8} s^9 \\
& + 0.000001995370766 s^{\frac{42}{5}} + 0.0004694480457 s^{\frac{32}{5}} + 2.217078629\ 10^{-8} s^{10} \\
& + 0.000008383000821 s^8 + 1.803089322\ 10^{-10} s^{\frac{58}{5}} + 1.015382851\ 10^{-7} s^{\frac{48}{5}},
\end{aligned}
$$

$$x_7^{0.8}(s) = 2 + s^2 + 0.09398496239s^{\frac{19}{5}} + 1.388888889s^{\frac{9}{5}} + 0.003648484567s^{\frac{28}{5}}$$
$$+ 0.1483855651s^{\frac{18}{5}} + 0.1483855651s^{\frac{37}{5}} + 0.006245183709s^{\frac{27}{5}}$$
$$+ 0.000001021172568s^{\frac{46}{5}} + 0.000001021172568s^{\frac{36}{5}} + 9.283386987\ 10^{-9}s^{11}$$
$$+ 0.000001943070402s^9 + 6.146310236\ 10^{-11}s^{\frac{64}{5}} + 1.835856387\ 10^{-8}s^{\frac{54}{5}},$$

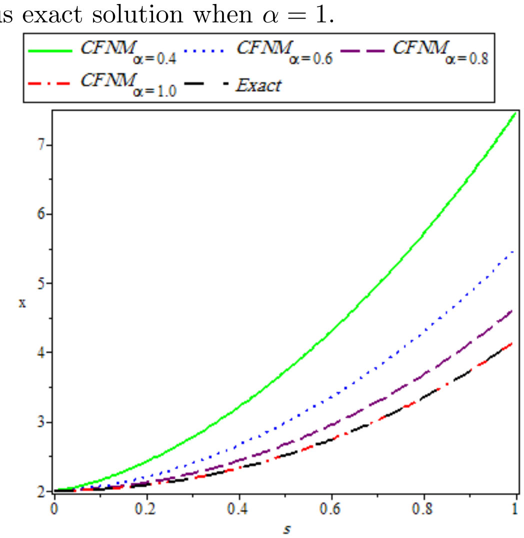
$$x_7^{1.0}(s) = 2 + 2s^2 + 0.1666666667s^4 + 0.005555555556s^6 + 0.00009920634921s^8$$
$$+ 0.000001102292769s^{10} + 8.350702795\ 10^{-9}s^{12} + 2.294149120\ 10^{-11}s^{14}.$$

According to Taylor expansion of $x(s)$, it clearly has seen that

$$\lim_{n\to\infty} x_n^{1.0}(s) = 4\cosh s - 2.$$

In Figure 1, the seventh-order approximate solution of conformable fractional Volterra integral equation for $\alpha = 0.4, 0.6, 0.8, 1.0$ and exact solution for $\alpha = 1$ are plotted.

Figure 1. The 7th-order approximation of CFNM for different values α versus exact solution when $\alpha = 1$.



Example 4.4. Consider the CFVIEs as follows (see [25])

$$x(s) = 3\sin\left(\frac{2}{\alpha}s^{\alpha}\right) - I^{\alpha}\left(\left(\frac{1}{\alpha}s^{\alpha} - \frac{1}{\alpha}t^{\alpha}\right)x(t)\right), \tag{4.8}$$

with the exact solution

$$x(s) = 4\sin\left(\frac{2}{\alpha}s^{\alpha}\right) - 2\sin\left(\frac{1}{\alpha}s^{\alpha}\right).$$

By using the proposed CFNM approach, we gain

$$
\begin{aligned}
x_0\left(s\right) &= 3\sin\left(\frac{2}{\alpha}s^{\alpha}\right),\\
x_1\left(s\right) &= 3\sin\left(\frac{2}{\alpha}s^{\alpha}\right) + I_{\alpha}\left(\left(\frac{1}{\alpha}s^{\alpha} - \frac{1}{\alpha}t^{\alpha}\right)3\sin\left(\frac{2}{\alpha}t^{\alpha}\right)\right),\\
x_2\left(s\right) &= 3\sin\left(\frac{2}{\alpha}s^{\alpha}\right) + I_{\alpha}\left(\left(\frac{1}{\alpha}s^{\alpha} - \frac{1}{\alpha}t^{\alpha}\right)3\sin\left(\frac{2}{\alpha}t^{\alpha}\right)\right)\\
&\quad + I_{\alpha}\left(\left(\frac{1}{\alpha}s^{\alpha} - \frac{1}{\alpha}t^{\alpha}\right)I_{\alpha}\left(\left(\frac{1}{\alpha}s^{\alpha} - \frac{1}{\alpha}t_1^{\alpha}\right)3\sin\left(\frac{2}{\alpha}t_1^{\alpha}\right)\right)\right),\\
x_3\left(s\right) &= 3\sin\left(\frac{2}{\alpha}s^{\alpha}\right) + I_{\alpha}\left(\left(\frac{1}{\alpha}s^{\alpha} - \frac{1}{\alpha}t^{\alpha}\right)3\sin\left(\frac{2}{\alpha}t^{\alpha}\right)\right)\\
&\quad + I_{\alpha}\left(\left(\frac{1}{\alpha}s^{\alpha} - \frac{1}{\alpha}t^{\alpha}\right)I_{\alpha}\left(\left(\frac{1}{\alpha}s^{\alpha} - \frac{1}{\alpha}t_1^{\alpha}\right)3\sin\left(\frac{2}{\alpha}t_1^{\alpha}\right)\right)\right)\\
&\quad + I_{\alpha}\left(\left(\frac{1}{\alpha}s^{\alpha} - \frac{1}{\alpha}t^{\alpha}\right)I_{\alpha}\left(\left(\frac{1}{\alpha}t^{\alpha} - \frac{1}{\alpha}t_1^{\alpha}\right)I_{\alpha}\left(\left(\frac{1}{\alpha}{t_1}^{\alpha} - \frac{1}{\alpha}t_2^{\alpha}\right)3\sin\left(\frac{2}{\alpha}t_2^{\alpha}\right)\right)\right)\right),\\
&\ \ \vdots
\end{aligned}
\tag{4.9}
$$

By solving above sequences of integral equations, the 6th-order approximate solution of equation (4.8), can be presented as follows

$$
\begin{aligned}
x_6(s) = \frac{4029}{1012}\sin\left(\frac{2}{\alpha}s^{\alpha}\right) - \frac{993}{512}\left(\frac{1}{\alpha}s^{\alpha}\right) + \frac{225}{768}\left(\frac{1}{\alpha}s^{\alpha}\right)^3 - \frac{33}{1280}\left(\frac{1}{\alpha}s^{\alpha}\right)^5\\
+ \frac{1}{40320}\left(\frac{1}{\alpha}s^{\alpha}\right)^7 - \frac{1}{241920}\left(\frac{1}{\alpha}s^{\alpha}\right)^9.
\end{aligned}
$$

In Figures 2, 3, 4, the 6th-order approximate and exact solutions of Volterra conformable fractional integral equation (4.8), for $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$ are plotted.

FIGURE 2. The 6th-order approximation of CFNM for different values α versus exact solution.

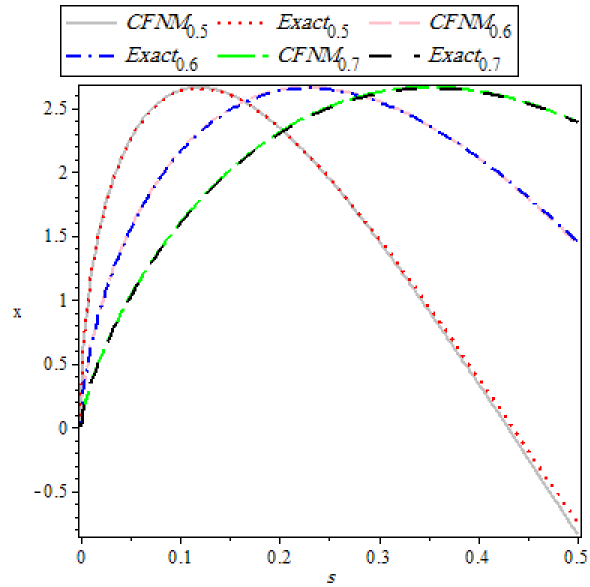


FIGURE 3. The 6th-order approximation of CFNM for different values α versus exact solution.

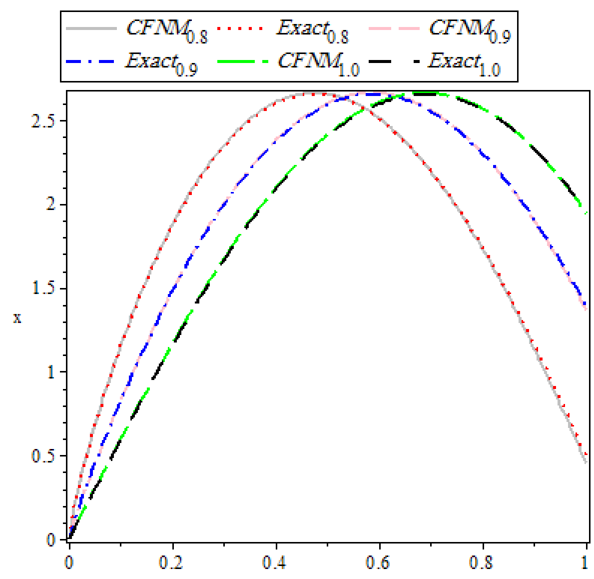
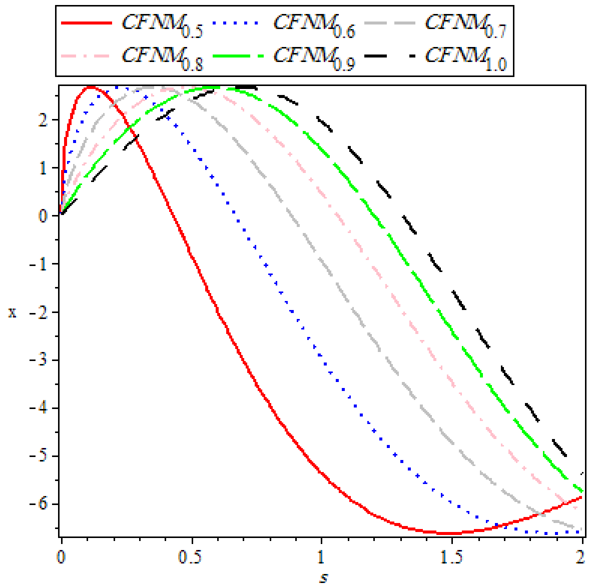


FIGURE 4. The 6th-order approximation of CFNM for different values α.



5. Conclusion

Under investigation in the present paper was been offered the solution of an important class of Volterra integral equations in the sense of the conformable fractional derivative. For this purpose, the well-established Neumann method was successfully developed and some theorems related to existence, uniqueness, and sufficient condition of convergence were been provided. Some illustrative examples also were been presented, confirming the super performance of the method in solving conformable fractional Volterra integral equations.

References

[1] T. Abdeljawad, *On conformable fractional calculus, Journal of Computational and Applied Mathematics*, *279* (2015), 57-66.

[2] R. Agarwal, S. Jain, and R. P. Agarwal, *Solution of fractional Volterra integral equation and non-homogeneous time fractional heat equation using integral transform of pathway type*, Progress in Fractional Differentiation and Applications, *1* (2015), 145-155.

[3] A. Aghili and H. Zeinali, *Solution to Volterra singular integral equations and nonhomogeneous time fractional PDEs*, General Mathematics Notes, *14* (2013), 6-20.

[4] W. A. Ahmood and A. Kilicman, *Solutions of linear multi-dimensional fractional order Volterra integral equations*, Journal of Theoretical and Applied Information Technology, *89* (2016), 381-388.

[5] A. Atangana and N. Bildik, *Existence and numerical solution of the Volterra fractional integral equations of the second kind*, Mathematical Problems in Engineering, *2013* (2013), 981526.

[6] M. R. A. Darani and S. Bagheri, *Fractional type of flatlet oblique multiwavelet for solving fractional differential and integro-differential equations*, Computational Methods for Differential Equations, *2*(4) (2014), 268-282.

[7] L. M. Delves and J. L. Mohamed, *Computational methods for integral equations*, Cambridge University Press, 1985.

[8] K. Diethelm and N. J. Ford, *Volterra integral equations and fractional calculus: Do neighbouring solutions intersect?*, Journal of Integral Equations and Applications, *24* (2012), 25-37.

[9] R. M. Evans, U. N. Katugampola, and D. A. Edwards, *Applications of fractional calculus in solving Abel-typeintegral equations: Surface-volume reaction problem*, Computers and Mathematics with Applications, *73* (2017), 1346-1362.

[10] K. Hosseini and R. Ansari, *New exact solutions of nonlinear conformable time-fractional Boussinesq equations using the modified Kudryashov method*, Waves in Random and Complex Media, (2017), DOI: 10.1080/17455030.2017.1296983.

[11] K. Hosseini, A. Bekir, M. Kaplan, and Ö. Guner, *On a new technique for solving the nonlinear conformable time-fractional differential equations*, Optical and Quantum Electronics, *49* (2017), 343.

[12] K. Hosseini, P. Mayeli, and R. Ansari, *Bright and singular soliton solutions of the conformable time-fractional KleinGordon equations with different nonlinearities*, Waves in Random and Complex Media, (2018), DOI: 10.1080/17455030.2017.1362133.

[13] K. Hosseini, P. Mayeli, A. Bekir, and Ö. Guner, *Density-Dependent Conformable Space-time Fractional Diffusion-Reaction Equation and Its Exact Solutions*, Communications in Theoretical Physics, *69*(1) (2018), 1-4.

[14] K. Hosseini, Yun-Jie Xu, P. Mayeli, A. Bekir, Ping Yao, Qin Zhou, Ö. Guner, *A study on the conformable time-fractional KleinGordon equations with quadratic and cubic nonlinearities*, OPTOELECTRONICS AND ADVANCED MATERIALS RAPID COMMUNICATIONS, *11* (7-8) (2017), 423-429.

[15] M. Ilie, J. Biazar, and Z. Ayati, *General solution of Bernoulli and Riccati fractional differential equations based on conformable fractional derivative*, International Journal of Applied Mathematical Research, *6*(2) (2017), 49-51.

[16] M. Ilie, J. Biazar, and Z. Ayati, *Application of the Lie Symmetry Analysis for second-order fractional differential equations*, Iranian Journal of Optimization, *9*(2) (2017), 79-83.

[17] M. Ilie, J. Biazar, and Z. Ayati, *Optimal homotopy asymptotic method for conformable fractional Volterra integral equations of the second kind*, 49^{th} Annual Iranian Mathematics Conference, August 23-26, 2018, ISC 97180-51902.

[18] M. Ilie, J. Biazar, and Z. Ayati, *General solution of second order fractional differential equations*, International Journal of Applied Mathematical Research, *7*(2) (2018), 56-61.

[19] M. Ilie, J. Biazar, and Z. Ayati, *Lie Symmetry Analysis for the solution of first-order linear and nonlinear fractional differential equations*, International Journal

of Applied Mathematical Research, *7*(2) (2018), 37-41.

[20] M. Ilie, J. Biazar, and Z. Ayati, *The first integral method for solving some conformable fractional differential equations*, Optical and Quantum Electronics, *50*(2) (2018), https://doi.org/10.1007/s11082-017-1307-x.

[21] M. Ilie, J. Biazar, and Z. Ayati, *Resonant solitons to the nonlinear Schrdinger equation with different forms of nonlinearities*, Optik, *164* (2018), 201-209.

[22] M. Ilie, J. Biazar, and Z. Ayati, *Analytical solutions for conformable fractional Bratu-type equations*, International Journal of Applied Mathematical Research, *7*(1) (2018), 15-19.

[23] M. Ilie, J. Biazar, and Z. Ayati, *Optimal Homotopy Asymptotic Method for first-order conformable fractional differential equations*, Journal of Fractional Calculus and Applications, *10*(1) (2019), 33-45.

[24] M. Ilie, J. Biazar, and Z. Ayati, *Analytical solutions for second-order fractional differential equations via OHAM*, Journal of Fractional Calculus and Applications, *10*(1) (2019), 105-119.

[25] M. Ilie, M. Navidi, and A. Khoshkenar, *Analytical solutions for conformable fractional Volterra integral equations of the second kind*, 49^{th} Annual Iranian Mathematics Conference, August 23-26, 2018, ISC 97180-51902.

[26] H. K. Jassim, *The approximate solutions of Fredholm integral equations on Cantor sets within local fractional operators*, Sahand Communications in Mathematical Analysis, *3* (2016), 13-20.

[27] R. Khalil, M. A. Horani, A. Yousef, and M. Sababheh, *A new definition of fractional derivative*, Journal of Computational and Applied Mathematics, *264* (2014), 65-70.

[28] U. Lepik, *Solving fractional integral equations by the Haar wavelet method*, Applied Mathematics and Computation, *214* (2009), 468-478.

[29] E. C. d. Oliveira and J. A. Tenreiro Machado, *A review of definitions for fractional derivatives and integral*, Mathematical Problems in Engineering, *2014* (2014), 238459.

[30] K. Sharma, R. Jain, and V. S. Dhakar, *A solution of generalized fractional Volterra type integral equation involving 4 function*, General Mathematics Notes, *8* (2012), 15-22.

[31] F. Smithies, *Integral equations*, Cambridge University Press, 1958.

[32] A. M. Wazwaz, *Linear and nonlinear integral equations methods and applications*, Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg, 2011.