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New extended direct algebraic method for the Tzitzéica type evolution equations arising in nonlinear optics

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Abstract In this study, the new extended direct algebraic method is exerted for constructing more general exact solutions of the three nonlinear evolution equations with physical interest namely, the Tzitzéica equation, the Dodd-Bullough-Mikhailor equation and the Liouville equation. By using of an appropriate traveling wave transformation reduces these equations to ODE. We state that this method is excellently a generalized form to obtain solitary wave solutions of the nonlinear evolution equations that are widely used in theoretical physics. The method appears to be easier and faster by means of symbolic computation system.

Keywords. Nonlinear evolution equation, Tzitzéica type evolution equations, New extended direct algebraic method, Traveling wave solutions.

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1. Introduction

Nonlinear evolution equations (NEEs) and partial derivatives are one of the most important mathematical tools in basic sciences and engineering. NEEs have been the subject of study in various basis of mathematical physical sciences such as physics, biology, chemistry, engineering. The Tzitzéica-type nonlinear evolution equations in nonlinear optics, including Tzitzéica, Dodd-Bullough-Mikhailov (DBM) and Liouville equations are a class of such equations which have gained significant attention during the last few years, which can written as follows [21]:

$$u_{tt} - u_{xx} - e^{u} + e^{-2u} = 0, \tag{1.1}$$

$$u_{xt} + e^{u} + e^{-2u} = 0, \tag{1.2}$$

$$u_{xt} + e^{u} = 0. \tag{1.3}$$

Many different methods have recently been introduced to solve Tzitzéica evolution equations, for example, Wazwaz [22] introduced the tanh method to obtain the solitons and periodic solutions of the DBM and Tzitzéica-Dodd-Bullough (TDB) equations. Abazari [1] used the $(\frac{G'}{G})$-expansion method to find more general exact solutions of the Tzitzica, DBM, and TDB equations. Manafian and Lakestani [17] used the improved $\tan(\frac{\phi(\xi)}{2})$-expansion method to show new and more general exact traveling wave solutions of the Tzitzéica, DBM, and TDB equations. Islam and Roshid [11] employed the $exp(-\phi(\xi))$-expansion method to find new profuse exact traveling wave solutions of the Tzitzéica-type non-linear evolution equations. Kumar et al. [16] applied sine-Gordon expansion method to solve the Tzitzéica type equations. Hosseini et al. [7] applied the modified Kudryashov method to get several new traveling wave solutions of the Tzitzéica, DBM, and TDB equations. In the past several decades, there is a wide variety of another approaches to nonlinear PDEs for constructing traveling wave solutions, such as Khater et. al. [12] for solving nonlinear partial differential equations. Bibi et. al. [3] presented the Khater method to obtain the solutions of the nonlinear Sharma Tasso-Olver (STO) equation. Eslami and Rezazadeh [9] employed the functional variable method for time-fractional Schrodinger-Hirto equation. Wakil et al. [8] presented modified extended tanh-function method for PDEs. The other popular methods have also been used to solve various problems [18]-[15].

The main aim in this article is to effectively employ new method to obtain exact solution of nonlinear partial differential equations, namely the generalized algebra method. This method is based on a suitable variables change, which transforms convert the main problem into ODE, and then, with the substitution of a hypothetical solution in the ODE, and by calculating the unknown coefficients in solution, the main solution of the Tzitzéica evolution equation is obtained.

The paper is organized as follows: In Section 2, the generalized algebra method is implemented to obtain the exact traveling wave solutions of the Tzitzéica type equations in nonlinear optics. Finally, we end this paper with conclusion in Section 3.

2. Application

In this section, we will employ the new extended direct algebraic method according to [20] to solve three well-known nonlinear evolution equations, namely, the Tzitzéica equation, the Dodd-Bullough-Mikhailov equation, the Liouville equation.

2.1. The Tzitzéica equation

Consider the Tzitzéica equation described by:

$$u_{tt} - u_{xx} - e^{u} + e^{-2u} = 0. \tag{2.1}$$

By using the Painleve transformation $u = Ln(v)$, Eq. (2.1) can be written as:

$$vv_{tt} - v_t^2 - vv_{xx} + v_x^2 - v^3 + 1 = 0. \tag{2.2}$$

Execute certain variable transformation

$$v(x,t) = U(\xi), \qquad \xi = kx + \lambda t, \tag{2.3}$$

Eq. (2.2) becomes

$$(\lambda^2 - k^2)(UU'' - (U')^2) - U^3 + 1 = 0. \tag{2.4}$$

Balancing the highest order derivative term and the highest order nonlinear term in (2.4), we find $N = 2$. Thus, we choose solution of Eq. (2.4) as follows:

$$U(\xi) = b_0 + b_1 Q(\xi) + b_2 Q^2(\xi). \tag{2.5}$$

By substituting (2.5) into Eq. (2.4) and collecting all terms with the same order of $Q(\xi)$ together, the left-hand-side of (2.4) are converted into polynomial in $Q(\xi)$. Setting each coefficient of each polynomial to zero, we arrive a set of algebraic system for b_0, b_1, b_2 and λ as follows:

$$b_0 = -\frac{\beta^2 + 2\alpha\sigma}{-\beta^2 + 4\alpha\sigma}, \quad b_1 = -\frac{6\beta\sigma}{-\beta^2 + 4\alpha\sigma}, \quad b_2 = -\frac{6\sigma^2}{-\beta^2 + 4\alpha\sigma}, \tag{2.6}$$

$$\lambda = \frac{\sqrt{(-\beta^2 + 4\alpha\sigma)(k^2 Ln^2 A(4\alpha\sigma - \beta^2) - 3)}}{(-\beta^2 + 4\alpha\sigma) Ln A}. \tag{2.7}$$

By using of Eqs. (2.5) and (2.6), becomes:
1) If $\gamma = \beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$, then

$$\begin{aligned} u_1(x,t) = Ln\Bigg(& \frac{-2\beta^2 + 2\alpha\sigma}{\gamma} \\ & + \frac{3\beta\sqrt{-\gamma}}{\gamma} tan_A\left(\frac{\sqrt{-\gamma}}{2}\left(kx - \sqrt{k^2 + \frac{3}{\gamma Ln^2 A}}t\right)\right) \\ & + \frac{6}{\gamma}\left[-\frac{\beta}{2} + \frac{\sqrt{-\gamma}}{2} tan_A\left(\frac{\sqrt{-\gamma}}{2}\left(kx - \sqrt{k^2 + \frac{3}{\gamma Ln^2 A}}t\right)\right)\right]^2\Bigg), \end{aligned}$$

$$u_2(x,t) = Ln\Bigg(\frac{-2\beta^2+2\alpha\sigma}{\gamma} - \frac{3\beta\sqrt{-\gamma}}{\gamma} cot_A\left(\frac{\sqrt{-\gamma}}{2}\left(kx - \sqrt{k^2+\frac{3}{\gamma Ln^2 A}t}\right)\right)$$

$$+\frac{6}{\gamma}\left[-\frac{\beta}{2} + \frac{\sqrt{-\gamma}}{2} cot_A\left(\frac{\sqrt{-\gamma}}{2}\left(kx - \sqrt{k^2+\frac{3}{\gamma Ln^2 A}t}\right)\right)\right]^2\Bigg),$$

$$u_3(x,t) = Ln\Bigg(\frac{-2\beta^2+2\alpha\sigma}{\gamma}$$

$$+\frac{3\beta\sqrt{-\gamma}}{\gamma} tan_A\left(\sqrt{-\gamma}\left(kx - \sqrt{k^2+\frac{3}{\gamma Ln^2 A}t}\right)\right)$$

$$\pm\frac{3\beta\sqrt{-pq\gamma}}{\gamma} sec_A\left(\sqrt{-\gamma}\left(kx - \sqrt{k^2+\frac{3}{\gamma Ln^2 A}t}\right)\right)$$

$$+\frac{6}{\gamma}\left[-\frac{\beta}{2} + \frac{\sqrt{-\gamma}}{2} tan_A\left(\sqrt{-\gamma}\left(kx - \sqrt{k^2+\frac{3}{\gamma Ln^2 A}t}\right)\right)\right.$$

$$\left.\pm\frac{\sqrt{-pq\gamma}}{2} sec_A\left(\sqrt{-\gamma}\left(kx - \sqrt{k^2+\frac{3}{\gamma Ln^2 A}t}\right)\right)\right]^2\Bigg),$$

$$u_4(x,t) = Ln\Bigg(\frac{-2\beta^2+2\alpha\sigma}{\gamma} - \frac{3\beta\sqrt{-\gamma}}{\gamma} cot_A\left(\sqrt{-\gamma}\left(kx - \sqrt{k^2+\frac{3}{\gamma Ln^2 A}t}\right)\right)$$

$$\pm\frac{3\beta\sqrt{-pq\gamma}}{\gamma} csc_A\left(\sqrt{-\gamma}\left(kx - \sqrt{k^2+\frac{3}{\gamma Ln^2 A}t}\right)\right)$$

$$+\frac{6}{\gamma}\left[-\frac{\beta}{2} - \frac{\sqrt{-\gamma}}{2} cot_A\left(\sqrt{-\gamma}\left(kx - \sqrt{k^2+\frac{3}{\gamma Ln^2 A}t}\right)\right)\right.$$

$$\left.\pm\frac{\sqrt{-pq\gamma}}{2} csc_A\left(\sqrt{-\gamma}\left(kx - \sqrt{k^2+\frac{3}{\gamma Ln^2 A}t}\right)\right)\right]^2\Bigg),$$

$$u_5(x,t) = Ln\Bigg(\tfrac{-2\beta^2+2\alpha\sigma}{\gamma} + \tfrac{3\beta\sqrt{-\gamma}}{2\gamma} tan_A\left(\tfrac{\sqrt{-\gamma}}{4}\left(kx - \sqrt{k^2+\tfrac{3}{\gamma Ln^2 A}t}\right)\right)$$
$$-\tfrac{3\beta\sqrt{-\gamma}}{2\gamma} cot_A\left(\tfrac{\sqrt{-\gamma}}{4}\left(kx - \sqrt{k^2+\tfrac{3}{\gamma Ln^2 A}t}\right)\right)$$
$$+\tfrac{6}{\gamma}\Bigg[-\tfrac{\beta}{2} + \tfrac{\sqrt{-\gamma}}{4} tan_A\left(\tfrac{\sqrt{-\gamma}}{4}\left(kx - \sqrt{k^2+\tfrac{3}{\gamma Ln^2 A}t}\right)\right)$$
$$-\tfrac{\sqrt{-\gamma}}{4} cot_A\left(\tfrac{\sqrt{-\gamma}}{4}\left(kx - \sqrt{k^2+\tfrac{3}{\gamma Ln^2 A}t}\right)\right)\Bigg]^2\Bigg).$$

2) If $\gamma = \beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$, then

$$u_6(x,t) = Ln\Bigg(\tfrac{-2\beta^2+2\alpha\sigma}{\gamma} - \tfrac{3\beta\sqrt{\gamma}}{\gamma} tanh_A\left(\tfrac{\sqrt{\gamma}}{2}\left(kx - \sqrt{k^2+\tfrac{3}{\gamma Ln^2 A}t}\right)\right)$$
$$+\tfrac{6}{\gamma}\left[-\tfrac{\beta}{2} - \tfrac{\sqrt{\gamma}}{2} tanh_A\left(\tfrac{\sqrt{\gamma}}{2}\left(kx - \sqrt{k^2+\tfrac{3}{\gamma Ln^2 A}t}\right)\right)\right]^2\Bigg),$$

$$u_7(x,t) = Ln\Bigg(\tfrac{-2\beta^2+2\alpha\sigma}{\gamma} - \tfrac{3\beta\sqrt{\gamma}}{\gamma} coth_A\left(\tfrac{\sqrt{\gamma}}{2}\left(kx - \sqrt{k^2+\tfrac{3}{\gamma Ln^2 A}t}\right)\right)$$
$$+\tfrac{6}{\gamma}\left[-\tfrac{\beta}{2} - \tfrac{\sqrt{\gamma}}{2} coth_A\left(\tfrac{\sqrt{\gamma}}{2}\left(kx - \sqrt{k^2+\tfrac{3}{\gamma Ln^2 A}t}\right)\right)\right]^2\Bigg),$$

$$u_8(x,t) = Ln\Bigg(\tfrac{-2\beta^2+2\alpha\sigma}{\gamma} - \tfrac{3\beta\sqrt{\gamma}}{\gamma} tanh_A\left(\sqrt{\gamma}\left(kx - \sqrt{k^2+\tfrac{3}{\gamma Ln^2 A}t}\right)\right)$$
$$\pm i\tfrac{3\beta\sqrt{pq\gamma}}{\gamma} sech_A\left(\sqrt{\gamma}\left(kx - \sqrt{k^2+\tfrac{3}{\gamma Ln^2 A}t}\right)\right)$$
$$+\tfrac{6}{\gamma}\Bigg[-\tfrac{\beta}{2} - \tfrac{\sqrt{\gamma}}{2} tanh_A\left(\sqrt{\gamma}\left(kx - \sqrt{k^2+\tfrac{3}{\gamma Ln^2 A}t}\right)\right)$$
$$\pm i\tfrac{\sqrt{pq\gamma}}{2} sech_A\left(\sqrt{\gamma}\left(kx - \sqrt{k^2+\tfrac{3}{\gamma Ln^2 A}t}\right)\right)\Bigg]^2\Bigg),$$

$$u_9(x,t) = Ln\Bigg(\tfrac{-2\beta^2+2\alpha\sigma}{\gamma} - \tfrac{3\beta\sqrt{\gamma}}{\gamma}cot_A\left(\sqrt{\gamma}\left(kx-\sqrt{k^2+\tfrac{3}{\gamma Ln^2A}}t\right)\right)$$

$$\pm\tfrac{3\beta\sqrt{pq\gamma}}{\gamma}csc_A\left(\sqrt{\gamma}\left(kx-\sqrt{k^2+\tfrac{3}{\gamma Ln^2A}}t\right)\right)$$

$$+\tfrac{6}{\gamma}\left[-\tfrac{\beta}{2}-\tfrac{\sqrt{\gamma}}{2}cot_A\left(\sqrt{\gamma}\left(kx-\sqrt{k^2+\tfrac{3}{\gamma Ln^2A}}t\right)\right)\right.$$

$$\left.\pm\tfrac{\sqrt{pq\gamma}}{2}csch_A\left(\sqrt{\gamma}\left(kx-\sqrt{k^2+\tfrac{3}{\gamma Ln^2A}}t\right)\right)\right]^2\Bigg),$$

$$u_{10}(x,t) = Ln\Bigg(\tfrac{-2\beta^2+2\alpha\sigma}{\gamma} - \tfrac{3\beta\sqrt{\gamma}}{2\gamma}tanh_A\left(\tfrac{\sqrt{\gamma}}{4}\left(kx-\sqrt{k^2+\tfrac{3}{\gamma Ln^2A}}t\right)\right)$$

$$-\tfrac{3\beta\sqrt{\gamma}}{2\gamma}coth_A\left(\tfrac{\sqrt{\gamma}}{4}\left(kx-\sqrt{k^2+\tfrac{3}{\gamma Ln^2A}}t\right)\right)$$

$$+\tfrac{6}{\gamma}\left[-\tfrac{\beta}{2}-\tfrac{\sqrt{\gamma}}{4}tanh_A\left(\tfrac{\sqrt{\gamma}}{4}\left(kx-\sqrt{k^2+\tfrac{3}{\gamma Ln^2A}}t\right)\right)\right.$$

$$\left.-\tfrac{\sqrt{\gamma}}{4}coth_A\left(\tfrac{\sqrt{\gamma}}{4}\left(kx-\sqrt{k^2+\tfrac{3}{\gamma Ln^2A}}t\right)\right)\right]^2\Bigg).$$

3) If $\alpha\sigma > 0, \sigma \neq 0$ and $\beta = 0$, then

$$u_{11}(x,t) = Ln\left(-\tfrac{1}{2}-\tfrac{3}{2}tan_A^2\left(\sqrt{\alpha\sigma}\left(kx+\sqrt{k^2-\tfrac{3}{4\alpha^2\sigma^2Ln^2A}}t\right)\right)\right),$$

$$u_{12}(x,t) = Ln\left(-\tfrac{1}{2}-\tfrac{3}{2}cot_A^2\left(\sqrt{\alpha\sigma}\left(kx+\sqrt{k^2-\tfrac{3}{4\alpha^2\sigma^2Ln^2A}}t\right)\right)\right).$$

$$u_{13}(x,t) = Ln\Bigg(-\tfrac{1}{2}-\tfrac{3}{2}\left[tan_A\left(2\sqrt{\alpha\sigma}\left(kx+\sqrt{k^2-\tfrac{3}{4\alpha^2\sigma^2Ln^2A}}t\right)\right)\right.$$

$$\left.\pm\sqrt{pq}sec_A\left(2\sqrt{\alpha\sigma}\left(kx+\sqrt{k^2-\tfrac{3}{4\alpha^2\sigma^2Ln^2A}}t\right)\right)\right]^2\Bigg),$$

$$u_{14}(x,t) = Ln\Bigg(-\tfrac{1}{2}-\tfrac{3}{2}\left[-cot_A\left(2\sqrt{\alpha\sigma}\left(kx+\sqrt{k^2-\tfrac{3}{4\alpha^2\sigma^2Ln^2A}}t\right)\right)\right.$$

$$\left.\pm\sqrt{pq}csc_A\left(2\sqrt{\alpha\sigma}\left(kx+\sqrt{k^2-\tfrac{3}{4\alpha^2\sigma^2Ln^2A}}t\right)\right)\right]^2\Bigg),$$

$$u_{15}(x,t) = Ln\left(-\frac{1}{2} - \frac{3}{8}\left[tan_A\left(\frac{\sqrt{\alpha\sigma}}{2}\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2\sigma^2 Ln^2 A}}t\right)\right)\right.\right.$$

$$\left.\left. -cot_A\left(\frac{\sqrt{\alpha\sigma}}{2}\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2\sigma^2 Ln^2 A}}t\right)\right)\right]^2\right).$$

4) If $\alpha\sigma < 0, \sigma \neq 0$ and $\beta = 0$, then

$$u_{16}(x,t) = Ln\left(-\frac{1}{2} + \frac{3}{2}tanh_A^2\left(\sqrt{-\alpha\sigma}\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2\sigma^2 Ln^2 A}}t\right)\right)\right),$$

$$u_{17}(x,t) = Ln\left(-\frac{1}{2} + \frac{3}{2}coth_A^2\left(\sqrt{-\alpha\sigma}\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2\sigma^2 Ln^2 A}}t\right)\right)\right).$$

$$u_{18}(x,t) = Ln\left(-\frac{1}{2} + \frac{3}{2}\left[-tanh_A\left(2\sqrt{-\alpha\sigma}\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2\sigma^2 Ln^2 A}}t\right)\right)\right.\right.$$

$$\left.\left. \pm i\sqrt{pq}sech_A\left(2\sqrt{-\alpha\sigma}\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2\sigma^2 Ln^2 A}}t\right)\right)\right]^2\right),$$

$$u_{19}(x,t) = Ln\left(-\frac{1}{2} + \frac{3}{2}\left[-coth_A\left(2\sqrt{-\alpha\sigma}\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2\sigma^2 Ln^2 A}}t\right)\right)\right.\right.$$

$$\left.\left. \pm \sqrt{pq}csch_A\left(2\sqrt{-\alpha\sigma}\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2\sigma^2 Ln^2 A}}t\right)\right)\right]^2\right),$$

$$u_{20}(x,t) = Ln\left(-\frac{1}{2} + \frac{3}{8}\left[tanh_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2\sigma^2 Ln^2 A}}t\right)\right)\right.\right.$$

$$\left.\left. +coth_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2\sigma^2 Ln^2 A}}t\right)\right)\right]^2\right).$$

5) If $\beta = 0$ and $\sigma = \alpha$, then

$$u_{21}(x,t) = Ln\left(-\frac{1}{2} - \frac{3}{2}tan_A^2\left(\alpha\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2 Ln^2 A}}t\right)\right)\right),$$

$$u_{22}(x,t) = Ln\left(-\frac{1}{2} - \frac{3}{2}cot_A^2\left(\alpha\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2 Ln^2 A}}t\right)\right)\right),$$

$$u_{23}(x,t) = Ln\left(-\frac{1}{2} - \frac{3}{2}\left[tan_A\left(2\alpha\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2 Ln^2 A}}t\right)\right)\right.\right.$$

$$\left.\left. \pm \sqrt{pq}sec_A\left(2\alpha\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2 Ln^2 A}}t\right)\right)\right]^2\right),$$

$$u_{24}(x,t) = Ln\left(-\frac{1}{2} - \frac{3}{2}\left[-cot_A\left(2\alpha\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2 Ln^2 A}}t\right)\right)\right.\right.$$

$$\left.\left.\pm\sqrt{pq}csc_A\left(2\alpha\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2 Ln^2 A}}t\right)\right)\right]^2\right),$$

$$u_{25}(x,t) = Ln\left(-\frac{1}{2} - \frac{3}{2}\left[tan_A\left(\frac{\alpha}{2}\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2 Ln^2 A}}t\right)\right)\right.\right.$$

$$\left.\left.-cot_A\left(\frac{\alpha}{2}\left(kx + \sqrt{k^2 - \frac{3}{4\alpha^2 Ln^2 A}}t\right)\right)\right]^2\right).$$

6) If $\beta = 0$ and $\sigma = -\alpha$, then

$$u_{26}(x,t) = Ln\left(-\frac{1}{2} + \frac{3}{2}tanh_A^2\left(\alpha\left(kx - \sqrt{k^2 - \frac{3}{4\alpha^2 Ln^2 A}}t\right)\right)\right),$$

$$u_{27}(x,t) = Ln\left(-\frac{1}{2} + \frac{3}{2}coth_A^2\left(\alpha\left(kx - \sqrt{k^2 - \frac{3}{4\alpha^2 Ln^2 A}}t\right)\right)\right),$$

$$u_{28}(x,t) = Ln\left(-\frac{1}{2} + \frac{3}{2}\left[-tanh_A\left(2\alpha\left(kx - \sqrt{k^2 - \frac{3}{4\alpha^2 Ln^2 A}}t\right)\right)\right.\right.$$

$$\left.\left.\pm i\sqrt{pq}sech_A\left(2\alpha\left(kx - \sqrt{k^2 - \frac{3}{4\alpha^2 Ln^2 A}}t\right)\right)\right]^2\right),$$

$$u_{29}(x,t) = Ln\left(-\frac{1}{2} + \frac{3}{2}\left[-coth_A\left(2\alpha\left(kx - \sqrt{k^2 - \frac{3}{4\alpha^2 Ln^2 A}}t\right)\right)\right.\right.$$

$$\left.\left.\pm\sqrt{pq}csch_A\left(2\alpha\left(kx - \sqrt{k^2 - \frac{3}{4\alpha^2 Ln^2 A}}t\right)\right)\right]^2\right),$$

$$u_{30}(x,t) = Ln\left(-\frac{1}{2} + \frac{3}{8}\left[tanh_A\left(\frac{\alpha}{2}\left(kx - \sqrt{k^2 - \frac{3}{4\alpha^2 Ln^2 A}}t\right)\right)\right.\right.$$

$$\left.\left.+coth_A\left(\frac{\alpha}{2}\left(kx - \sqrt{k^2 - \frac{3}{4\alpha^2 Ln^2 A}}t\right)\right)\right]^2\right).$$

7) If $\alpha = 0$ and $\beta \neq 0$, then

$$u_{31}(x,t) = Ln\,(1- \left[\frac{6p}{cosh_A\left(\beta kx-\sqrt{k^2\beta^2+\frac{3}{Ln^2A}}t\right)-sinh_A\left(\beta kx-\sqrt{k^2\beta^2+\frac{3}{Ln^2A}}t\right)+p}\right]$$
$$+\left[\frac{6p}{cosh_A\left(\beta kx-\sqrt{k^2\beta^2+\frac{3}{Ln^2A}}t\right)-sinh_A\left(\beta kx-\sqrt{k^2\beta^2+\frac{3}{Ln^2A}}t\right)+p}\right]^2\Bigg),$$

$$u_{32}(x,t) = Ln\,(1- 6\left[\frac{sinh_A\left(\beta kx-\sqrt{k^2\beta^2+\frac{3}{Ln^2A}}t\right)+cosh_A\left(\beta kx-\sqrt{k^2\beta^2+\frac{3}{Ln^2A}}t\right)}{cosh_A\left(\beta kx-\sqrt{k^2\beta^2+\frac{3}{Ln^2A}}t\right)+sinh_A\left(\beta kx-\sqrt{k^2\beta^2+\frac{3}{Ln^2A}}t\right)+q}\right]$$
$$+6\left[\frac{sinh_A\left(\beta kx-\sqrt{k^2\beta^2+\frac{3}{Ln^2A}}t\right)+cosh_A\left(\beta kx-\sqrt{k^2\beta^2+\frac{3}{Ln^2A}}t\right)}{cosh_A\left(\beta kx-\sqrt{k^2\beta^2+\frac{3}{Ln^2A}}t\right)+sinh_A\left(\beta kx-\sqrt{k^2\beta^2+\frac{3}{Ln^2A}}t\right)+q}\right]^2\Bigg).$$

8) If $\beta = k, \sigma = mk \ \ (m \neq 0)$ and $\alpha = 0$, then

$$u_{33}(x,t) = Ln\left(1+6m\left[\frac{pA^{\left(k^2x-\sqrt{k^4+\frac{3}{Ln^2A}}t\right)}}{q-mpA^{\left(kx-\sqrt{k^4+\frac{3}{Ln^2A}}t\right)}}\right]\right.$$
$$+6m^2\left[\frac{A^{\left(kx-\sqrt{k^4+\frac{3}{Ln^2A}}t\right)}}{q-mpA^{\left(kx-\sqrt{k^4+\frac{3}{Ln^2A}}t\right)}}\right]^2.$$

2.2. The Dodd-Bullough-Mikhailov equation

Consider the Dodd-Bullough-Mikhailov equation described by:

$$u_{xt} + e^u + e^{-2u} = 0. \tag{2.8}$$

By using the Painleve transformation $u = ln(v)$, Eq. (2.7) can be written as:

$$vv_{xt} - v_x v_t + v^3 + 1 = 0. \tag{2.9}$$

Execute certain variable transformation

$$v(x,t) = U(\xi), \qquad \xi = kx + \lambda t, \tag{2.10}$$

Eq. (2.8) becomes

$$k\lambda(UU'' - (U')^2) + U^3 + 1 = 0. \tag{2.11}$$

Balancing the highest order derivative term and the highest order nonlinear term in (2.10), we find $N = 2$. Thus, we choose solution of Eq. (2.10) as follows:

$$U(\xi) = b_0 + b_1 Q(\xi) + b_2 Q^2(\xi), \tag{2.12}$$

By substituting (2.11) into Eq. (2.10) and collecting all terms with the same order of $Q(\xi)$ together, the left-handside of (2.10) are converted into polynomial in $Q(\xi)$. Setting each coefficient of each polynomial to zero, we arrive a set of algebraic system

for b_0, b_1, b_2 and λ as follows:
Case 1:

$$b_0 = \frac{\beta^2 + 2\alpha\sigma}{-\beta^2 + 4\alpha\sigma}, \qquad b_1 = \frac{6\beta\sigma}{-\beta^2 + 4\alpha\sigma},$$
$$b_2 = \frac{6\sigma^2}{-\beta^2 + 4\alpha\sigma}, \qquad \lambda = \frac{-3}{kLn^2A(-\beta^2 + 4\alpha\sigma)}. \tag{2.13}$$

By using of Eqs. (2.11) and (2.12), becomes:
1) If $\gamma = \beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$, then

$$u_1(x,t) = Ln\Bigg(\frac{2\beta^2 - 2\alpha\sigma}{\gamma} - \frac{3\beta\sqrt{-\gamma}}{\gamma} tan_A\left(\frac{\sqrt{-\gamma}}{2}\left(kx + \frac{3}{k\gamma Ln^2A}t\right)\right)$$
$$- \frac{6}{\gamma}\left[-\frac{\beta}{2} + \frac{\sqrt{-\gamma}}{2} tan_A\left(\frac{\sqrt{-\gamma}}{2}\left(kx + \frac{3}{k\gamma LnA}t\right)\right)\right]^2\Bigg),$$

$$u_2(x,t) = Ln\Bigg(\frac{2\beta^2 - 2\alpha\sigma}{\gamma} + \frac{3\beta\sqrt{-\gamma}}{\gamma} cot_A\left(\frac{\sqrt{-\gamma}}{2}\left(kx + \frac{3}{k\gamma Ln^2A}t\right)\right)$$
$$- \frac{6}{\gamma}\left[-\frac{\beta}{2} + \frac{\sqrt{-\gamma}}{2} cot_A\left(\frac{\sqrt{-\gamma}}{2}\left(kx + \frac{3}{k\gamma Ln^2A}t\right)\right)\right]^2\Bigg),$$

$$u_3(x,t) = Ln\Bigg(\frac{2\beta^2 - 2\alpha\sigma}{\gamma} - \frac{3\beta\sqrt{-\gamma}}{\gamma} tan_A\left(\sqrt{-\gamma}\left(kx + \frac{3}{k\gamma Ln^2A}t\right)\right)$$
$$\pm \frac{3\beta\sqrt{-pq\gamma}}{\gamma} sec_A\left(\sqrt{-\gamma}\left(kx + \frac{3}{k\gamma Ln^2A}t\right)\right)$$
$$- \frac{6}{\gamma}\left[-\frac{\beta}{2} + \frac{\sqrt{-\gamma}}{2} tan_A\left(\sqrt{-\gamma}\left(kx + \frac{3}{k\gamma Ln^2A}t\right)\right)\right.$$
$$\left.\pm \frac{\sqrt{-pq\gamma}}{2} sec_A\left(\sqrt{-\gamma}\left(kx + \frac{3}{k\gamma Ln^2A}t\right)\right)\right]^2\Bigg),$$

$$u_4(x,t) = Ln\Bigg(\frac{2\beta^2 - 2\alpha\sigma}{\gamma} + \frac{3\beta\sqrt{-\gamma}}{\gamma} cot_A\left(\sqrt{-\gamma}\left(kx + \frac{3}{k\gamma Ln^2A}t\right)\right)$$
$$\pm \frac{3\beta\sqrt{-pq\gamma}}{\gamma} csc_A\left(\sqrt{-\gamma}\left(kx + \frac{3}{k\gamma Ln^2A}t\right)\right)$$
$$- \frac{6}{\gamma}\left[-\frac{\beta}{2} - \frac{\sqrt{-\gamma}}{2} cot_A\left(\sqrt{-\gamma}\left(kx - \frac{3}{k\gamma Ln^2A}t\right)\right)\right.$$
$$\left.\pm \frac{\sqrt{-pq\gamma}}{2} csc_A\left(\sqrt{-\gamma}\left(kx + \frac{3}{k\gamma Ln^2A}t\right)\right)\right]^2\Bigg),$$

$$
\begin{aligned}
u_5(x,t) = Ln\Bigg(& \frac{2\beta^2 - 2\alpha\sigma}{\gamma} - \frac{3\beta\sqrt{-\gamma}}{2\gamma} tan_A\left(\frac{\sqrt{-\gamma}}{4}\left(kx + \frac{3}{k\gamma Ln^2 A}t\right)\right) \\
& + \frac{3\beta\sqrt{-\gamma}}{2\gamma} cot_A\left(\frac{\sqrt{-\gamma}}{4}\left(kx + \frac{3}{k\gamma Ln^2 A}t\right)\right) \\
& - \frac{6}{\gamma}\left[-\frac{\beta}{2} + \frac{\sqrt{-\gamma}}{4} tan_A\left(\frac{\sqrt{-\gamma}}{4}\left(kx + \frac{3}{k\gamma Ln^2 A}t\right)\right)\right. \\
& \left. - \frac{\sqrt{-\gamma}}{4} cot_A\left(\frac{\sqrt{-\gamma}}{4}\left(kx + \frac{3}{k\gamma Ln^2 A}t\right)\right)\right]^2\Bigg).
\end{aligned}
$$

2) If $\gamma = \beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$, then

$$
\begin{aligned}
u_6(x,t) = Ln\Bigg(& \frac{2\beta^2 - 2\alpha\sigma}{\gamma} + \frac{3\beta\sqrt{\gamma}}{\gamma} tanh_A\left(\frac{\sqrt{\gamma}}{2}\left(kx + \frac{3}{k\gamma Ln^2 A}t\right)\right) \\
& - \frac{6}{\gamma}\left[-\frac{\beta}{2} - \frac{\sqrt{\gamma}}{2} tanh_A\left(\frac{\sqrt{\gamma}}{2}\left(kx + \frac{3}{k\gamma Ln^2 A}t\right)\right)\right]^2\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
u_7(x,t) = Ln\Bigg(& \frac{2\beta^2 - 2\alpha\sigma}{\gamma} + \frac{3\beta\sqrt{\gamma}}{\gamma} coth_A\left(\frac{\sqrt{\gamma}}{2}\left(kx + \frac{3}{k\gamma Ln^2 A}t\right)\right) \\
& - \frac{6}{\gamma}\left[-\frac{\beta}{2} + \frac{\sqrt{\gamma}}{2} coth_A\left(\frac{\sqrt{\gamma}}{2}\left(kx + \frac{3}{k\gamma Ln^2 A}t\right)\right)\right]^2\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
u_8(x,t) = Ln\Bigg(& \frac{2\beta^2 - 2\alpha\sigma}{\gamma} + \frac{3\beta\sqrt{\gamma}}{\gamma} tanh_A\left(\sqrt{\gamma}\left(kx + \frac{3}{k\gamma Ln^2 A}t\right)\right) \\
& \pm i\frac{3\beta\sqrt{pq\gamma}}{\gamma} sech_A\left(\sqrt{\gamma}\left(kx + \frac{3}{k\gamma Ln^2 A}t\right)\right) \\
& - \frac{6}{\gamma}\left[-\frac{\beta}{2\sigma} - \frac{\sqrt{\gamma}}{2} tanh_A\left(\sqrt{\gamma}\left(kx + \frac{3}{k\gamma Ln^2 A}t\right)\right)\right. \\
& \left. \pm i\frac{\sqrt{pq\gamma}}{2} sech_A\left(\sqrt{\gamma}\left(kx + \frac{3}{k\gamma Ln A}t\right)\right)\right]^2\Bigg),
\end{aligned}
$$

$$\begin{aligned} u_9(x,t) = Ln\Bigg(& \frac{2\beta^2-2\alpha\sigma}{\gamma} + \frac{3\beta\sqrt{\gamma}}{\gamma} coth_A\left(\sqrt{\gamma}\left(kx+\frac{3}{k\gamma Ln^2 A}t\right)\right) \\ & \pm \frac{3\beta\sqrt{pq\gamma}}{\gamma} csch_A\left(\sqrt{\gamma}\left(kx+\frac{3}{k\gamma Ln^2 A}t\right)\right) \\ & - \frac{6}{\gamma}\Bigg[-\frac{\beta}{2}-\frac{\sqrt{\gamma}}{2}cot_A\left(\sqrt{\gamma}\left(kx+\frac{3}{k\gamma Ln^2 A}t\right)\right) \\ & \pm \frac{\sqrt{pq\gamma}}{2} csch_A\left(\sqrt{\gamma}\left(kx+\frac{3}{k\gamma Ln^2 A}t\right)\right)\Bigg]^2\Bigg), \end{aligned}$$

$$\begin{aligned} u_{10}(x,t) = Ln\Bigg(& \frac{\beta^2-2\alpha\sigma}{\gamma} + \frac{3\beta\sqrt{\gamma}}{2\gamma} tanh_A\left(\frac{\sqrt{\gamma}}{4}\left(kx+\frac{3}{k\gamma Ln^2 A}t\right)\right) \\ & + \frac{3\beta\sqrt{\gamma}}{2\gamma} coth_A\left(\frac{\sqrt{\gamma}}{4}\left(kx+\frac{3}{k\gamma Ln^2 A}t\right)\right) \\ & - \frac{6}{\gamma}\Bigg[-\frac{\beta}{2}-\frac{\sqrt{\gamma}}{4}tanh_A\left(\frac{\sqrt{\gamma}}{4}\left(kx+\frac{3}{k\gamma Ln^2 A}t\right)\right) \\ & - \frac{\sqrt{\gamma}}{4} coth_A\left(\frac{\sqrt{\gamma}}{4}\left(kx+\frac{3}{k\gamma Ln^2 A}t\right)\right)\Bigg]^2\Bigg). \end{aligned}$$

3) If $\alpha\sigma > 0, \sigma \neq 0$ and $\beta = 0$, then

$$u_{11}(x,t) = Ln\left(\frac{1}{2}+\frac{3}{2}tan_A^2\left(\sqrt{\alpha\sigma}\left(kx-\frac{3}{4k\alpha\sigma Ln^2 A}t\right)\right)\right),$$

$$u_{12}(x,t) = Ln\left(\frac{1}{2}+\frac{3}{2}cot_A^2\left(\sqrt{\alpha\sigma}\left(kx-\frac{3}{4k\alpha\sigma Ln^2 A}t\right)\right)\right).$$

$$\begin{aligned} u_{13}(x,t) = Ln\Bigg(& \frac{1}{2}+\frac{3}{2}\Bigg[tan_A\left(2\sqrt{\alpha\sigma}\left(kx-\frac{3}{4k\alpha\sigma Ln^2 A}t\right)\right) \\ & \pm \sqrt{pq} sec_A\left(2\sqrt{\alpha\sigma}\left(kx-\frac{3}{4k\alpha\sigma Ln^2 A}t\right)\right)\Bigg]^2\Bigg), \end{aligned}$$

$$\begin{aligned} u_{14}(x,t) = Ln\Bigg(& \frac{1}{2}+\frac{3}{2}\Bigg[-cot_A\left(2\sqrt{\alpha\sigma}\left(kx-\frac{3}{4k\alpha\sigma Ln^2 A}t\right)\right) \\ & \pm \sqrt{pq} csc_A\left(2\sqrt{\alpha\sigma}\left(kx-\frac{3}{4k\alpha\sigma Ln^2 A}t\right)\right)\Bigg]^2\Bigg), \end{aligned}$$

$$u_{15}(x,t) = Ln\left(\frac{1}{2} + \frac{3}{2}\left[tan_A\left(\frac{\sqrt{\alpha\sigma}}{2}\left(kx - \frac{3}{4k\alpha\sigma Ln^2 A}t\right)\right)\right.\right.$$
$$\left.\left. - cot_A\left(\frac{\sqrt{\alpha\sigma}}{2}\left(kx - \frac{3}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]^2\right).$$

4) If $\alpha\sigma < 0, \sigma \neq 0$ and $\beta = 0$, then

$$u_{16}(x,t) = Ln\left(\frac{1}{2} + \frac{3}{2}tanh_A^2\left(\sqrt{-\alpha\sigma}\left(kx - \frac{3}{4k\alpha\sigma Ln^2 A}t\right)\right)\right),$$

$$u_{17}(x,t) = Ln\left(\frac{1}{2} + \frac{3}{2}coth_A^2\left(\sqrt{-\alpha\sigma}\left(kx - \frac{3}{4k\alpha\sigma Ln^2 A}t\right)\right)\right).$$

$$u_{18}(x,t) = Ln\left(\frac{1}{2} + \frac{3}{2}\left[-tanh_A\left(2\sqrt{-\alpha\sigma}\left(kx - \frac{3}{4k\alpha\sigma Ln^2 A}t\right)\right)\right.\right.$$
$$\left.\left. \pm i\sqrt{pq}sech_A\left(2\sqrt{-\alpha\sigma}\left(kx - \frac{3}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]^2\right),$$

$$u_{19}(x,t) = Ln\left(\frac{1}{2} + \frac{3}{2}\left[-coth_A\left(2\sqrt{-\alpha\sigma}\left(kx - \frac{3}{4k\alpha\sigma Ln^2 A}t\right)\right)\right.\right.$$
$$\left.\left. \pm \sqrt{pq}csch_A\left(2\sqrt{-\alpha\sigma}\left(kx - \frac{3}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]^2\right),$$

$$u_{20}(x,t) = Ln\left(\frac{1}{2} + \frac{3}{8}\left[tanh_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\left(kx - \frac{3}{4k\alpha\sigma Ln^2 A}t\right)\right)\right.\right.$$
$$\left.\left. + coth_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\left(kx - \frac{3}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]^2\right).$$

5) If $\beta = 0$ and $\sigma = \alpha$, then

$$u_{21}(x,t) = Ln\left(\frac{1}{2} + \frac{3}{2}tan_A^2\left(\alpha kx - \frac{3}{4k\alpha Ln^2 A}t\right)\right),$$

$$u_{22}(x,t) = Ln\left(\frac{1}{2} + \frac{3}{2}cot_A^2\left(\alpha kx - \frac{3}{4k\alpha Ln^2 A}t\right)\right),$$

$$u_{23}(x,t) = Ln\left(\frac{1}{2} + \frac{3}{2}\left[tan_A\left(2\alpha kx - \frac{3}{2k\alpha Ln^2 A}t\right)\right.\right.$$
$$\left.\left. \pm \sqrt{pq}sec_A\left(2\alpha kx - \frac{3}{2k\alpha Ln^2 A}t\right)\right]^2\right),$$

$$u_{24}(x,t) = Ln\left(\frac{1}{2} + \frac{3}{2}\left[-cot_A\left(2\alpha kx - \frac{3}{2k\alpha Ln^2 A}t\right)\right.\right.$$
$$\left.\left.\pm\sqrt{pq}csc_A\left(2\alpha kx - \frac{3}{2k\alpha Ln^2 A}t\right)\right]^2\right),$$

$$u_{25}(x,t) = Ln\left(\frac{1}{2} + \frac{3}{8}\left[tan_A\left(\frac{\alpha k}{2}x - \frac{3}{8k\alpha Ln^2 A}t\right)\right.\right.$$
$$\left.\left.-cot_A\left(\frac{\alpha k}{2}x - \frac{3}{8k\alpha Ln^2 A}t\right)\right]^2\right).$$

6) If $\beta = 0$ and $\sigma = -\alpha$, then

$$u_{26}(x,t) = Ln\left(\frac{1}{2} - \frac{3}{2}tanh_A^2\left(\alpha kx + \frac{3}{4k\alpha Ln^2 A}t\right)\right),$$

$$u_{27}(x,t) = Ln\left(\frac{1}{2} - \frac{3}{2}coth_A\left(\alpha kx + \frac{3}{4k\alpha Ln^2 A}t\right)\right),$$

$$u_{28}(x,t) = Ln\left(\frac{1}{2} - \frac{3}{2}\left[-tanh_A\left(2\alpha kx + \frac{3}{2k\alpha Ln^2 A}t\right)\right.\right.$$
$$\left.\left.\pm i\sqrt{pq}sech_A\left(2\alpha kx + \frac{3}{2k\alpha Ln^2 A}t\right)\right]^2\right),$$

$$u_{29}(x,t) = Ln\left(\frac{1}{2} - \frac{3}{2}\left[-coth_A\left(2\alpha kx + \frac{3}{2k\alpha Ln^2 A}t\right)\right.\right.$$
$$\left.\left.\pm\sqrt{pq}csch_A\left(2\alpha kx + \frac{3}{2k\alpha Ln^2 A}t\right)\right]^2\right),$$

$$u_{30}(x,t) = Ln\left(\frac{1}{2} - \frac{3}{8}\left[tanh_A\left(\frac{\alpha k}{2}x + \frac{3}{8k\alpha Ln^2 A}t\right)\right.\right.$$
$$\left.\left.+coth_A\left(\frac{\alpha k}{2}x + \frac{3}{8k\alpha Ln^2 A}t\right)\right]^2\right).$$

7) If $\alpha = 0$ and $\beta \neq 0$, then

$$u_{31}(x,t) = Ln\left(-1 + \left[\frac{6p}{cosh_A\left(\beta kx + \frac{3}{k\beta Ln^2 A}t\right) - sinh_A\left(\beta kx + \frac{3}{k\beta Ln^2 A}t\right) + p}\right]\right.$$
$$\left.-\left[\frac{6p}{cosh_A\left(\beta kx + \frac{3}{k\beta Ln^2 A}t\right) - sinh_A\left(\beta kx + \frac{3}{k\beta Ln^2 A}t\right) + p}\right]^2\right),$$

$$u_{32}(x,t) = Ln\left(1+6\left[\frac{sinh_A\left(\beta kx+\frac{3}{k\beta Ln^2A}t\right)+cosh_A\left(\beta kx+\frac{3}{k\beta Ln^2A}t\right)}{cosh_A\left(\beta kx+\frac{3}{k\beta Ln^2A}t\right)+sinh_A\left(\beta kx+\frac{3}{k\beta Ln^2A}t\right)+q}\right]\right.$$
$$\left.-6\left[\frac{sinh_A\left(\beta kx+\frac{3}{k\beta Ln^2A}t\right)+cosh_A\left(\beta kx+\frac{3}{k\beta Ln^2A}t\right)}{cosh_A\left(\beta kx+\frac{3}{k\beta Ln^2A}t\right)+sinh_A\left(\beta kx+\frac{3}{k\beta Ln^2A}t\right)+q}\right]^2\right).$$

8) If $\beta = k, \sigma = mk \ \ (m \neq 0)$ and $\alpha = 0$, then

$$u_{33}(x,t) = Ln\left(-1-6m\left[\frac{pA^{\left(k^2x+\frac{3}{k^2Ln^2A}t\right)}}{q-mpA^{\left(k^2x+\frac{3}{k^2Ln^2A}t\right)}}\right]\right.$$
$$\left.-6m^2\left[\frac{A^{\left(k^2x+\frac{3}{k^2Ln^2A}t\right)}}{q-mpA^{\left(k^2x+\frac{3}{k^2Ln^2A}t\right)}}\right]^2\right).$$

Case 2:

$$\begin{aligned} b_0 &= -\frac{(\frac{1}{2}\pm\frac{1}{2}i\sqrt{3})(\beta^2+2\alpha\sigma)}{-\beta^2+4\alpha\sigma}, \quad b_1 = -\frac{6(\frac{1}{2}\pm\frac{1}{2}i\sqrt{3})\beta\sigma}{-\beta^2+4\alpha\sigma}, \\ b_2 &= -\frac{6(\frac{1}{2}\pm\frac{1}{2}i\sqrt{3})\sigma^2}{-\beta^2+4\alpha\sigma}, \qquad \lambda = \frac{3(\frac{1}{2}\pm\frac{1}{2}i\sqrt{3})}{k(-\beta^2+4\alpha\sigma)Ln^2A}. \end{aligned} \tag{2.14}$$

Let us suppose that $\varphi = \frac{1}{2}\pm\frac{1}{2}i\sqrt{3}$. Therefore, by using of Eqs. (2.11) and (2.13), becomes:

1) If $\gamma = \beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$, then

$$u_1(x,t) = Ln\left(\frac{\varphi(-2\beta^2+2\alpha\sigma)}{\gamma}+\frac{3\varphi\beta\sqrt{-\gamma}}{\gamma}tan_A\left(\frac{\sqrt{-\gamma}}{2}\left(kx-\frac{3\varphi}{k\gamma Ln^2A}t\right)\right)\right.$$
$$\left.+\frac{6\varphi}{\gamma}\left[-\frac{\beta}{2}+\frac{\sqrt{-\gamma}}{2}tan_A\left(\frac{\sqrt{-\gamma}}{2}\left(kx-\frac{3\varphi}{k\gamma Ln^2A}t\right)\right)\right]^2\right),$$

$$u_2(x,t) = Ln\left(\frac{\varphi(-2\beta^2+2\alpha\sigma)}{\gamma}-\frac{3\varphi\beta\sqrt{-\gamma}}{\gamma}cot_A\left(\frac{\sqrt{-\gamma}}{2}\left(kx-\frac{3\varphi}{k\gamma Ln^2A}t\right)\right)\right.$$
$$\left.+\frac{6\varphi}{\gamma}\left[-\frac{\beta}{2}+\frac{\sqrt{-\gamma}}{2}cot_A\left(\frac{\sqrt{-\gamma}}{2}\left(kx-\frac{3\varphi}{k\gamma Ln^2A}t\right)\right)\right]^2\right),$$

$$u_3(x,t) = Ln\Bigg(\frac{\varphi(-2\beta^2+2\alpha\sigma)}{\gamma} + \frac{3\varphi\beta\sqrt{-\gamma}}{\gamma} tan_A\left(\sqrt{-\gamma}\left(kx - \frac{3\varphi}{k\gamma Ln^2 A}t\right)\right)$$

$$\pm \frac{3\varphi\beta\sqrt{-pq\gamma}}{\gamma} sec_A\left(\sqrt{-\gamma}\left(kx - \frac{3\varphi}{k\gamma Ln^2 A}t\right)\right)$$

$$+\frac{6\varphi}{\gamma}\Bigg[-\frac{\beta}{2} + \frac{\sqrt{-\gamma}}{2} tan_A\left(\sqrt{-\gamma}\left(kx - \frac{3\varphi}{k\gamma Ln^2 A}t\right)\right)$$

$$\pm \frac{\sqrt{-pq\gamma}}{2} sec_A\left(\sqrt{-\gamma}\left(kx - \frac{3\varphi}{k\gamma Ln^2 A}t\right)\right)\Bigg]^2\Bigg),$$

$$u_4(x,t) = Ln\Bigg(\frac{\varphi(-2\beta^2+2\alpha\sigma)}{\gamma} - \frac{3\varphi\beta\sqrt{-\gamma}}{\gamma} cot_A\left(\sqrt{-\gamma}\left(kx - \frac{3\varphi}{k\gamma Ln^2 A}t\right)\right)$$

$$\pm \frac{3\varphi\beta\sqrt{-pq\gamma}}{\gamma} csc_A\left(\sqrt{-\gamma}\left(kx - \frac{3\varphi}{k\gamma Ln^2 A}t\right)\right)$$

$$+\frac{6\varphi}{\gamma}\Bigg[-\frac{\beta}{2} - \frac{\sqrt{-\gamma}}{2} cot_A\left(\sqrt{-\gamma}\left(kx - \frac{3\varphi}{k\gamma Ln^2 A}t\right)\right)$$

$$\pm \frac{\sqrt{-pq\gamma}}{2} csc_A\left(\sqrt{-\gamma}\left(kx - \frac{3\varphi}{k\gamma Ln^2 A}t\right)\right)\Bigg]^2\Bigg),$$

$$u_5(x,t) = Ln\Bigg(\frac{\varphi(-2\beta^2+2\alpha\sigma)}{\gamma} + \frac{3\varphi\beta\sqrt{-\gamma}}{2\gamma} tan_A\left(\frac{\sqrt{-\gamma}}{4}\left(kx - \frac{3\varphi}{k\gamma Ln^2 A}t\right)\right)$$

$$-\frac{3\varphi\beta\sqrt{-\gamma}}{2\gamma} cot_A\left(\frac{\sqrt{-\gamma}}{4}\left(kx - \frac{3\varphi}{k\gamma Ln^2 A}t\right)\right)$$

$$+\frac{6\varphi}{\gamma}\Bigg[-\frac{\beta}{2} + \frac{\sqrt{-\gamma}}{4} tan_A\left(\frac{\sqrt{-\gamma}}{4}\left(kx - \frac{3\varphi}{k\gamma Ln^2 A}t\right)\right)$$

$$-\frac{\sqrt{-\gamma}}{4} cot_A\left(\frac{\sqrt{-\gamma}}{4}\left(kx - \frac{3\varphi}{k\gamma Ln^2 A}t\right)\right)\Bigg]^2\Bigg).$$

2) If $\gamma = \beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$, then

$$u_6(x,t) = Ln\Bigg(\frac{\varphi(-2\beta^2+2\alpha\sigma)}{\gamma} - \frac{3\varphi\beta\sqrt{\gamma}}{\gamma} tanh_A\left(\frac{\sqrt{\gamma}}{2}\left(kx - \frac{3\varphi}{k\gamma Ln^2 A}t\right)\right)$$

$$+\frac{6\varphi}{\gamma}\left[-\frac{\beta}{2} - \frac{\sqrt{\gamma}}{2} tanh_A\left(\frac{\sqrt{\gamma}}{2}\left(kx - \frac{3\varphi}{k\gamma Ln^2 A}t\right)\right)\right]^2\Bigg),$$

$$u_7(x,t) = Ln\left(\frac{\varphi(-2\beta^2+2\alpha\sigma)}{\gamma} - \frac{3\varphi\beta\sqrt{\gamma}}{\gamma}coth_A\left(\frac{\sqrt{\gamma}}{2}\left(kx - \frac{3\varphi}{k\gamma Ln^2A}t\right)\right)\right.$$
$$\left.+\frac{6\varphi}{\gamma}\left[-\frac{\beta}{2} + \frac{\sqrt{\gamma}}{2}coth_A\left(\frac{\sqrt{\gamma}}{2}\left(kx - \frac{3\varphi}{k\gamma Ln^2A}t\right)\right)\right]^2\right),$$

$$u_8(x,t) = Ln\left(\frac{\varphi(-2\beta^2+2\alpha\sigma)}{\gamma} - \frac{3\varphi\beta\sqrt{\gamma}}{\gamma}tanh_A\left(\sqrt{\gamma}\left(kx - \frac{3\varphi}{k\gamma Ln^2A}t\right)\right)\right.$$
$$\pm i\frac{3\varphi\beta\sqrt{pq\gamma}}{\gamma}sech_A\left(\sqrt{\gamma}\left(kx - \frac{3\varphi}{k\gamma Ln^2A}t\right)\right)$$
$$+\frac{6\varphi}{\gamma}\left[-\frac{\beta}{2} - \frac{\sqrt{\gamma}}{2}tanh_A\left(\sqrt{\gamma}\left(kx - \frac{3\varphi}{k\gamma Ln^2A}t\right)\right)\right.$$
$$\left.\left.\pm i\frac{\sqrt{pq\gamma}}{2}sech_A\left(\sqrt{\gamma}\left(kx - \frac{3\varphi}{k\gamma Ln^2A}t\right)\right)\right]^2\right),$$

$$u_9(x,t) = Ln\left(\frac{\varphi(-2\beta^2+2\alpha\sigma)}{\gamma} - \frac{3\varphi\beta\sqrt{\gamma}}{\gamma}cot_A\left(\sqrt{\gamma}\left(kx - \frac{3\varphi}{k\gamma Ln^2A}t\right)\right)\right.$$
$$\pm\frac{3\varphi\beta\sqrt{pq\gamma}}{\gamma}csc_A\left(\sqrt{\gamma}\left(kx - \frac{3\varphi}{k\gamma Ln^2A}t\right)\right)$$
$$+\frac{6\varphi}{\gamma}\left[-\frac{\beta}{2} - \frac{\sqrt{-\gamma}}{2}cot_A\left(\sqrt{\gamma}\left(kx - \frac{3\varphi}{k\gamma Ln^2A}t\right)\right)\right.$$
$$\left.\left.\pm\frac{\sqrt{pq\gamma}}{2}csch_A\left(\sqrt{\gamma}\left(kx - \frac{3\varphi}{k\gamma Ln^2A}t\right)\right)\right]^2\right),$$

$$u_{10}(x,t) = Ln\left(\frac{\varphi(-2\beta^2+2\alpha\sigma)}{\gamma} - \frac{3\varphi\beta\sqrt{\gamma}}{2\gamma}tanh_A\left(\frac{\sqrt{\gamma}}{4}\left(kx - \frac{3\varphi}{k\gamma Ln^2A}t\right)\right)\right.$$
$$-\frac{3\varphi\beta\sqrt{\gamma}}{2\gamma}coth_A\left(\frac{\sqrt{\gamma}}{4}\left(kx - \frac{3\varphi}{k\gamma Ln^2A}t\right)\right)$$
$$+\frac{6\varphi}{\gamma}\left[-\frac{\beta}{2} - \frac{\sqrt{\gamma}}{4}tanh_A\left(\frac{\sqrt{\gamma}}{4}\left(kx - \frac{3\varphi}{k\gamma Ln^2A}t\right)\right)\right.$$
$$\left.\left.-\frac{\sqrt{\gamma}}{4}coth_A\left(\frac{\sqrt{\gamma}}{4}\left(kx - \frac{3\varphi}{k\gamma Ln^2A}t\right)\right)\right]^2\right),$$

3) If $\alpha\sigma > 0, \sigma \neq 0$ and $\beta = 0$, then

$$u_{11}(x,t) = Ln\left(-\frac{\varphi}{2} - \frac{3\varphi}{2}tan_A^2\left(\sqrt{\alpha\sigma}\left(kx + \frac{3\varphi}{4k\alpha\sigma Ln^2 A}t\right)\right)\right),$$

$$u_{12}(x,t) = Ln\left(-\frac{\varphi}{2} - \frac{3\varphi}{2}cot_A^2\left(\sqrt{\alpha\sigma}\left(kx + \frac{3\varphi}{4k\alpha\sigma Ln^2 A}t\right)\right)\right).$$

$$u_{13}(x,t) = Ln\left(-\frac{\varphi}{2} - \frac{3\varphi}{2}\left[tan_A\left(2\sqrt{\alpha\sigma}\left(kx + \frac{3\varphi}{4k\alpha\sigma Ln^2 A}t\right)\right)\right.\right.$$
$$\left.\left.\pm\sqrt{pq}sec_A\left(2\sqrt{\alpha\sigma}\left(kx - \frac{3\varphi}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]^2\right),$$

$$u_{14}(x,t) = Ln\left(-\frac{\varphi}{2} - \frac{3\varphi}{2}\left[-cot_A\left(2\sqrt{\alpha\sigma}\left(kx + \frac{3\varphi}{4k\alpha\sigma Ln^2 A}t\right)\right)\right.\right.$$
$$\left.\left.\pm\sqrt{pq}csc_A\left(2\sqrt{\alpha\sigma}\left(kx - \frac{3\varphi}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]^2\right),$$

$$u_{15}(x,t) = Ln\left(-\frac{\varphi}{2} - \frac{3\varphi}{8}\left[tan_A\left(\frac{\sqrt{\alpha\sigma}}{2}\left(kx + \frac{3\varphi}{4k\alpha\sigma Ln^2 A}t\right)\right)\right.\right.$$
$$\left.\left.-cot_A\left(\frac{\sqrt{\alpha\sigma}}{2}\left(kx + \frac{3\varphi}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]^2\right),$$

4) If $\alpha\sigma < 0, \sigma \neq 0$ and $\beta = 0$, then

$$u_{16}(x,t) = Ln\left(-\frac{\varphi}{2} - \frac{3\varphi}{2}tanh_A^2\left(\sqrt{-\alpha\sigma}\left(kx + \frac{3\varphi}{4k\alpha\sigma Ln^2 A}t\right)\right)\right),$$

$$u_{17}(x,t) = Ln\left(-\frac{\varphi}{2} - \frac{3\varphi}{2}coth_A^2\left(\sqrt{-\alpha\sigma}\left(kx + \frac{3\varphi}{4k\alpha\sigma Ln^2 A}t\right)\right)\right).$$

$$u_{18}(x,t) = Ln\left(-\frac{\varphi}{2} - \frac{3\varphi}{2}\left[-tanh_A\left(2\sqrt{-\alpha\sigma}\left(kx + \frac{3\varphi}{4k\alpha\sigma Ln^2 A}t\right)\right)\right.\right.$$
$$\left.\left.\pm i\sqrt{pq}sech_A\left(2\sqrt{-\alpha\sigma}\left(kx + \frac{3\varphi}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]^2\right),$$

$$u_{19}(x,t) = Ln\left(-\frac{\varphi}{2} - \frac{3\varphi}{2}\left[-coth_A\left(2\sqrt{-\alpha\sigma}\left(kx + \frac{3\varphi}{4k\alpha\sigma Ln^2 A}t\right)\right)\right.\right.$$
$$\left.\left.\pm\sqrt{pq}csch_A\left(2\sqrt{-\alpha\sigma}\left(kx + \frac{3\varphi}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]^2\right),$$

$$u_{20}(x,t) = Ln\left(-\frac{\varphi}{2} - \frac{3\varphi}{8}\left[tanh_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\left(kx + \frac{3\varphi}{4k\alpha\sigma Ln^2 A}t\right)\right)\right.\right.$$
$$\left.\left. + coth_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\left(kx + \frac{3\varphi}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]^2\right).$$

5) If $\beta = 0$ and $\sigma = \alpha$, then

$$u_{21}(x,t) = Ln\left(-\frac{\varphi}{2} - \frac{3\varphi}{2}tan_A^2\left(\alpha kx + \frac{3\varphi}{4k\alpha Ln^2 A}t\right)\right),$$

$$u_{22}(x,t) = Ln\left(-\frac{\varphi}{2} - \frac{3\varphi}{2}cot_A^2\left(\alpha kx + \frac{3\varphi}{4k\alpha Ln^2 A}t\right)\right),$$

$$u_{23}(x,t) = Ln\left(-\frac{\varphi}{2} - \frac{3\varphi}{2}\left[tan_A\left(2\alpha kx + \frac{3\varphi}{2k\alpha Ln^2 A}t\right)\right.\right.$$
$$\left.\left. \pm \sqrt{pq}sec_A\left(2\alpha kx + \frac{3\varphi}{2k\alpha Ln^2 A}t\right)\right]^2\right),$$

$$u_{24}(x,t) = Ln\left(-\frac{\varphi}{2} - \frac{3\varphi}{2}\left[-cot_A\left(2\alpha kx + \frac{3\varphi}{2k\alpha Ln^2 A}t\right)\right.\right.$$
$$\left.\left. \pm \sqrt{pq}csc_A\left(2\alpha kx + \frac{3\varphi}{2k\alpha Ln^2 A}t\right)\right]^2\right),$$

$$u_{25}(x,t) = Ln\left(-\frac{\varphi}{2} - \frac{3\varphi}{2}\left[tan_A\left(\frac{\alpha k}{2}x + \frac{3\varphi}{8k\alpha Ln^2 A}t\right)\right.\right.$$
$$\left.\left. - cot_A\left(\frac{\alpha k}{2}x + \frac{3\varphi}{8k\alpha Ln^2 A}t\right)\right]^2\right).$$

6) If $\beta = 0$ and $\sigma = -\alpha$, then

$$u_{26}(x,t) = Ln\left(-\frac{\varphi}{2} + \frac{3\varphi}{2}tanh_A^2\left(\alpha kx - \frac{3\varphi}{4k\alpha Ln^2 A}t\right)\right),$$

$$u_{27}(x,t) = Ln\left(-\frac{\varphi}{2} + \frac{3\varphi}{2}coth_A^2\left(\alpha kx - \frac{3\varphi}{4k\alpha Ln^2 A}t\right)\right),$$

$$u_{28}(x,t) = Ln\left(-\frac{\varphi}{2} + \frac{3\varphi}{2}\left[-tanh_A\left(2\alpha kx - \frac{3\varphi}{2k\alpha Ln^2 A}t\right)\right.\right.$$
$$\left.\left. \pm i\sqrt{pq}sech_A\left(2\alpha kx - \frac{3\varphi}{2k\alpha Ln^2 A}t\right)\right]^2\right),$$

$$u_{29}(x,t) = Ln\left(-\frac{\varphi}{2} + \frac{3\varphi}{2}\left[-coth_A\left(2\alpha kx - \frac{3\varphi}{2k\alpha Ln^2 A}t\right)\right.\right.$$
$$\left.\left.\pm\sqrt{pq}csch_A\left(2\alpha kx - \frac{3\varphi}{2k\alpha Ln^2 A}t\right)\right]^2\right),$$

$$u_{30}(x,t) = Ln\left(-\frac{\varphi}{2} + \frac{3\varphi}{8}\left[tanh_A\left(\frac{\alpha k}{2}x - \frac{3\varphi}{8k\alpha Ln^2 A}t\right)\right.\right.$$
$$\left.\left.+ coth_A\left(\frac{\alpha k}{2}x - \frac{3\varphi}{8k\alpha Ln^2 A}t\right)\right]^2\right).$$

7) If $\alpha = 0$ and $\beta \neq 0$, then

$$u_{31}(x,t) = Ln\left(\varphi - \left[\frac{6\varphi p}{cosh_A\left(\beta kx - \frac{3\varphi}{k\beta Ln^2 A}t\right) - sinh_A\left(\beta kx - \frac{3\varphi}{k\beta Ln^2 A}t\right) + p}\right]\right.$$
$$\left.+\left[\frac{6\varphi p}{cosh_A\left(\beta kx - \frac{3\varphi}{k\beta Ln^2 A}t\right) - sinh_A\left(\beta kx - \frac{3\varphi}{k\beta Ln^2 A}t\right) + p}\right]^2\right),$$

$$u_{32}(x,t) = Ln\left(\varphi - 6\varphi\left[\frac{sinh_A\left(\beta kx - \frac{3\varphi}{k\beta Ln^2 A}t\right) + cosh_A\left(\beta kx - \frac{3\varphi}{k\beta Ln^2 A}t\right)}{cosh_A\left(\beta kx - \frac{3\varphi}{k\beta Ln^2 A}t\right) + sinh_A\left(\beta kx - \frac{3\varphi}{k\beta Ln^2 A}t\right) + q}\right]\right.$$
$$\left.+6\varphi\left[\frac{sinh_A\left(\beta kx - \frac{3\varphi}{k\beta Ln^2 A}t\right) + cosh_A\left(\beta kx - \frac{3\varphi}{k\beta Ln^2 A}t\right)}{cosh_A\left(\beta kx - \frac{3\varphi}{k\beta Ln^2 A}t\right) + sinh_A\left(\beta kx - \frac{3\varphi}{k\beta Ln^2 A}t\right) + q}\right]^2\right).$$

8) If $\beta = k, \sigma = mk(m \neq 0)$ and $\alpha = 0$, then

$$u_{33}(x,t) = Ln\left(\varphi + 6m\varphi\left[\frac{pA^{\left(k^2 x - \frac{3\varphi}{k^2 Ln^2 A}t\right)}}{q - mpA^{\left(k^2 x - \frac{3\varphi}{k^2 Ln^2 A}t\right)}}\right]\right.$$
$$\left.+6m^2\varphi\left[\frac{A^{\left(k^2 x - \frac{3\varphi}{k^2 Ln^2 A}t\right)}}{q - mpA^{\left(k^2 x - \frac{3\varphi}{k^2 Ln^2 A}t\right)}}\right]^2\right.$$

2.3. The Liouville equation

Consider the Liouville equation described by:

$$u_{xt} + e^u = 0. \tag{2.15}$$

By using the Painleve transformation $u = Ln(v)$, Eq. (2.15) can be written as:

$$vv_{xt} - v_x v_t + v^3 = 0. \tag{2.16}$$

Execute certain variable transformation

$$v(x,t) = U(\xi), \qquad \xi = kx + \lambda t, \tag{2.17}$$

Eq. (2.15) becomes

$$k\lambda(UU'' - (U')^2) + U^3 = 0. \tag{2.18}$$

Balancing the highest order derivative term and the highest order nonlinear term in (2.17), we find $N = 1$. Thus, we choose solution of Eq. (2.17) as follows:

$$U(\xi) = b_0 + b_1 Q(\xi), \tag{2.19}$$

By substituting (2.18) into Eq. (2.17) and collecting all terms with the same order of $Q(\xi)$ together, the left-handside of (2.17) are converted into polynomial in $Q(\xi)$. Setting each coefficient of each polynomial to zero, we arrive a set of algebraic system for b_0, b_1 and λ as follows:

$$\begin{aligned} &b_0 = \frac{-4\alpha\sigma + \beta^2 + \sqrt{\beta^2(-4\alpha\sigma + \beta^2)}}{2(-\beta^2 + 4\alpha\sigma)}, \quad b_1 = -\frac{\sigma}{\sqrt{\beta^2 - 4\alpha\sigma}}, \\ &\lambda = \frac{1}{k(-\beta^2 + 4\alpha\sigma)Ln^2 A}. \end{aligned} \tag{2.20}$$

By using of Eqs. (2.18) and (2.19), becomes:

1) If $\gamma = \beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$, then

$$u_1(x,t) = Ln\left(-\frac{1}{2} + \frac{\beta}{\sqrt{\gamma}} - \frac{i}{2}tan_A\left(\frac{\sqrt{-\gamma}}{2}\left(kx - \frac{1}{k\gamma Ln^2 A}t\right)\right)\right),$$

$$u_2(x,t) = Ln\left(-\frac{1}{2} + \frac{\beta}{\sqrt{\gamma}} + \frac{i}{2}cot_A\left(\frac{\sqrt{-\gamma}}{2}\left(kx - \frac{1}{k\gamma Ln^2 A}t\right)\right)\right),$$

$$\begin{aligned} u_3(x,t) = Ln\Bigg(&-\frac{1}{2} + \frac{\beta}{\sqrt{\gamma}} - \frac{i}{2}tan_A\left(\sqrt{-\gamma}\left(kx\frac{1}{k\gamma Ln^2 A}t\right)\right) \\ &\pm \frac{\sqrt{-pq}}{2}sec_A\left(\sqrt{-\gamma}\left(kx - \frac{1}{k\gamma Ln^2 A}t\right)\right)\Bigg), \end{aligned}$$

$$\begin{aligned} u_4(x,t) = Ln\Bigg(&-\frac{1}{2} + \frac{\beta}{\sqrt{\gamma}} + \frac{i}{2}cot_A\left(\sqrt{-\gamma}\left(kx - \frac{1}{k\gamma Ln^2 A}t\right)\right) \\ &\pm \frac{\sqrt{-pq}}{2}csc_A\left(\sqrt{-\gamma}\left(kx - \frac{1}{k\gamma Ln^2 A}t\right)\right)\Bigg), \end{aligned}$$

$$\begin{aligned} u_5(x,t) = Ln\Bigg(&-\frac{1}{2} + \frac{\beta}{\sqrt{\gamma}} - \frac{i}{4}tan_A\left(\frac{\sqrt{-\gamma}}{4}\left(kx - \frac{1}{k\gamma Ln^2 A}t\right)\right) \\ &+ \frac{i}{4}cot_A\left(\frac{\sqrt{-\gamma}}{4}\left(kx - \frac{1}{k\gamma Ln^2 A}t\right)\right)\Bigg). \end{aligned}$$

2) If $\gamma = \beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$, then

$$u_6(x,t) = Ln\left(-\frac{1}{2} + \frac{\beta}{\sqrt{\gamma}} + \frac{1}{2}tanh_A\left(\frac{\sqrt{\gamma}}{2}\left(kx - \frac{1}{k\gamma Ln^2 A}t\right)\right)\right),$$

$$u_7(x,t) = Ln\left(-\frac{1}{2} + \frac{\beta}{\sqrt{\gamma}} + \frac{1}{2}coth_A\left(\frac{\sqrt{\gamma}}{2}\left(kx - \frac{1}{k\gamma Ln^2 A}t\right)\right)\right),$$

$$u_8(x,t) = Ln\left(-\frac{1}{2} + \frac{\beta}{\sqrt{\gamma}} + \frac{1}{2}tanh_A\left(\sqrt{\gamma}\left(kx - \frac{1}{k\gamma Ln^2 A}t\right)\right)\right.$$
$$\left.\pm i\frac{\sqrt{pq}}{2}sech_A\left(\sqrt{\gamma}\left(kx - \frac{1}{k\gamma Ln^2 A}t\right)\right)\right),$$

$$u_9(x,t) = Ln\left(-\frac{1}{2} + \frac{\beta}{\sqrt{\gamma}} + \frac{1}{2}cot_A\left(\sqrt{\gamma}\left(kx - \frac{1}{k\gamma Ln^2 A}t\right)\right)\right.$$
$$\left.\pm \frac{\sqrt{pq}}{2}csc_A\left(\sqrt{\gamma}\left(kx - \frac{1}{k\gamma Ln^2 A}t\right)\right)\right),$$

$$u_{10}(x,t) = Ln\left(-\frac{1}{2} + \frac{\beta}{\sqrt{\gamma}} + \frac{1}{4}tanh_A\left(\frac{\sqrt{\gamma}}{4}\left(kx - \frac{1}{k\gamma Ln^2 A}t\right)\right)\right.$$
$$\left.+ \frac{1}{4}coth_A\left(\frac{\sqrt{\gamma}}{4}\left(kx - \frac{1}{k\gamma Ln^2 A}t\right)\right)\right).$$

3) If $\alpha\sigma > 0, \sigma \neq 0$ and $\beta = 0$, then

$$u_{11}(x,t) = Ln\left(\frac{1}{2} \pm \frac{i}{2}\left[tan_A\left(\sqrt{\alpha\sigma}\left(kx + \frac{1}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]\right),$$

$$u_{12}(x,t) = Ln\left(\frac{1}{2} \pm \frac{i}{2}\left[cot_A\left(\sqrt{\alpha\sigma}\left(kx + \frac{1}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]\right).$$

$$u_{13}(x,t) = Ln\left(\frac{1}{2} \pm \frac{i}{2}\left[tan_A\left(2\sqrt{\alpha\sigma}\left(kx + \frac{1}{4k\alpha\sigma Ln^2 A}t\right)\right)\right.\right.$$
$$\left.\left.\pm \sqrt{pq}sec_A\left(2\sqrt{\alpha\sigma}\left(kx + \frac{1}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]\right),$$

$$u_{14}(x,t) = Ln\left(\frac{1}{2} \pm \frac{i}{2}\left[-cot_A\left(2\sqrt{\alpha\sigma}\left(kx + \frac{1}{4k\alpha\sigma Ln^2 A}t\right)\right)\right.\right.$$
$$\left.\left.\pm \sqrt{pq}csc_A\left(2\sqrt{\alpha\sigma}\left(kx + \frac{1}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]\right),$$

$$u_{15}(x,t) = Ln\left(\frac{1}{2} \pm \frac{i}{4}\left[tan_A\left(\frac{\sqrt{\alpha\sigma}}{2}\left(kx + \frac{1}{4k\alpha\sigma Ln^2 A}t\right)\right)\right.\right.$$
$$\left.\left.- cot_A\left(\frac{\sqrt{\alpha\sigma}}{2}\left(kx + \frac{1}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]\right).$$

4) If $\alpha\sigma < 0, \sigma \neq 0$ and $\beta = 0$, then

$$u_{16}(x,t) = Ln\left(\frac{1}{2} \pm \frac{i}{2}\left[tanh_A\left(\sqrt{-\alpha\sigma}\left(kx + \frac{1}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]\right),$$

$$u_{17}(x,t) = Ln\left(\frac{1}{2} \pm \frac{i}{2}\left[coth_A\left(\sqrt{-\alpha\sigma}\left(kx + \frac{1}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]\right).$$

$$u_{18}(x,t) = Ln\left(\frac{1}{2} \pm \frac{i}{2}\left[-tanh_A\left(2\sqrt{-\alpha\sigma}\left(kx + \frac{1}{4k\alpha\sigma Ln^2 A}t\right)\right)\right.\right.$$
$$\left.\left. \pm\, i\sqrt{pq}sech_A\left(2\sqrt{-\alpha\sigma}\left(kx + \frac{1}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]\right),$$

$$u_{19}(x,t) = Ln\left(\frac{1}{2} \pm \frac{i}{2}\left[-coth_A\left(2\sqrt{-\alpha\sigma}\left(kx + \frac{1}{4k\alpha\sigma Ln^2 A}t\right)\right)\right.\right.$$
$$\left.\left. \pm \sqrt{pq}csch_A\left(2\sqrt{-\alpha\sigma}\left(kx + \frac{1}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]\right),$$

$$u_{20}(x,t) = Ln\left(\frac{1}{2} \pm \frac{i}{4}\left[tanh_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\left(kx + \frac{1}{4k\alpha\sigma Ln^2 A}t\right)\right)\right.\right.$$
$$\left.\left. + coth_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\left(kx + \frac{1}{4k\alpha\sigma Ln^2 A}t\right)\right)\right]\right).$$

5) If $\beta = 0$ and $\sigma = \alpha$, then

$$u_{21}(x,t) = Ln\left(-\frac{1}{2} + \frac{1}{2}\left[tan_A\left(\alpha kx + \frac{1}{4k\alpha Ln^2 A}t\right)\right)\right),$$

$$u_{22}(x,t) = Ln\left(-\frac{1}{2} + \frac{1}{2}\left[cot_A\left(\alpha kx + \frac{1}{4k\alpha Ln^2 A}t\right)\right)\right),$$

$$u_{23}(x,t) = Ln\left(-\frac{1}{2} + \frac{1}{2}\left[tan_A\left(2\alpha kx + \frac{1}{2k\alpha Ln^2 A}t\right)\right.\right.$$
$$\left.\left. \pm \sqrt{pq}sec_A\left(2\alpha kx + \frac{3}{2k\alpha Ln^2 A}t\right)\right]\right),$$

$$u_{24}(x,t) = Ln\left(-\frac{1}{2} + \frac{1}{2}\left[-cot_A\left(2\alpha kx + \frac{1}{2k\alpha Ln^2 A}t\right)\right.\right.$$
$$\left.\left. \pm \sqrt{pq}csc_A\left(2\alpha kx + \frac{3}{2k\alpha Ln^2 A}t\right)\right]\right),$$

$$u_{25}(x,t) = Ln\left(-\frac{1}{2} + \frac{1}{4}\left[tan_A\left(\frac{\alpha k}{2}x + \frac{1}{8k\alpha Ln^2 A}t\right) - cot_A\left(\frac{\alpha k}{2}x + \frac{1}{8k\alpha Ln^2 A}t\right)\right]\right).$$

6) If $\beta = 0$ and $\sigma = -\alpha$, then

$$u_{26}(x,t) = Ln\left(\frac{1}{2} + \frac{1}{2}tanh_A^2\left(\alpha kx - \frac{1}{4k\alpha Ln^2 A}t\right)\right),$$

$$u_{27}(x,t) = Ln\left(\frac{1}{2} + \frac{1}{2}coth_A^2\left(\alpha kx - \frac{1}{4k\alpha Ln^2 A}t\right)\right),$$

$$u_{28}(x,t) = Ln\left(\frac{1}{2} + \frac{1}{2}\left[-tanh_A\left(2\alpha kx - \frac{1}{2k\alpha Ln^2 A}t\right) \pm i\sqrt{pq}sech_A\left(2\alpha kx - \frac{1}{2k\alpha Ln^2 A}t\right)\right]^2\right),$$

$$u_{29}(x,t) = Ln\left(\frac{1}{2} + \frac{1}{2}\left[-coth_A\left(2\alpha kx - \frac{1}{2k\alpha Ln^2 A}t\right) \pm \sqrt{pq}csch_A\left(2\alpha kx - \frac{1}{2k\alpha Ln^2 A}t\right)\right]^2\right),$$

$$u_{30}(x,t) = Ln\left(\frac{1}{2} - \frac{1}{4}\left[tanh_A\left(\frac{\alpha k}{2}x - \frac{1}{8k\alpha Ln^2 A}t\right) + coth_A\left(\frac{\alpha k}{2}x - \frac{1}{8k\alpha Ln^2 A}t\right)\right]\right).$$

7) If $\alpha = 0$ and $\beta \neq 0$, then

$$u_{31}(x,t) = Ln\left(-1 + \left[\frac{p}{cosh_A\left(\beta kx - \frac{1}{k\beta Ln^2 A}t\right) - sinh_A\left(\beta kx - \frac{1}{k\beta Ln^2 A}t\right) + p}\right]\right),$$

$$u_{32}(x,t) = Ln\left(-1 + \left[\frac{sinh_A\left(\beta kx - \frac{1}{k\beta Ln^2 A}t\right) + cosh_A\left(\beta kx - \frac{1}{k\beta Ln^2 A}t\right)}{cosh_A\left(\beta kx - \frac{1}{k\beta Ln^2 A}t\right) + sinh_A\left(\beta kx - \frac{1}{k\beta Ln^2 A}t\right) + q}\right]\right).$$

8) If $\beta = k, \sigma = mk \ \ (m \neq 0)$ and $\alpha = 0$, then

$$u_{33}(x,t) = Ln\left(-1 - m\left[\frac{pA^{\left(k^2 x - \frac{1}{k^2 Ln^2 A}t\right)}}{q - mpA^{\left(k^2 x - \frac{1}{k^2 Ln^2 A}t\right)}}\right]\right).$$

3. Conclusions

In this work, the new approach of expansion method samely the extended direct algebraic method has successfully been implemented to investigate Tzitzéica equation, Dodd-Bullough-Mikhailor equation and Liouville equation. Using of an appropriate traveling wave transformation reduces these equations to ODE. With the aid of Maple, we have assured the correctness of the obtained solutions by putting them back into the original equation. we hope that the will be useful for further studies in applied sciences.

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