Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 8, No. 1, 2020, pp. 85-98
DOI:10.22034/cmde.2019.9463

Self-adjointness, conservation laws and invariant solutions of the Buckmaster equation

Saeede Rashidi*
Faculty of mathematical sciences, Shahrood university of technology,
Shahrood, Semnan, Iran.
E-mail: saeederashidi@yahoo.com

Seyed Reza Hejazi
Faculty of mathematical sciences, Shahrood university of technology,
Shahrood, Semnan, Iran.
E-mail: ra.hejazi@gmail.com

Abstract The present paper considers the group analysis of extended $(1+1)$-dimensional Buckmaster equation and its conservation laws. Symmetry operators of Buckmaster equation are found via Lie algorithm of differential equations. The method of non-linear self-adjointness is applied to the considered equation. The infinite set of conservation laws associated with the finite algebra of Lie point symmetries of the Buckmaster equation is computed. The corresponding conserved quantities are obtained from their respective densities. Furthermore, the similarity reductions corresponding to the symmetries of the equation are constructed.

Keywords. Buckmaster equation, Lie point symmetry, Direct method, Homotopy operator, Similarity Reductions.

2010 Mathematics Subject Classification. 58D19, 58J70, 76M60, 70S10, 70H33.

1. Introduction

Partial differential equations (PDEs) form the basis of many mathematical models of the physical, chemical and biological phenomenon, and more recently their applications have spread into economics, financial forecasting, image processing and other fields. Symmetries and conservation laws belong to the central studies of non-linear evolutional equations. Specially, one non-linear PDE is believed to be integral in the sense that it possesses an infinite number of symmetries or conservation laws. Besides, one can construct one or more conservation laws from one known symmetry, but almost all the conservation laws of PDEs may not have physical interpretations except for several well-known cases, such as the invariance of the spatial transformation ensures the conservation of momentum and the invariance of the temporal transformation guarantees the energy conservation.

In this paper, the Lie point symmetry method is used for solving a non-linear PDE. In fact, some linear and most non-linear differential equations are virtually impossible

Received: 21 February 2018 ; Accepted: 24 September 2018.
* Corresponding author.

to solve using exact solutions, so it is often possible to find numerical or approximate solutions for such type of problems. The main goal of this paper is to analyze the symmetry properties of the (1+1)-dimensional Buckmaster equation:

$$u_t = (u^4)_{xx} + (u^3)_x. \tag{1.1}$$

The Buckmaster equation is used in thin viscous fluid sheet flows and has been widely studied by the various methods [6]. The dynamics of thin viscous sheets reveals a number of interesting phenomena such as draw resonance or buckling. Buckling is an instability that occurs in thin bodies when longitudinal compression exceeds a well-defined threshold and makes the body bend out-of-plane. Purely viscous two-dimensional sheets have been considered by Buckmaster, Nachman, and Ting (1975) (referred to hereafter as BNT) and Wilmott (1989), and the onset of buckling in a two-dimensional sheet impinging on a plate is considered by Yarin and Tchavdarov (1994). The linear stability of a viscous sheet subjected to shear was treated by Benjamin and Mullin (1988). Models for axisymmetric viscous sheets have been derived by Pearson and Petrie (1970) and Yarin, Gospodinov, and Roussinov (1994). A fully non-linear model for the evolution of a three-dimensional sheet of arbitrary geometry has been derived by Howell (1994) and applied to the blowing of glass sheets by van de Fliert, Howell and Ockendon (1995).

The goal of this paper is to illustrate available methods of flux construction for Eq. (1.1). For this purpose, three different methods are applied including Noether's theorem, direct and Herman-Poole method. The paper is organized in the following manner. It is shown that the Buckmaster equation is non-linearly self-adjoint and then the Lie point symmetries are computed. Using this property and applying the theorem on non-local conservation laws the conservation laws corresponding to the symmetries of the equation in question are calculated. Finally, similarity reductions and explicit solutions are derived.

2. Non-linear Self-Adjointness

The construction of conservation laws demonstrates a practical significance of the non-linear self-adjointness [5, 9, 10]. The general concept of non-linear self-adjointness is suggested here. Let us consider a system of $m-$differential equations (linear or non-linear)

$$F_\alpha \left(x, u, u_{(1)}, \cdots, u_{(s)}\right) = 0, \quad \alpha = 1, \cdots, m, \tag{2.1}$$

with m-dependent variables $u = (u^1, \cdots, u^m)$. Eqs. (2.1) involve the partial derivatives $u_{(1)}, \cdots, u_{(s)}$ up to order s.

Definition 2.1. The adjoint equations to Eqs. (2.1) are given by

$$F^*_\alpha \left(x, u, v, u_{(1)}, v_{(1)}, \cdots, u_{(s)}, v_{(s)}\right) = \frac{\delta \mathcal{L}}{\delta u^\alpha}, \quad \alpha = 1, \cdots, m, \tag{2.2}$$

where $\mathcal{L}$ is the formal Lagrangian for Eqs. (2.1) defined by

$$\mathcal{L} = v^\beta F_\beta \equiv \sum_{\beta=1}^{m} v^\beta F_\beta. \tag{2.3}$$

Here $v = (v^1, \cdots, v^m)$ are new dependent variables and $v_{(1)}, \cdots, v_{(s)}$ are their derivatives, e.g. $v_{(1)} = v_i^\alpha$, $\quad v_i^\alpha = D_i(v^\alpha)$. We use $\delta/\delta u^\alpha$ for the Euler-Lagrange operator

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}, \quad \alpha = 1, \cdots, m. \tag{2.4}$$

The total differentiation is extended to the new dependent variables:

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + v_i^\alpha \frac{\partial}{\partial v^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + v_{ij}^\alpha \frac{\partial}{\partial v_j^\alpha} + \cdots .$$

For a linear equation $\mathcal{L}[u] = 0$, the adjoint operator defined by (2.2) is identical with the classical adjoint operator $\mathcal{L}^*[v]$ determined by the equation $v\mathcal{L}[u] - u\mathcal{L}^*[v] = D_i(p^i)$. Now let us investigate the Eq. (1.1), for non-linear self-adjointness. Consider the Buckmaster equation in the expanded form

$$F \equiv u_t - 4u^3 u_{xx} - 12u^2 u_x^2 - 3u^2 u_x. \tag{2.5}$$

Using its formal Lagrangian

$$\mathcal{L} = v \left(u_t - 4u^3 u_{xx} - 12u^2 u_x^2 - 3u^2 u_x \right). \tag{2.6}$$

The following adjoint equation is obtained to Eq. (1.1)

$$F^* \equiv -v_t - 4u^3 v_{xx} + 3u^2 v_x. \tag{2.7}$$

Eq. (1.1) is said to be non-linearly self-adjoint if the Eqs. (2.5) and (1.1) can be related by the equation $F^* = \lambda F$ after the substitution $v = \varphi(t, x, u)$ with a certain function $\varphi \neq 0$. Here λ is an undetermined variable coefficient; it will be found in the process of calculations. Thus, the non-linear self-adjointness condition is written by

$$- v_t - 4u^3 v_{xx} + 3u^2 v_x = \lambda \left(u_t - 4u^3 u_{xx} - 12u^2 u_x^2 - 3u^2 u_x \right), \tag{2.8}$$

where one makes the following replacements of v and its derivatives:

$$v = \varphi(t, x, u), \quad v_t = D_t(\varphi), \quad v_x = D_x(\varphi), \quad v_{xx} = D_x^2(\varphi).$$

After this replacement Eq. (2.8) should be satisfied identically in the variables $t, x, u, u_t, u_x, u_{xx}$. Let us express the derivatives of v involved in the adjoint equation (2.7) in the expanded form,

$$v_t = \varphi_u u_t + \varphi_t, \quad v_x = \varphi_u u_x + \varphi_x,$$
$$v_{xx} = \varphi_{uu} u_x^2 + \varphi_u u_{xx} + 2\varphi_{xu} u_x + \varphi_{xx},$$

and substitute them in the left side of Eq. (2.8). The comparison of the coefficients for u_t in both side of Eq. (2.8) yields $\lambda = -\varphi_u$. Then the comparison of the coefficients for u_{xx} leads to the equation $-4u^3 u_{xx} \varphi_u = -4u^3 u_{xx} \lambda$. It follows from these equations that $\varphi_u = 0$, hence, $\varphi = \varphi(t, x)$, and $\lambda = 0, v_t = \varphi_t, v_{xx} = \varphi_{xx}$. Now Eq. (2.8) becomes to $-\varphi_t - 4u^3 \varphi_{xx} + 3u^2 \varphi_x = 0$. It yields that $\varphi_t = 0, \varphi_x = 0, \varphi_{xx} = 0$. The general solution of the above system is easily found and provides the following substitution $v = C$. We demonstrated that Eq. (1.1) is non-linearly self-adjoint via Definition 2.1 and that the substitution v has the form $v = C$.

3. Lie Symmetry Analysis of The Buckmaster Equation

Symmetry plays a very important role in various fields of nature. Lie method is an effective method and a large number of equations are solved with the aid of this method. There are still many authors who use this method to find the exact solutions of non-linear differential equations. Since this method has powerful tools to find exact solutions of non-linear problems. For example, when we are confronted with a complicated system of PDEs or ODEs, it is interesting to find a vast set of exact solutions for the given system via a systematic method with no limitation, this would be done by using Lie's symmetry as an analytic applicable method. The general procedure to obtain Lie symmetries of differential equations, and their applications to find analytic solutions of the equations are described in detail in several monographs on the subject (e.g. [8, 13]) and in numerous papers in the literature (e.g. [11, 12, 15, 16, 17]).

In this section, the Lie point symmetries of the Eq. (1.1) are obtained by using the standard Lie algorithm. The one-parameter continuous groups of equivalence transformations have the form:

$$\bar{t} = \bar{t}(t, x, u), \qquad \bar{x} = \bar{x}(t, x, u), \qquad \bar{u} = \bar{u}(t, x, u),$$

and map the Eq. (1.1) into the following equation:

$$\bar{u}_{\bar{t}} = (\bar{u}^4)_{\bar{x}\bar{x}} + (\bar{u}^3)_{\bar{x}}. \tag{3.1}$$

The vector field associated with the above group transformations can be written as

$$X = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}.$$

The second prolonged generator of X with respect to the derivatives involved in Eq. (2.5) is

$$X^{(2)} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \eta_t\frac{\partial}{\partial u_t} + \eta_x\frac{\partial}{\partial u_x} + \eta_{xx}\frac{\partial}{\partial u_{xx}},$$

where

$$\eta_i = D_i(\eta) - u_x(D_i\xi) - u_t(D_i\tau), \quad \eta_{ij} = D_j(\eta_i) - u_{xj}(D_j\xi) - u_{tj}(D_j\tau),$$

are the coefficients of prolongation. Eq. (2.5) admits X as a symmetry if the following condition is satisfied

$$X^{(2)}F\Big|_{F=0} = 0. \tag{3.2}$$

The invariance condition (3.2) gives the following determining equation:

$$\eta_t - 6\eta u_x u - 24\eta u u_x^2 - 24\eta_x u_x u^2 - 12\eta u_{xx}u^2 - 3\eta_x u^2 - 4\eta_{xx}u^3 = 0. \tag{3.3}$$

The solution of the expanded form of this linear system of PDEs gives the symmetry Lie algebra of the Buckmaster equation which is spanned by the following generators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}. \tag{3.4}$$

The one-parameter groups G_i generated by the X_i are given in the following table. The entries give the transformed point $\exp(\epsilon X_i)(t,x,u) = (\tilde{t},\tilde{x},\tilde{u})$:

$$G_1 : (t+\epsilon, x, u), \quad G_2 : (t, x+\epsilon, u), \quad G_3 : (te^{\epsilon}, xe^{-\epsilon}, ue^{-\epsilon}).$$

Since each group G_i is a symmetry group, exponentiation shows that if $u = f(t,x)$ is a solution of the Buckmaster equation, so are the functions

$$u^{(1)} = f(t-\epsilon, x), \quad u^{(2)} = f(t, x-\epsilon), \quad u^{(3)} = e^{\epsilon} f(e^{-\epsilon}t, e^{\epsilon}x),$$

where ϵ is any real number. The groups G_1 and G_2 demonstrate the time and space-invariance of the equation, reflecting the fact that the Buckmaster equation has constant coefficients. The well-known scaling symmetry turns up in G_3.

4. Conservation Laws Provided by Lie Point Symmetries

Noether showed that if one PDE has a point symmetry of the action functional (action integral), then one obtains the fluxes of a local conservation law through an explicit formula that involves the infinitesimals of the point symmetry and the Lagrangian (Lagrangian density) of the action functional [3, 4].

Consider the Lagrangian action principle of the form:

$$J[u] = \int_R \mathcal{L}(x^i, u^{\alpha}, u_i^{\alpha}, u_{ij}^{\alpha}, \cdots) dx. \tag{4.1}$$

At a critical point, the action is stationary, i.e.;

$$\delta J = J[u+\epsilon v] - J[u] = \int_R \delta\mathcal{L} dx = 0,$$

where

$$\delta\mathcal{L} \equiv \mathcal{L}[u+\epsilon v] - \mathcal{L}[u] = \epsilon \left(v^{\gamma} E_{\gamma}(\mathcal{L}) + D_i C^i[u,v] \right) + O(\epsilon^2). \tag{4.2}$$

Thus, the critical point requirement $\delta J = 0$ is satisfied if the u^{α}s satisfy the Euler-Lagrange equations:

$$E_{\alpha}[\mathcal{L}] = \frac{\partial\mathcal{L}}{\partial u^{\alpha}} - D_i\left(\frac{\partial\mathcal{L}}{\partial u_i^{\alpha}}\right) + D_iD_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}^{\alpha}}\right) - D_iD_jD_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^{\alpha}}\right)$$
$$+ \cdots + (-1)^k D_{j_1}\cdots D_{j_k}\left(\frac{\partial\mathcal{L}}{\partial u_{j_1\cdots j_k}^{\alpha}}\right) = 0, \quad \alpha = 1, \cdots m, \tag{4.3}$$

where D_iC^i's vanish on the boundary ∂R of the domain R. In (4.2) the boundary vector $C^i[u,v]$ is given by

$$\begin{aligned} C^i[u,v] &= v^\gamma\left[\frac{\partial \mathcal{L}}{\partial u_i^\gamma} - D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\gamma}\right) + D_iD_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\gamma}\right) - \cdots\right] \\ &\quad + v_j^\gamma\left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\gamma} - D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\gamma}\right) + D_lD_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijkl}^\gamma}\right) - \cdots\right] \\ &\quad + v_{jk}^\gamma\left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\gamma} - D_s\left(\frac{\partial \mathcal{L}}{\partial u_{ijks}^\gamma}\right) + \cdots\right] + \cdots . \end{aligned}$$

Theorem 4.1. ***(Noether's theorem):*** *If the action (4.1) is invariant under a generalized Lie transformation $\tilde{x} = x + \epsilon\xi^i + O(\epsilon^2), \tilde{u^\alpha} = u^\alpha + \epsilon\phi^\alpha + O(\epsilon^2)$, then for any solution u of the Euler-Lagrange equation $E_\gamma[\mathcal{L}] = 0$, there is a corresponding conservation law:*

$$D_i(C^i[u, W^\alpha] + \xi\mathcal{L}) = 0. \tag{4.4}$$

Let us introduce the notation $x^1 = t, x^2 = x$ and $u^1 = u$. Thus, a conservation law corresponding to the operator (2.4) has the form

$$\left[D_t(C^1) + D_x(C^2)\right]\Big|_{(1.1)} = 0, \tag{4.5}$$

where $\Big|_{(1.1)}$ means that the equation holds on the solutions of the Eq. (1.1). Since the maximum order of derivatives involved in formal Lagrangian $\mathcal{L}$ given by Eq. (2.6) is equal to two, this formula is reduced to

$$C^i = \xi^i\mathcal{L} + W^\alpha\left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\right)\right] + D_j(W^\alpha)\left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\right],$$

where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$, is the characteristic of the symmetries. The above formula must be applied to the symmetries (3.4) for computations. Invoking that the Eq. (1.1) is non-linearly self-adjoint with the substitution $v = \varphi(t,x,u) = A$, we will replace in C^i the variable v with u. Thus, two conserved vectors arrived for the Eq. (1.1). Since the formal Lagrangian (2.6) vanishes on the solutions of the Eq. (1.1), we can omit the term $\xi^i\mathcal{L}$ and the take formula for the conserved vector in the following form:

$$C^i = W^\alpha\left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\right)\right] + D_j(W^\alpha)\left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\right]. \tag{4.6}$$

Using in (4.6) the expression (2.6) for $\mathcal{L}$ and eliminating v with constants A, we obtain

$$\begin{aligned} C^1 &= WA, \qquad (4.7)\\ C^2 &= W[-24u^2u_xA - 3u^2A - D_x(-4u^3A)] + D_x(W)(-4u^3A). \end{aligned}$$

4.1. **Conservation Laws via** X_1**.** The operator $X = \frac{\partial}{\partial t}$ with characteristic $W = -u_t$ gives the conserved vector $C = (C^1, C^2)$ such as

$$C^1 = -Au_t, \qquad C^2 = A\left(12u^2u_xu_t + 3u^2u_t + 4u^3u_{tx}\right). \tag{4.8}$$

Now we continue with the following less trivial and useful operation with conserved vectors. Let

$$C^1|_{(1.1)} = \tilde{C^1} + D_2(H^2) + \cdots + D_n(H^n). \tag{4.9}$$

Thus, the conserved vector $C = (C^1, C^2, \cdots, C^m)$ can be replaced with the equivalent conserved vector

$$\tilde{C} = (\tilde{C^1}, \tilde{C^2}, \cdots, \tilde{C^m}) = 0, \tag{4.10}$$

with the components

$$\tilde{C^1}, \quad \tilde{C^2} = C^2 + D_1(H^2), \cdots, \quad \tilde{C^m} = C^m + D_1(H^m). \tag{4.11}$$

The passage from $C = (C^1, C^2, \cdots, C^m)$ to the vector (4.10) is based on the commutativity of the total differentiation. Namely, we have

$$D_1D_2(H^2) = D_2D_1(H^2), \quad D_1D_n(H^n) = D_nD_1(H^n),$$

and therefore the conservation equation (4.5) for the vector
$C = (C^1, C^2, \cdots, C^m)$ is equivalent to the conservation equation

$$[D_i(\tilde{C^i})]_{(1.1)} = 0.$$

Now the differential conservation equations (4.8) will be written to an equivalent form by using the operations (4.9)-(4.11) of the conserved vectors. Namely, let us apply these operations to the conserved vector (4.8). Noting that $C^1 = -Au_t - u_{tx} + D_x(u_t)$, and using the operations (4.9)-(4.11) we transform the vector (4.8) to the form

$$\tilde{C^1} = -Au_t - u_{tx}, \quad \tilde{C^2} = A\left(12u^2u_xu_t + 3u^2u_t + 4u^3u_{tx}\right) + u_{tt}, \tag{4.12}$$

if the differential conservation equation rewritten with the vector (4.12), the following equation is obtained:

$$\begin{aligned} D_t(\tilde{C^1}) + D_x(\tilde{C^2}) = {} & Au_{tt} - u_{ttx} + 24Auu_tu_x^2 + 12Au^2u_tu_{xx} + 24Au^2u_xu_{tx} \\ & + 6Auu_xu_t + 3Au^2u_{tx} + 12Au^2u_xu_{tx} + 4Au^3u_{txx} + u_{ttx}\Big|_{(2.5)} = 0. \end{aligned}$$

4.2. **Conservation Laws via** X_2**.** Similarly the symmetry $X = \frac{\partial}{\partial x}$ with characteristic $W = -u_x$ of the Eq. (1.1) provides the conserved vector $C = (C^1, C^2)$ with components

$$C^1 = -Au_x, \qquad C^2 = A\left(12u^2u_x^2 + 3u^2u_x + 4u^3u_{xx}\right). \tag{4.13}$$

Note that if $C^1 = -Au_x - u_{xx} + D_x(u_x)$, by transferring the terms of the form $D_x(\cdots)$ from C^1 to C^2 we obtain

$$\tilde{C^1} = -Au_x - u_{xx}, \quad \tilde{C^2} = A\left(12u^2u_x^2 + 3u^2u_x + 4u^3u_{xx}\right) + u_{xx}. \tag{4.14}$$

4.3. **Conservation Laws via X_3.** Finally the symmetry $X = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}$ of Eq. (1.1), gives the characteristic function $W = -u - tu_t + xu_x$. So, one can obtain the conserved vector $C = (C^1, C^2)$ with components

$$\begin{aligned} C^1 &= A(-u - tu_t + xu_x), \\ C^2 &= A(12u^3u_x + 3u^3 - 12xu^2u_x^2 - 3xu^2u_x \\ &\quad +12tu^2u_xu_t + 3tu^2u_t - 4xu^3u_{xx} + 4tu^3u_{xt}). \end{aligned} \tag{4.15}$$

It is clear that they are involve an arbitrary solution A of the adjoint system (2.7), and they present an infinite number of the conservation laws.

5. Direct Method for Construction of Conservation Laws

A more general systematic method of constructing local conservation is laws called the direct method [1]. Within this method, one seeks a set of local multipliers (also called integrating factors or characteristics) depending on independent and dependent variables of a given system of PDEs and derivatives of dependent variables up to some fixed order.

Consider a system (2.1) of $N-$PDEs of order k with $n-$independent variables $x = (x^1, \cdots, x^n)$ and $m-$dependent variables $u(x) = (u^1(x), \cdots, u^m(x))$, given by

$$F^\sigma[u] = F^\sigma(x, u, \partial u, \cdots, u_{(k)}) = 0, \quad \sigma = 1, \cdots, N. \tag{5.1}$$

In general for a given system of PDEs, non-trivial local conservation laws arise from linear combinations of the equations of the PDEs with multipliers that yield non-trivial divergence expressions. By their construction, such divergence expressions

$$D_i\Phi^i[u] = D_1\Phi^1[u] + \cdots + D_n\Phi^n[u] = 0, \tag{5.2}$$

vanish on all solutions of the system of PDEs. In (5.2), D_i is the total derivatives with respect to x_i and $\Phi^i[u] = \Phi^i(x, u, \partial u, \cdots, u_{(r)}), i = 1, \cdots, n$, are called the fluxes of conservation laws. In particular, a set of multipliers $\{\Lambda_\sigma[U]\}_{\sigma=1}^N = \{\Lambda_\sigma(x, U, \partial U, \cdots, U_{(\ell)})\}_{\sigma=1}^N$ yields a divergence expressions for system of PDEs $F[u]$ if the identity $\Lambda_\sigma[U]F^\sigma[U] \equiv D_i\Phi^i[U]$, holds for arbitrary functions $U(x)$. Then, on the solutions $U(x) = u(x)$ of the system (5.1), if $\Lambda_\sigma[u]$ is non-singular, one has the local conservation laws

$$\Lambda_\sigma[U]F^\sigma[U] \equiv D_i\Phi^i[U] = 0.$$

Theorem 5.1. *A set of non-singular local multipliers $\{\Lambda_\sigma(x, U, \partial U, \cdots, U_{(\ell)})\}_{\sigma=1}^N$ yields a local conservation law for the system (5.1) if and only if the set of identities*

$$E_{U^j}(\Lambda_\sigma(x, U, \partial U, \cdots, U_{(\ell)})F^\sigma(x, U, \partial U, \cdots, U_{(k)})) \equiv 0, \quad j = 1, \cdots m, \tag{5.3}$$

holds for arbitrary functions $U(x)$.

The set of Eqs. (5.3) yields the set of linear determining equations in order to find all sets of local conservation law multipliers of the system (5.1) by considering multipliers of all orders $\ell = 1, 2, \cdots$. Since Eqs. (5.3) hold for arbitrary $U(x)$, it follows that one can treat each U^μ and each of its derivatives U_i^μ, U_{ij}^μ, etc. as independent variables along with x^i, and consequently the linear system of PDEs (5.3) splits into an over-determined linear system of determining equations whose solutions are the sets of local multipliers $\{\Lambda_\sigma(x, U, \partial U, \cdots, U_{(\ell)})\}_{\sigma=1}^{N}$ of the system (5.1).

This method is applied to obtain the local conservation laws of Eq. (1.1). Suppose

$$F[u] = u_t - 4u^3 u_{xx} - 12u^2 u_x^2 - 3u^2 u_x. \tag{5.4}$$

We seek all local conservation law multipliers of the form $\Lambda = \xi(t, x, U)$ of the PDE system (5.4). In terms of the Euler operators

$$E_U = \frac{\partial}{\partial U} - D_t \frac{\partial}{\partial U_t} - D_x \frac{\partial}{\partial U_x} + D_{xx} \frac{\partial}{\partial U_{xx}},$$

the determining Eqs. (5.3) for the multipliers (5.5) become

$$E_U\left[\xi(t, x, U)(u_t - 4u^3 u_{xx} - 12u^2 u_x^2 - 3u^2 u_x)\right] \equiv 0, \tag{5.5}$$

where $U(t, x)$ is an arbitrary function. The solution $\xi(t, x, U)$ is the local multipliers of all non-trivial local conservation laws of zero-th order of the Buckmaster Eq. (5.4). Solving the above determining equation yields $\Lambda = C$ where C is constant. The zero-th order conservation laws are true and we avoid to express them. Next consider first order multipliers, i.e., $\Lambda = \Lambda(t, x, U, U_t, U_x)$ and second order $\Lambda = \Lambda(t, x, U, U_t, U_x, U_{tt}, U_{tx}, U_{tt})$. We can find the first and second order conservation laws from these multipliers by the amended method. See Tables 1 and 2 for the results.

Table 1. Fisrt order local conservation laws

Density	Fluxes	First order conservation laws
$-xu_x - u$	xu_t	$D_t(-xu_x - u) + D_x(xu_t) = 0$
$-tu_x$	$tu_t + u$	$D_t(-tu_x) + D_x(tu_t + u) = 0$
$-2xt$	x^2	$D_t(-2xt) + D_x(x^2) = 0$
x^2	t^2	$D_t(x^2) + D_x(t^2) = 0$
t^2	$-2xt$	$D_t(t^2) + D_x(-2xt) = 0$
$x - t$	$-t + x$	$D_t(x - t) + D_x(-t + x) = 0$
$-u_x$	u_t	$D_t(-u_x) + D_x(u_t) = 0$
$-uu_x$	uu_t	$D_t(-uu_x) + D_x(uu_t) = 0$

6. Hereman-Poole method

One method to investigate the complete integrability of a system of PDEs is to determine whether the system has infinitely many conservation laws. While developing a method for computing conservation laws of non-linear PDEs in (1+1)-dimensions, the authors needed a technique to symbolically integrate expressions involving unspecified functions. Mathematica's integrate function often failed to integrate such integrands, in particular, when transcendental functions were present. The computation of conservation laws of non-linear PDEs in multiple space variables requires a tool to invert total divergences. The homotopy operator used in the proof of the exactness of the (bi-)variational complex can do the required integrations. In the case of complicated forms of multipliers and equations, for the inversion of divergence operators, one can use homotopy operators that arise in differential geometry and

TABLE 2. Second order local conservation laws

Density	Fluxes	Second order conservation laws
$-u_{tx}$	u_{tt}	$D_t(-u_{tx}) + D_x(u_{tt}) = 0$
$-u_x u_{xx}$	$2u_x u_{tx}$	$D_t(-u_x u_{xx}) + D_x(2u_x u_{tx}) = 0$
$-xu_{xx} - u_x$	xu_{tx}	$D_t(-xu_{xx} - u_x) + D_x(xu_{tx}) = 0$
tu_{xx}	$tu_{tx} + u_x$	$D_t(u_{xx}) + D_x(u_{tx} + u_x) = 0$
$-u_x u_{tx} - u_t u_{xx}$	$-u_x u_{tt} + u_t u_{tx}$	$D_t(-u_x u_{tx} - u_t u_{xx}) + D_x(u_x u_{tt} + u_t u_{tx}) = 0$
$-xu_{tx} - u_t$	xu_{tt}	$D_t(-xu_{tx} - u_t) + D_x(xu_{tt}) = 0$
$-tu_{tx} + u_x$	tu_{tt}	$D_t(-tu_{tx} + u_x) + D_x(tu_{tt}) = 0$
$-uu_{xx} - u_x^2$	$uu_{tx} + u_t u_x$	$D_t(-uu_{xx} - u_x^2) + D_x(uu_{tx} + u_t u_x) = 0$
$-uu_{tx} - u_x u_t$	$uu_{tt} + u_t^2$	$D_t(-uu_{tx} - u_x u_t) + D_x(uu_{tt} + u_t^2) = 0$
$-xu_x - u$	xu_t	$D_t(-xu_x - u) + D_x(xu_t) = 0$
$-tu_x$	$tu_t + u$	$D_t(-tu_x) + D_x(tu_t + u) = 0$
$u_t u_{tx}$	$-u_t u_{tt}$	$D_t(u_t u_{tx}) + D_x(-u_t u_{tt}) = 0$
$-u_{xx}$	u_{tx}	$D_t(-u_{xx}) + D_x(u_{tx}) = 0$

reduce the problem of finding fluxes to a problem of integration in single-variable calculus. We begin this part by a brief definition of a homotopy operator [7, 14].

Definition 6.1. Let x be the independent variable and $f = f(x, u^{(M)}(x))$ be an exact differential function, i.e. there exist a function F such that $F = D_x^{-1} f$. Thus, F is the integral of f. The First homotopy operator is defined as

$$\mathcal{H}_{\mathbf{u}(x)}(f) = \int_{\lambda_0}^{1} \left(\sum_{j=1}^{N} \mathcal{I}_{u^j(x)}(f) \right) [\lambda u] \frac{d\lambda}{\lambda},$$

where $u = (u^1, \cdots, u^j, \cdots, u^N)$. The integrand $\mathcal{I}_{u^j(x)}(f)$, is defined by

$$\mathcal{I}_{u^j(x)}(f) = \sum_{k=1}^{M_1^j} \left(\sum_{i=0}^{k-1} u^j_{ix} (-D_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u^j_{kx}},$$

where M_1^j is the order of f in dependent variable u^j with respect to x. The homotopy with $\lambda_0 = 0$ is used, except when singularities at $\lambda = 0$ occur.

Definition 6.2. Let $f = f(t, x, u^{(M)}(t, x))$ be an exact differential function involving two independent variable (t, x). The second homotopy operator is a vector operator with two components $\left(\mathcal{H}^x_{u(t,x)}(f), \mathcal{H}^t_{u(t,x)}(f) \right)$, where

$$H^t_{u(t,x)}(f) = \int_{\lambda_0}^{1} \left(\sum_{j=1}^{N} \mathcal{I}^t_{u^j(t,x)}(f) \right) [\lambda u] \frac{d\lambda}{\lambda}, \tag{6.1}$$

$$H^x_{u(t,x)}(f) = \int_{\lambda_0}^{1} \left(\sum_{j=1}^{N} \mathcal{I}^x_{u^j(t,x)}(f) \right) [\lambda u] \frac{d\lambda}{\lambda}. \tag{6.2}$$

The $t-$integrand, $\mathcal{I}^t_{u^j(t,x)}(f)$, is defined as

$$\mathcal{I}^t_{u^j(t,x)}(f) =$$

$$\sum_{k_1=1}^{M_1^j}\sum_{k_2=0}^{M_2^j}\left(\sum_{i_1=0}^{k_1-1}\sum_{i_2=0}^{k_2} B^t u^j_{t^{i_1}x^{i_2}}(-D_t)^{k_1-i_1-1}(-D_x)^{k_2-i_2}\right)\frac{\partial f}{\partial u^j_{t^{k_1}x^{k_2}}}, \tag{6.3}$$

with combinatorial coefficient

$$B^t = B(i_1,i_2,k_1,k_2) = \frac{\binom{i_1+i_2}{i_1}\binom{k_1+k_2-i_1-i_2-1}{k_1-i_1-1}}{\binom{k_1+k_2}{k_1}}.$$

Similarly, the $x-$integrand $\mathcal{I}^x_{u^j(t,x)}(f)$, is given by

$$\mathcal{I}^x_{u^j(t,x)}(f) = \sum_{k_1=0}^{M_1^j}\sum_{k_2=1}^{M_2^j}\left(\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{k_2-1} B^x u^j_{t^{i_1}x^{i_2}}(-D_t)^{k_1-i_1}(-D_x)^{k_2-i_2-1}\right)\frac{\partial f}{\partial u^j_{t^{k_1}x^{k_2}}}, \tag{6.4}$$

where

$$B^x = B(i_1,i_2,k_1,k_2) = \frac{\binom{i_1+i_2}{i_2}\binom{k_1+k_2-i_1-i_2-1}{k_2-i_2-1}}{\binom{k_1+k_2}{k_2}}.$$

The usual homotopy operator with $\lambda_0 = 0$ applies when (6.1) and (6.2) converge. However, due to a possible singularity at $\lambda = 0$; (6.1) and (6.2) might diverge for $\lambda_0 \to 0$: This can occur with rational as well as irrational integrands. In such cases, one can take $\lambda_0 \to \infty$ or, alternatively, evaluate the indefinite integral and let $\lambda_0 \to 1$.

Using homotopy operator Div^{-1} is guaranteed by the following theorem.

Theorem 6.3. *Let $f = f(t,x,u^{(M)}(t,x))$ be exact, i.e. $f = DivF$ for some $F = F(t,x,u^{(M-1)}(t,x))$. Then, $F = Div^{-1}(f) = \left(\mathcal{H}^x_{u(t,x)}(f), \mathcal{H}^t_{u(t,x)}(f)\right)$.*

Each of the multipliers $\Lambda_\sigma = \xi_\sigma$ found in the direct method produces a conservation law as $D_t\Psi^i + D_x\Phi^i = 0$ with the characteristic

$$D_t\Psi^i + D_x\Phi^i = \xi_\sigma(F).$$

So we can find flux and density of the Eq. (1.1) by using the multiplier $\Lambda = \xi = C$ found in the direct method. Once we have one component of the conservation law, e.g., the density or a component of the flux, the remaining components could be computed using the homotopy operator. For calculating flux (Φ) and density (Ψ) for each coefficient Λ_i use second homotopy operator $\left(\mathcal{H}^t_{u(t,x)}(f), \mathcal{H}^x_{u(t,x)}(f)\right)$. For $\xi = 1$, by using (6.3), we compute $\mathcal{I}^t_{u^j(t,x)}(f) = u$. Likewise, by considering (6.4), we compute $\mathcal{I}^x_{u^j(t,x)}(f) = -16u^3u_x - 3u^3$. Then by using (6.1) and (6.2), we have

$$\Psi = \mathcal{H}^t_{\mathrm{u}(t,x)}(f) = \int_0^1 \mathcal{I}^t_{u^j(t,x)}(f)[\lambda u]\frac{d\lambda}{\lambda} = \int_0^1 \lambda u\frac{d\lambda}{\lambda} = u, \tag{6.5}$$

$$\begin{aligned}\Phi = \mathcal{H}^x_{\mathrm{u}(t,x)}(f) &= \int_0^1 \mathcal{I}^x_{u^j(t,x)}(f)[\lambda u]\frac{d\lambda}{\lambda} = \int_0^1 (-16\lambda^3u^3\lambda u_x - 3\lambda^3u^3)\frac{d\lambda}{\lambda}\\ &= -4u^3u_x - u^3. \end{aligned}\tag{6.6}$$

Now we see that the conservation laws corresponding to multipliers $\xi = 1$ for system Eq. (1.1) is given by: $D_t(\Psi) + D_x(\Phi) = u_t - 12u^2u_x^2 - 3u^2u_x - 4u^3u_{xx}$.

7. Similarity reductions

The first advantage of symmetry group method is to construct new solutions from known solutions. To do this, the infinitesimals are considered and their corresponding invariants are determined. The Buckmaster equation is expressed in the coordinates (t, x, u), so the problem of reduction this equation is to search for its form in specific coordinates. Those coordinates will be constructed by searching for independent invariants (r, V) corresponding to an infinitesimal generator. So using the chain rule, the expression of the system in the new coordinate allows us to the reduced system. Since our original PDE has two independent variables, then this equation transforms into an ODE. Here we will compute some invariant solutions with respect to symmetries. At first, the similarity variables for each term of the Lie algebra of symmetries are obtained by integrating the characteristic equations. Then this method is used to reduce the system for finding the invariant solutions [2, 13].

7.1. **Time translation invariance X_1.** The classical similarity solution of Eq. (1.1) for this symmetry is obtained by integrating the group trajectories $\frac{dt}{d\epsilon} = 1$, where ϵ is a parameter along the trajectories. Integration of $\frac{dt}{d\epsilon} = 1$ yields the invariant transformation $x = r, u(x,t) = V(r)$, thus, the reduced equation with respect to above invariants is

$$-V(r)^2\Big(4V(r)(V''(r)) + 12(V'(r))^2 + 3(V'(r))\Big) = 0, \tag{7.1}$$

and the similarity solutions are $V(r) = 0$ and

$$\begin{aligned}&r + 4V(r) - 4/3C_1\sqrt[3]{4}\ln\Big(V(r) + \sqrt[3]{4}C_1\Big)\\ &+2/3\ln\Big((V(r))^2 - V(r)\sqrt[3]{4}C_1 + 2\sqrt[3]{2}C_1^2\Big)\sqrt[3]{4}C_1\\ &-4/3C_1\sqrt[3]{4}\sqrt{3}\arctan\left[\frac{1}{\sqrt{3}}\left(1/2\frac{\sqrt[3]{4^2}V(r)}{C_1} - 1\right)\right] + C_2 = 0.\end{aligned}$$

With subsititution $u(t,x) = V(r)$ and $x = r$ we have $u(t,x) = 0$ as a trivial solution and

$$\begin{aligned}
&x + 4u - 4/3C_1\sqrt[3]{4}\ln\left(u + \sqrt[3]{4}C_1\right)\\
&+2/3\ln\left(u^2 - u\sqrt[3]{4}C_1 + 2\sqrt[3]{2}C_1^2\right)\sqrt[3]{4}C_1\\
&-4/3C_1\sqrt[3]{4}\sqrt{3}\arctan\left[\frac{1}{\sqrt{3}}\left(1/2\frac{\sqrt[3]{4^2}u}{C_1} - 1\right)\right] + C_2 = 0.
\end{aligned}$$

7.2. **Solution translation invariance X_2.** For this symmetry every translated solution with any constant is a similarity solution. Because $\frac{dx}{d\epsilon} = 1$ by integration of that we have $t = r, u(t,x) = V(r)$. Then, the reduced equation is $V'(r) = 0$ so we have $u_t = 0$, thus $u(t,x) = C$.

7.3. **Solution translation invariance X_3.** The classical similarity solution of Eq. (1.1) for the last symmetry is obtained by integrating the group trajectories

$$\frac{dt}{d\epsilon} = t, \quad \frac{dx}{d\epsilon} = -x, \quad \frac{du}{d\epsilon} = -u. \tag{7.2}$$

Integration of (7.2) yields the invariant transformation

$$t = re^q, \quad x = 1/e^q, \quad u(x,t) = V(r)/e^q, \tag{7.3}$$

thus the reduced equation with respect to invariants (7.3) is

$$\begin{aligned}
&4V''(r)V(r)^3r^2 + 12V(r)^2V'(r)^2r^2 + 32V(r)^3V'(r)r\\
&\quad +3V(r)^2V'(r)r + 12V(r)^4 + 3V(r)^3 - V'(r) = 0.
\end{aligned} \tag{7.4}$$

Consequently, the Eq. (1.1) reduced to the ODE (7.4), but the solution is so tedious and could be found by some numerical methods.

8. Conclusion

In this paper, a Lie group analysis for an important PDE called Buckmaster equations is given. The Lie algebra of symmetries was found by a useful algorithm. Also, these operators are applied for finding conservation laws of the system due to Noether's method. Because of the limitation of this method, the direct method is used to obtain fluxes and densities for the system. In the case of complicated forms of multipliers and/or equations, for the inversion of divergence operators, one can use homotopy operators that arise in differential geometry and reduce the problem of finding fluxes and densities to a problem of integration in single-variable calculus. After computing multipliers, fluxes and densities of the corresponding divergence expression can be reconstructed via Hereman-Poole algorithm. Finally, the reduced forms of the equation are found via the similarity variables obtained from the symmetries.

Acknowledgment

The authors would like to thanks the reviewers for the useful and constructive comments.

References

[1] S. C. Anco and G. W. Bluman, *Direct construction of conservation laws from filed equation*, Phys. Rev. Lett, *78* (1997), 2869–2873.

[2] G. W. Bluman and J. D. Cole, *The general similarity solution of the heat equation*, J. Math. Mech, *18* (1969), 1025–1042.

[3] G. W. Bluman, *Connections Between Symmetries and Conservation Laws*, SIGMA, *1* (2005), Paper 011, 16 pages.

[4] G. W. Bluman, A. F. Cheviakov, and S. C. Anco, *Application of symmetry methods to partial differential equations*, Springer-Verlage, New York, 2000.

[5] G. W. Bluman, Alexei F. Cheviakov, and Stephen C. Anco, *Construction of conservation laws: How the direct method Generalizes Noether's theorem*, Proceeding of 4-th Workshop "Group analysis of differential equations and integribility", (2009), 1–23.

[6] J.D. Buckmaster, A. Nachman, and L. Ting, *The buckling and stretching of a viscida*, J. Fluid Mech, *69* (1975), 1–20, DOI: 10.1017/S0022112075001279.

[7] W. Hereman, B. Deconinck, and L. D. Poole, *Continuous and discrete homotopy operators: A theoretical approach made concrete*, Mathematics and Computers in Simulation, *74* (2007), 352–360, DOI: 10.1016/j.matcom.2006.10.012.

[8] P. E. Hydon, *Symmetry method for differential equations, Cambridge University Press*, Cambridge, UK, 2000.

[9] N. H. Ibragimov, *A new conservation theorem*, Journal of Mathematical Analysis and Applications, *333* (2007), 311–328, DOI: 10.1016/j.jmaa.2006.10.078.

[10] N. H. Ibragimov, *Nonlinear self-adjointness in constructing conservation laws*, Archives of ALGA *7/8* (2011), 1–99.

[11] E. Lashkarian and S. R. Hejazi, *Group analysis of the time fractional generalized diffusion equation*, Physica A, *479* (2017), 572–579, DOI: 10.1016/j.physa.2017.02.062.

[12] M. Nadjafikhah and S. R. Hejazi, *Symmetry analysis of telegraph equation*, Asian-European Journal of Mathematics, *4*(1) (2011), 117-126, DOI: 10.1142/S1793557111000101.

[13] P. J. Olver, *Applications of Lie groups to differential equations*, Second Edition, GTM, 107, Springer-Verlage, New York, 1993.

[14] D. Poole and W. Hereman, *Symbolic computation of conservation laws of non-linear partial differential equations in multi-dimensions*, Journal of Symbolic Computation, *46* (2011), 1355-1377, DOI: 10.1016/j.jsc.2011.08.014.

[15] S. Rashidi, S. R. Hejazi, and E. dastranj, *Approximate symmetry analysis of nonlinear Rayleigh-wave equation*, International Journal of Geometric Methods in Modern Physics, *15*(4) (2018), 1850055 (18 pages), DOI: 10.1142/S021988781850055X.

[16] S. Rashidi and S. R. Hejazi, *Symmetry properties, similarity reduction and exact solutions of fractional Boussinesq equation*, International Journal of Geometric Methods in Modern Physics, *14*(6) (2017) 1750083, (15 pages), DOI: 10.1142/S0219887817500839.

[17] S. Rashidi and S. R. Hejazi, *Analyzing Lie symmetry and constructing conservation laws for time-fractional Benny-Lin equation*, International Journal of Geometric Methods in Modern Physics, *14*(12) (2017) 1750170 (25 pages), DOI: 10.1142/S0219887817501705.