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Oscillatory behavior for nonlinear delay differential equation with several non-monotone arguments

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Abstract This paper is devoted to obtaining some new sufficient conditions for the oscillation of all solutions of first order nonlinear differential equations with several deviating arguments. Finally, an illustrative example related to our results is given.

Keywords. Nonlinear, Delay differential equation, Non-monotone argument, Oscillatory solutions, Nonoscillatory solutions.

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1. Introduction

Considering the retarded differential equation of form

$$x'(t) + \sum_{i=1}^{n} p_i(t) f_i(x(\tau_i(t))) = 0, \quad t \geq t_0, \tag{1.1}$$

where the functions $p_i(t),\ \tau_i(t) \in C\left([t_0, \infty), \mathbf{R}^+\right)$ for every $i = 1, 2, \cdots, n$ and $\tau_i(t)$ are non-monotone or nondecreasing such that

$$\tau_i(t) \leq t \ \text{ for } t \geq t_0 \text{ and } \lim_{t\to\infty} \tau_i(t) = \infty \text{ for } 1 \leq i \leq n \tag{1.2}$$

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and

$$f_i \in C(\mathbb{R}, \mathbb{R}) \text{ and } x f_i(x) > 0 \text{ for } x \neq 0, \tag{1.3}$$

for $1 \leq i \leq n$.

By a solution of (1.1) we mean a continuously differentiable function defined on $[\tau_i(T_0), \infty]$ for some $T_0 \geq t_0$ such that (1.1) is held for $t \geq T_0$. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is said to be nonoscillatory.

For $n = 1$, Eq. (1.1) reduces to

$$x'(t) + p(t) f(x(\tau(t))) = 0, \ t \geq t_0. \tag{1.4}$$

Recently, there has been an increasing interest in the study of the oscillatory behaviour of following special form of (1.4)

$$x'(t) + p(t) x(\tau(t)) = 0, \ t \geq t_0. \tag{1.5}$$

See, for example, [1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 18, 19, 20, 23] and references cited therein. The first systematic study for the oscillation of all solutions of equation (1.5) was made by Myshkis. In 1950 [19], the author proved that every solution of (1.5) oscillates if

$$\limsup_{t\to\infty} [t - \tau(t)] < \infty \text{ and } \liminf_{t\to\infty} [t - \tau(t)] \liminf_{t\to\infty} p(t) > \frac{1}{e}.$$

In 1972, Ladas, Lakshmikantham and Papadakis [17] proved that the same conlusions hold if $\tau(t)$ is nondecreasing and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s) ds > 1. \tag{1.6}$$

In 1982, Koplatadze and Canturija [15] proved that if $\tau(t)$ is not necessarily monotone and

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s) ds > \frac{1}{e}, \tag{1.7}$$

then all solutions of Eq. (1.5) oscillate, while if

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s) ds < \frac{1}{e}, \tag{1.8}$$

then Eq. (1.5) has a nonoscillatory solution.

Now, let us consider again Eq. (1.4). The problem of establishing sufficient conditions for oscillation of all solutions to (1.4) has been inspired of many authors. The following result was given by Ladde et al. in [18]. Assume that the f, p and τ in Eq. (1.4) satisfy the following conditions;

(i) $\tau(t) \leq t$, for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$ and let $\tau(t)$ be strictly increasing on $\mathbb{R}^+$,

(ii) $p(t)$ is locally integrable and $p(t) \geq 0$, *(iii)* $f \in C(\mathbb{R},\mathbb{R})$ and $xf(x) > 0$ for $x \neq 0$ and let f be nondecreasing and $\lim_{x\to 0} \frac{x}{f(x)} = M < \infty$.

Then the authors proved that if

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > M$$

or

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{M}{e},$$

then all solutions of Eq. (1.4) are oscillatory.

In 1984, Fukagai and Kusano [10] proved that the following result. Suppose that (1.2), (1.3) hold and that $\tau(t)$ is nondecreasing function. Suppose moreover that

$$\limsup_{x\to 0} \frac{|x|}{|f(x)|} = N < \infty.$$

If

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{N}{e},$$

then all solutions of Eq. (1.4) are oscillatory.

In 2016, Öcalan et al. [23] obtained that the following results. Assume that $\tau(t)$ is not necessarily monotone, $h(t) = \sup_{s\leq t} \tau(s)$, $t \geq t_0$ and $\limsup\limits_{x\to 0} \frac{x}{f(x)} = M$. Thus, if

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{M}{e}, \quad \text{where } 0 \leq M < \infty$$

or

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)ds > 2M, \quad \text{where } 0 < M < \infty,$$

then all solutions of Eq. (1.4) are oscillatory.

Now, we consider the following linear form of Eq. (1.1)

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(\tau_i(t)) = 0. \tag{1.9}$$

In 1996, Li [21] studied the equation (1.9) with constant delays of the form

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t - k_i) = 0, \tag{1.10}$$

and he proved that if

$$\liminf_{t\to\infty}\sum_{i=1}^{n}\int_{t-k_i}^{t} p_i(s)ds > \frac{1}{e}, \tag{1.11}$$

then all solutions of Eq. (1.10) oscillate.

In 2004, Tang [20] proved that if

$$\limsup_{t\to\infty}\sum_{i=1}^{n}\int_{t-k_i}^{t} p_i(s)ds > 1, \tag{1.12}$$

then all solutions of Eq. (1.10) oscillate.

In 1984, Hunt and Yorke [14] considered the following equation with variable delays of the form

$$x'(t)+\sum_{i=1}^{n}p_i(t)x(t-k_i(t)) = 0, \tag{1.13}$$

under the assumption that there is a uniform upper bound k_0 on the k_i's and proved that if

$$\liminf_{t\to\infty}\sum_{i=1}^{n}k_i(t)p_i(t) > \frac{1}{e}, \tag{1.14}$$

then all solutions of Eq. (1.13) oscillate.

In 1991, Győri and Ladas [13] proved that if $k_i(t)$ are nondecreasing functions and

$$\limsup_{t\to\infty}\sum_{i=1}^{n}\int_{t-k(t)}^{t} p_i(s)ds > 1, \tag{1.15}$$

where $k(t) = \min_{1\leq i\leq n}\{k_i(t)\}$ and $\lim_{t\to\infty}(t-k(t)) = \infty$, then all solutions of Eq. (1.13) oscillate.

In 1984, Fukagai and Kusano [10] established the following results for Eq. (1.9). Assume that $\tau_i(t)$ are nondecreasing functions, $\lim_{t\to\infty}\tau_i(t) = \infty$ and there is a continuous nondecreasing function $\tau^*(t)$ such that $\tau_i(t) \leq \tau^*(t) \leq t$ for $t \geq t_0$ $(1 \leq i \leq n)$. If

$$\liminf_{t\to\infty}\int_{\tau^*(t)}^{t}\sum_{i=1}^{n}p_i(s)ds > \frac{1}{e}, \tag{1.16}$$

then all solutions of Eq. (1.9) oscillate. If, on the other hand, there exists a continuous nondecreasing function $\tau_*(t)$ such that $\tau_*(t) \leq \tau_i(t)$ for $t \geq t_0$ $(1 \leq i \leq n)$, $\lim_{t\to\infty}\tau_*(t) = \infty$ and

$$\int_{\tau_*(t)}^{t}\sum_{i=1}^{n}p_i(s)ds \leq \frac{1}{e},$$

for all sufficiently large t, then Eq. (1.9) has a nonoscillatory solution.

In 2000, Grammatikopoulus et al. [12] improved the above results as follows; assume that the functions $\tau_i(t)$ are nondecreasing for all $i \in \{1, ..., n\}$,

$$\int_0^\infty |p_i(t) - p_j(t)| < +\infty, \quad i, j = 1, ..., n$$

and

$$\liminf_{t\to\infty} \int_{\tau_i(t)}^{t} p_i(s)ds = \beta_i > 0, \quad i = 1, ..., n.$$

If

$$\sum_{i=1}^{n} (\liminf_{t\to\infty} \int_{\tau_i(t)}^{t} p_i(s)ds) > \frac{1}{e}, \tag{1.17}$$

then all solutions of (1.9) oscillate.

In 2015, Chatzarakis, Öcalan and Öztürk [4] established that the following result. Assume that the functions $\tau_i(t)$ are strictly increasing functions for all $i \in \{1, ..., n\}$. If

$$\limsup_{t\to\infty} \sum_{i=1}^{n} \int_{\tau_i(t)}^{t} p_i(s)ds > 1 \tag{1.18}$$

or

$$\liminf_{t\to\infty} \sum_{i=1}^{n} \int_{t}^{\tau_i^{-1}(t)} p_i(s)ds > \frac{1}{e}, \tag{1.19}$$

then all solutions of Eq. (1.9) oscillate.

In 2015, Infante et al. [22] established the following result. Assume that there exist nondecreasing functions $\sigma_i(t) \in C([t_0, \infty))$ such that

$$\tau_i(t) \le \sigma_i(t) \le t, \quad 1 \le i \le n.$$

If

$$\limsup_{t\to\infty} \prod_{j=1}^{n} \left[\prod_{i=1}^{n} \int_{\sigma_j(t)}^{t} p_i(s) \exp(\int_{\tau_i(s)}^{\sigma_i(t)} \sum_{i=1}^{n} p_i(\xi) \times \exp\left(\int_{\tau_i(\xi)}^{\xi} \sum_{i=1}^{n} p_i(u)du \right) d\xi)ds \right]^{\frac{1}{m}} > \frac{1}{m^m}, \tag{1.20}$$

then all solutions of Eq. (1.9) oscillate.

To the best of our knowledge, there are few papers about oscillatory behavior of solutions of Eq. (1.1). See, for example, [5, 10, 18].

The following theorem was given by Ladde et al. in [18].

Theorem 1.1. *Assume that (1.2), (1.3) hold and* $\tau_i(t)$ *are strictly increasing on* $\mathbb{R}_+$, $p_i(t)$ $(1 \le i \le n)$ *are locally integrable,* f_i $(1 \le i \le n)$ *are nondecreasing functions and*

$$\lim_{x\to 0}\frac{x}{f_i(x)} = M_i < +\infty.$$

If τ_i *are nondecreasing functions for* $1 \le i \le n$, *and*

$$\liminf_{t\to\infty}\int_{\tau^*(t)}^{t}\sum_{i=1}^{n}p_i(s)ds > \frac{M^*}{e}$$

or

$$\limsup_{t\to\infty}\int_{\tau^*(t)}^{t}\sum_{i=1}^{n}p_i(s)ds > M^*,$$

where $M^* = \max_{1\le i\le n} M_i$ *and* $\tau^*(t) = \max_{1\le i\le n}\tau_i(t)$, *then every solution of Eq. (1.1) is oscillatory.*

The following theorem was given by Fukagai and Kusano in [10].

Theorem 1.2. *We consider the following equation with several deviating arguments of the type*

$$x'(t) + p(t)\sum_{i=1}^{n}f(x(\tau_i(t))) = 0, \qquad (1.21)$$

where $p(t)$ *and* $\tau_i(t)$ *are continuous on* $[a,\infty)$, *nondecreasing and* $\lim_{t\to\infty}\tau_i(t) = \infty$, $1 \le i \le n$. *Suppose that* $f(x_1, x_2, ..., x_n)$ *is a continuous function on* $\mathbb{R}^n$ *such that*

$$x_1 f(x_1, x_2, ..., x_n) > 0 \quad \text{and} \quad x_1 x_n > 0$$

and

$$M = \limsup_{x_i\to 0}\frac{|x_1|^{\alpha_1}\dots|x_n|^{\alpha_n}}{|f(x_1, x_2, ..., x_n)|} < \infty$$

for some nonnegative constants $\alpha_i, 1 \le i \le n$, *with* $\sum_{i=1}^{n}\alpha_i = 1$. *If there is a continuous nondecreasing function* $\tau^\star(t)$ *such that* $\tau_i(t) \le \tau^\star(t) \le t$ *for* $t \ge a, 1 \le i \le n$ *and*

$$\liminf_{t\to\infty}\int_{\tau^\star(t)}^{t}p(s)ds > \frac{M}{e},$$

then Eq. (1.21) is oscillatory.

Thus, in this paper, our aim is to obtain some oscillation criteria for all solutions of Eq. (1.1).

2. Main Results

In this section, we present a new sufficient conditions for the oscillation of all solutions Eq. (1.1), under the assumption that the arguments $\tau_i(t), 1 \leq i \leq n,$ are not necessarily monotone. Set

$$h_i(t) := \sup_{s \leq t} \tau_i(s),\ t \geq t_0. \tag{2.1}$$

Clearly, $h_i(t)$ are nondecreasing and $\tau_i(t) \leq h_i(t), 1 \leq i \leq n$ for all $t \geq t_0$. Also, we suppose that the function f holds the following condition

$$\limsup_{x \to 0} \frac{x}{f_i(x)} = M_i,\ 0 \leq M_i < \infty \ \text{ for } 1 \leq i \leq n. \tag{2.2}$$

Theorem 2.1. *Assume that (1.2), (1.3) and (2.2) hold. If $\tau_i(t)$ are not necessarily monotone and*

$$\liminf_{t \to \infty} \int_{h(t)}^{t} \sum_{i=1}^{n} p_i(s) ds > \frac{M^*}{e}, \tag{2.3}$$

where $M^* = \max_{1 \leq i \leq n} M_i$ and $h(t) = \min_{1 \leq i \leq n} h_i(t)$, then all solutions of Eq. (1.1) oscillate.

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of Eq. (1.1). Since $-x(t)$ is also a solution of Eq. (1.1), we can confine our discussion only to the case where the solution $x(t)$ is eventually positive. Then there exists a $t_1 \geq t_0$ such that $x(t), x(\tau_i(t)) > 0, 1 \leq i \leq n$, for all $t \geq t_1$. Thus from (1.1), we have

$$x'(t) = -\sum_{i=1}^{n} p_i(t) f_i(x(\tau_i(t))) \leq 0 \text{ for all } t \geq t_1.$$

Thus $x(t)$ is nonincreasing and has a limit $l \geq 0$ as $t \to \infty$. Now, we claim that $l = 0$. Condition (2.3) implies that

$$\int_{a}^{\infty} \sum_{i=1}^{n} p_i(t) dt = \infty. \tag{2.4}$$

In view of (2.4) and Theorem 3.1.5 in [18], we have $\lim_{t \to \infty} x(t) = 0$. Suppose $M_i > 0$ for $1 \leq i \leq n$. By the help of (2.2) we can choose $t_2 \geq t_1$ so large that

$$f_i(x(t)) \geq \frac{1}{2M_i} x(t) \geq \frac{1}{2M^\star} x(t) \ \text{ for } t \geq t_2. \tag{2.5}$$

Since $\tau_i(t) \leq h(t) \leq h_i(t)$ for $1 \leq i \leq n$, $x(t)$ is nonincreasing and $h_i(t)$ are nondecreasing, using (1.1) and (2.5), we have for $t_3 \geq t_2$

$$x'(t) + \frac{1}{2M^\star} \sum_{i=1}^{n} p_i(t) x(h(t)) \leq 0,\ t \geq t_3. \tag{2.6}$$

Also, from (2.3), it follows that there exists a constant $c > 0$ such that

$$\int_{h(t)}^{t} \sum_{i=1}^{n} p_i(s)ds \geq c > \frac{M^*}{e},\ t \geq t_3. \tag{2.7}$$

So, from (2.7), there exists a real number $t^* \in (h(t), t)$, for all $t \geq t_3$ such that

$$\int_{h(t)}^{t^*} \sum_{i=1}^{n} p_i(s)ds > \frac{M^*}{2e} \text{ and } \int_{t^*}^{t} \sum_{i=1}^{n} p_i(s)ds > \frac{M^*}{2e}. \tag{2.8}$$

Integrating (2.6) from $h(t)$ to t^* and using $x(t)$ is nonincreasing, then we get

$$x(t^*) - x(h(t)) + \frac{1}{2M^*} \int_{h(t)}^{t^*} \sum_{i=1}^{n} p_i(s)x(h(s))ds \leq 0$$

or

$$x(t^*) - x(h(t)) + \frac{1}{2M^*} x(h(t^*)) \int_{h(t)}^{t^*} \sum_{i=1}^{n} p_i(s)ds \leq 0.$$

With the help of (2.8), we have

$$-x(h(t)) + \frac{1}{2M^*} x(h(t^*)) \frac{M^*}{2e} < 0. \tag{2.9}$$

Integrating (2.6) from t^* to t and using the same facts, we get

$$x(t) - x(t^*) + \frac{1}{2M^*} \int_{t^*}^{t} \sum_{i=1}^{n} p_i(s)x(h(s))ds \leq 0$$

or

$$x(t) - x(t^*) + \frac{1}{2M^*} x(h(t)) \int_{t^*}^{t} \sum_{i=1}^{n} p_i(s)ds \leq 0.$$

Following from (2.8), we have

$$-x(t^*) + \frac{1}{2M^*} x(h(t)) \frac{M^*}{2e} < 0. \tag{2.10}$$

Then combining (2.9) and (2.10), we obtain

$$x(t^*) > \frac{1}{4e} x(h(t)) > \frac{1}{(4e)^2} x(h(t^*)).$$

Hence, we have for $t_4 \geq t_3$

$$\frac{x(h(t^*))}{x(t^*)} < (4e)^2 \text{ for } t \geq t_4.$$

Let

$$w = \frac{x(h(t^*))}{x(t^*)} \geq 1 \tag{2.11}$$

and since $1 \leq w < (4e)^2$, w is finite. Now, we divide (1.1) to $x(t)$ and then integrating from $h(t)$ to t we get

$$\int_{h(t)}^{t} \frac{x^{'}(s)}{x(s)} ds + \int_{h(t)}^{t} \sum_{i=1}^{n} p_i(s) \frac{f_i(x(\tau_i(s)))}{x(s)} ds = 0$$

and

$$\ln \frac{x(t)}{x(h(t))} + \int_{h(t)}^{t} \sum_{i=1}^{n} p_i(s) \frac{f_i(x(\tau_i(s)))}{x(s)} ds = 0.$$

Using the above equalities, we can write

$$\ln \frac{x(t)}{x(h(t))} + \int_{h(t)}^{t} \sum_{i=1}^{n} p_i(s) \frac{f_i(x(\tau_i(s)))}{x(\tau_i(s))} \frac{x(\tau_i(s))}{x(s)} ds = 0.$$

Since $\tau_i(t) \leq h(t) \leq h_i(t)$ for $1 \leq i \leq n$, we have

$$\ln \frac{x(t)}{x(h(t))} + \int_{h(t)}^{t} \sum_{i=1}^{n} p_i(s) \frac{f_i(x(\tau_i(s)))}{x(\tau_i(s))} \frac{x(h(s))}{x(s)} ds \leq 0.$$

It follows that

$$\ln \frac{x(h(t))}{x(t)} \geq \sum_{i=1}^{n} \frac{f_i(x(\tau_i(\xi)))}{x(\tau_i(\xi))} \frac{x(h(\xi))}{x(\xi)} \int_{h(t)}^{t} p_i(s) ds, \tag{2.12}$$

where ξ is defined by $h(t) < \xi < t$. From (2.2), (2.7), (2.11) and then taking the $\liminf$ of both sides of (2.12), we find $\ln w > \frac{w}{e}$. But this case is not possible since $\ln x \leq \frac{x}{e}$ for all $x > 0$. Now, we consider the case where $M^* = 0$. So, it is obvious from (2.2) that

Hence, we have

$$\lim_{x \to 0} \frac{x}{f_i(x)} = 0. \tag{2.13}$$

According to (2.13) and $\frac{x}{f_i(x)} > 0$, there exists a $t_4 \geq t_3$, we get

$$\frac{x}{f_i(x)} < \epsilon, \quad t \geq t_4$$

and

$$\frac{f_i(x)}{x} > \frac{1}{\epsilon},\ t \geq t_4 \tag{2.14}$$

where $\epsilon > 0$ is an arbitrary real number. Thus, from (1.1) and (2.14), we have

$$x'(t) + \frac{1}{\epsilon}\sum_{i=1}^{n} p_i(t)x(h(t)) < 0,\ t \geq t_4. \tag{2.15}$$

Integrating (2.15) from $h(t)$ to t, we obtain

$$x(t) - x(h(t)) + \frac{1}{\epsilon}\int_{h(t)}^{t}\sum_{i=1}^{n} p_i(s)x(h(s))ds < 0,$$

and

$$-x(h(t)) + \frac{1}{\epsilon}x(h(t))\int_{h(t)}^{t}\sum_{i=1}^{n} p_i(s)ds < 0. \tag{2.16}$$

By using (2.7) and (2.16), we can write

$$\frac{c}{\epsilon} < 1$$

or

$$\epsilon > c,$$

which contradicts to $\lim_{x\to 0}\frac{x}{f_i(x)} = 0$. Thus the proof of the theorem is completed. □

Theorem 2.2. *Assume that (1.2), (1.3), (2.2) and (2.4) hold with* $0 < M_i < \infty$. *If* $\tau_i(t)$ *are not necessarily monotone and*

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\sum_{i=1}^{n} p_i(s)ds > M^*, \tag{2.17}$$

where $h(t)$ *and* M^* *are defined by Theorem 2.1, then all solutions of Eq. (1.1) oscillate.*

Proof. Suppose the contrary. Then there exists a nonoscillatory solution $x(t)$ of Eq. (1.1). In view of (2.4), we know from Theorem 2.1 that $\lim_{t\to\infty} x(t) = 0$ for $t \geq t_1$. Again using (2.3), we have a constant $\theta > 1$ such that

$$f_i(x(t)) \geq \frac{1}{\theta M_i}x(t) \geq \frac{1}{\theta M^\star}x(t) \quad \text{for } t \geq t_2. \tag{2.18}$$

From Eq. (1.1) and (2.18), we get

$$x'(t) + \frac{1}{\theta M^*}\sum_{i=1}^{n} p_i(t)x(\tau_i(t)) \leq 0.$$

Since $\tau_i(t) \leq h(t)$ for $1 \leq i \leq n$ and $x(t)$ is nonincreasing, we have

$$x'(t) + \frac{1}{\theta M^*}\sum_{i=1}^{n} p_i(t)x(h(t)) \leq 0. \tag{2.19}$$

Integrating (2.19) from $h(t)$ to t and using the fact that $h(t)$ is nondecreasing, we get

$$x(t) - x(h(t)) + \frac{1}{\theta M^*}\int_{h(t)}^{t}\sum_{i=1}^{n} p_i(s)x(h(s))ds \leq 0$$

or

$$-x(h(t)) + \frac{1}{\theta M^*}x(h(t))\int_{h(t)}^{t}\sum_{i=1}^{n} p_i(s)ds \leq 0.$$

This implies

$$-x(h(t))\left[1 - \frac{1}{\theta M^*}\int_{h(t)}^{t}\sum_{i=1}^{n} p_i(s)ds\right] \leq 0$$

and hence

$$\int_{h(t)}^{t}\sum_{i=1}^{n} p_i(s)ds \leq \theta M^*$$

for sufficiently large t. Therefore, we get

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\sum_{i=1}^{n} p_i(s)ds \leq \theta M^*. \tag{2.20}$$

On the other hand, from (2.17), we can write

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\sum_{i=1}^{n} p_i(s)ds = K > M^*.$$

So, we get $M^* < \frac{K+M^*}{2} < K$. Therefore, if we choose $\theta = \frac{K+M^*}{2M^*} > 1$, then from (2.20), we get

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\sum_{i=1}^{n} p_i(s)ds = K \leq \theta M^* = \frac{K+M^*}{2}.$$

This is a contradiction to $K > \frac{K+M^*}{2}$. So, the proof is completed. $\square$

Remark 2.3. We remark that if $\tau_i(t)$ are nondecreasing, then for every $i = 1, 2, \cdots, n$, we have $\tau_i(t) = h_i(t)$ for all $t \geq t_0$. Therefore, the conditions (2.3) and (2.17), respectively, reduce to

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{n} p_i(s)ds > \frac{M^*}{e} \tag{2.21}$$

and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{n} p_i(s)ds > M^*, \tag{2.22}$$

where $\tau(t) = \max_{1\leq i\leq n} \tau_i(t)$.

Now, we have the following example.

Example 2.4. *Consider the retarded equation*

$$x'(t) + \frac{1}{e}x(\tau_1(t))\ln(10 + |x(\tau_1(t))|) + \frac{2}{e}x(\tau_2(t))\ln(8 + |x(\tau_2(t))|) = 0, \; t > 0, \tag{2.23}$$

where

$$\tau_1(t) = \begin{cases} t-1, & \text{if } t \in [3k, 3k+1] \\ -3t + 12k + 3, & \text{if } t \in [3k+1, 3k+2] \\ 5t - 12k - 13, & \text{if } t \in [3k+2, 3k+3] \end{cases}$$

and $\tau_2(t) = \tau_1(t) + 1, \;\; k \in \mathbb{N}_0$. *By* (2.1), *we see that*

$$h_1(t) := \sup_{s\leq t} \tau_1(s) = \begin{cases} t-1, & \text{if } t \in [3k, 3k+1] \\ 3k, & \text{if } t \in [3k+1, 3k+2.6] \\ 5t - 12k - 13, & \text{if } t \in [3k+2.6, 3k+3] \end{cases}$$

and

$$h_2(t) = h_1(t) + 1.$$

Therefore, $h(t) = \min_{1\leq i\leq n} h_i(t) = h_1(t)$. *If we put* $p_1(t) = \frac{1}{e}$, $p_2(t) = \frac{2}{e}$ *and* $f_1(x) = x\ln(10 + |x(\tau_1(t))|)$, $f_2(x) = x\ln(8 + |x(\tau_2(t))|)$. *Then, we have*

$$M_1 = \limsup_{x\to 0} \frac{x}{f_1(x)} = \limsup_{x\to 0} \frac{x}{x\ln(10 + |x(\tau_1(t))|)} = \frac{1}{\ln 10}$$

and

$$M_2 = \limsup_{x\to 0} \frac{x}{f_2(x)} = \limsup_{x\to 0} \frac{x}{x\ln(8 + |x(\tau_2(t))|)} = \frac{1}{\ln 8}.$$

So, we have

$$\max\{M_1, M_2\} = M^* = \frac{1}{\ln 8}.$$

Thus, we get

$$\liminf_{t\to\infty}\int_{h(t)}^{t}\sum_{i=1}^{n}p_i(s)ds=\frac{3}{e}>\frac{M^*}{e}=\frac{1}{e\ln 8}.$$

That is, all conditions of Theorem 2.1 are satisfied and therefore all solutions of (2.23) oscillate.

3. Conclusion

In this paper, we investigate the oscillatory behavior for first order nonlinear differential equation with several non-monotone arguments and we obtain some new sufficient conditions for this equation, involving $liminf$ and $limsup$. Also, an illustrative example related to our results is given.

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