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k-fractional integral inequalities of Hadamard type for $(h-m)-$convex functions

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Abstract In this paper, we establish Hadamard type fractional integral inequalities for a more general class of functions that is the class of $(h-m)-$convex functions. These results are due to Riemann-Liouville (RL) k-fractional integrals: a generalization of RL fractional integrals. Several known results are special cases of proved results.

Keywords. Riemann-Liouville fractional integrals, Riemann-Liouville $k-$fractional integrals, $(h-m)-$convex functions.

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1. Introduction and preliminaries

Mathematics is an art of giving things misleading names. The beautiful and at first glance mysterious name, the Fractional Calculus (FC) is just one of those misnomers. A misnomer for the theory of operators of integration and differentiation of arbitrary fractional order and their application. In 1871, Heaviside states that, there is universe of mathematics lying between the complete differentiation and integration, and fractional operators push themselves forward sometimes and are just as real as others. In recent decades, they have been found useful in various fields: rheology, quantitative biology, electro-chemistry, scattering theory, diffusion, transport theory, probability, statistics, potential theory and elasticity. Nowadays, there exits a great number of articles, surveys and several books, entirely devoted to FC (see, [4, 5, 6, 9, 17, 21, 22, 23]). The definition of RL fractional differentiation played an important role in the development of FC. Here we give definition of RL fractional integrals.

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Definition 1.1. Let $f \in L_1[a,b]$. We define left and right RL fractional integral $I^{\alpha}_{a+}f(x)$ and $I^{\alpha}_{b-}f(x)$ of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ by

$$I^{\alpha}_{a+}f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, \quad x > a,$$

and

$$I^{\alpha}_{b-}f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)dt}{(t-x)^{1-\alpha}}, \quad x < b$$

respectively. Here $\Gamma(\alpha)$ is the Euler's Gamma function and $I^{0}_{a+}f(x) = I^{0}_{b-}f(x) = f(x)$.

Many generalizations of RL fractional integral operators have been introduced. In [20], Mubeen and Habibullah gave a slight generalization as RL k-fractional integrals:

Definition 1.2. Let $f \in L_1[a,b]$. Then we define left and right k-fractional RL integrals $I^{\alpha,k}_{a^+}f(x)$ and $I^{\alpha,k}_{b^-}f(x)$ of order α and $k > 0$ by

$$I^{\alpha,k}_{a^+}f(x) := \frac{1}{k\Gamma_k(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{\frac{\alpha}{k}-1}}, \quad x > a,$$

$$I^{\alpha,k}_{b^-}f(x) := \frac{1}{k\Gamma_k(\alpha)} \int_x^b \frac{f(t)dt}{(t-x)^{\frac{\alpha}{k}-1}}, \quad x < b,$$

where $\Gamma_k(\alpha)$ is the $k-$Gamma function and $I^{0,1}_{a^+}f(x) = I^{0,1}_{b^-}f(x) = f(x)$.

In recent years the theory of inequalities in mathematical analysis via fractional integral operators of different kinds has been introduced in fractional calculus (FC) (see, [1, 4, 7, 8, 9, 12, 13, 11, 14, 15, 16, 26, 27] and references in there). Inequalities have a significant role in the field of convex analysis, while the classical Hadamard inequality is equivalent to the definition of convex functions.

Definition 1.3. A function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, is said to be convex function if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y),\ x, y \in I, \alpha \in [0,1].$$

Convex functions are generalized in many ways leading to very attractive results. In [24], Özdemir *et el.* introduced a generalized class of functions namely $(h-m)-$convex functions.

Definition 1.4. Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. We say that $f : [0,b] \to \mathbb{R}$ is a $(h-m)-$convex function if f is non-negative satisfying

$$f(\alpha x + m(1-\alpha)y) \leq h(\alpha)f(x) + mh(1-\alpha)f(y).$$

and for suitable choice of $h(\alpha)$ and $m = 1$, the class of $(h-m)-$convex functions reduces to the different known classes of convex functions.

In Section 2, we prove $k-$fractional integral inequality of Hadamard type for $(h-m)-$convex functions and deduce some related results. In Section 3, we prove a version of $k-$fractional integral inequality of Hadamard type for functions like f so that $|f'|$ is $(h-m)-$convex whose derivative in absolute values are $(h-m)-$convex. In Section 4, we prove $k-$fractional integral inequality of Hadamard type for product of two $(h-m)-$convex functions and also for the product of $(\alpha, h-m)-$convex functions. Also we find connection with some well known results.

2. Hadamard type inequalities for $(h-m)-$convex functions via RL $k-$fractional integrals

In the following we give $k-$fractional integral inequalities of Hadamard type for $(h-m)-$convex functions.

Theorem 2.1. *Let $f:[0,\infty)\to\mathbb{R}$ be a $(h-m)-$convex function with $m\in(0,1]$, $f\in L_1[a,b]$, $a,b\in[0,\infty)$ where $\frac{a}{m}, mb\in[a,b]$. Then we will have*

$$\begin{aligned} &f\left(\frac{bm+a}{2}\right)\leq h\left(\frac{1}{2}\right)\frac{\Gamma_k(\alpha+k)}{(mb-a)^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{a^+}f(mb)+I^{\alpha,k}_{b^-}f\left(\frac{a}{m}\right)\right]\\ &\leq\frac{\alpha}{k}h\left(\frac{1}{2}\right)\left\{\left[m^2f\left(\frac{a}{m^2}\right)+mf(b)\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t)dt\right.\\ &\left.+[mf(b)+f(a)]\int_0^1 t^{\frac{\alpha}{k}-1}h(t)dt\right\}. \end{aligned} \tag{2.1}$$

Proof. Since f is $(h-m)-$convex on $[a,b]$, then

$$f\left(\frac{um+v}{2}\right)\leq h\left(\frac{1}{2}\right)(mf(u)+f(v)), u,v\in[a,b].$$

Since $\frac{a}{m}, mb\in[a,b]$, then $(1-t)\frac{a}{m}+tb\leq b$ and $(1-t)mb+ta\geq a$. By setting $u=(1-t)\frac{a}{m}+tb$ and $v=m(1-t)b+ta$ in the above inequality for $t\in[0,1]$, then by integrating over $[0,1]$ after multiplying with $t^{\frac{\alpha}{k}-1}$ we have

$$\begin{aligned} &f\left(\frac{bm+a}{2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}dt\\ &\leq h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\frac{\alpha}{k}-1}mf\left((1-t)\frac{a}{m}+tb\right)dt\right.\\ &\left.+\int_0^1 t^{\frac{\alpha}{k}-1}f(m(1-t)b+ta)dt\right]. \end{aligned}$$

Now if we let $w=(1-t)\frac{a}{m}+tb$ and $z=m(1-t)b+ta$ in right side of above inequality, we get

$$\begin{aligned} &f\left(\frac{bm+a}{2}\right)\leq\\ &h\left(\frac{1}{2}\right)\frac{\Gamma_k(\alpha+k)}{(mb-a)^{\frac{\alpha}{k}}}\left[m^{\alpha+1}I^{\alpha,k}_{b^-}f\left(\frac{a}{m}\right)+I^{\alpha,k}_{a^+}f(mb)\right]. \end{aligned} \tag{2.2}$$

Which completes the proof of first inequality in (2.1). On the other hand by using the $(h-m)-$convexity of f, we have

$$\begin{aligned}&mf\left((1-t)\frac{a}{m}+tb\right)+f(m(1-t)b+ta)\\&\leq m^2h(1-t)f\left(\frac{a}{m^2}\right)+mh(t)f(b)+mh(1-t)f(b)+h(t)f(a).\end{aligned}$$

By multiplying both sides of above inequality with $\alpha h\left(\frac{1}{2}\right)t^{\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, after some calculations we get

$$\begin{aligned}&h\left(\frac{1}{2}\right)\frac{\Gamma_k(\alpha+k)}{(mb-a)^{\frac{\alpha}{k}}}\left[m^{\alpha+1}I^{\alpha,k}_{b-}f\left(\frac{a}{m}\right)+I^{\alpha,k}_{a+}f(mb)\right]\\&\leq\frac{\alpha}{k}h\left(\frac{1}{2}\right)\left\{\left[m^2f\left(\frac{a}{m^2}\right)+mf(b)\right]\int_0^1t^{\frac{\alpha}{k}-1}h(1-t)dt\right.\\&\left.+[mf(b)+f(a)]\int_0^1t^{\frac{\alpha}{k}-1}h(t)dt\right\}\end{aligned}$$

Which is the other side of the inequality (2.1). □

Corollary 2.2. *In Theorem 2.1 with $k=1$, leads to the following inequality for $(h-m)-$convex function via RL fractional integrals*

$$\begin{aligned}&f\left(\frac{bm+a}{2}\right)\leq h\left(\frac{1}{2}\right)\frac{\Gamma(\alpha+1)}{(mb-a)^{\alpha}}\left[I^{\alpha}_{a+}f(mb)+I^{\alpha}_{b-}f\left(\frac{a}{m}\right)\right]\\&\leq\frac{\alpha}{k}h\left(\frac{1}{2}\right)\left\{\left[m^2f\left(\frac{a}{m^2}\right)+mf(b)\right]\int_0^1t^{\alpha-1}h(1-t)dt\right.\\&\left.+[mf(b)+f(a)]\int_0^1t^{\alpha-1}h(t)dt\right\}\end{aligned}$$

Which leads to in above theorem we get [26, Theorem 2] stated in following corollary if we take $k=1$, $h(t)=t$ and $m=1$.

Corollary 2.3. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then we will have*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[I^{\alpha}_{a+}f(b)+I^{\alpha}_{b-}f(a)\right]\leq\frac{f(a)+f(b)}{2}.$$

Another $k-$fractional integral inequality of Hadamard type for $(h-m)-$convex function is obtained as follows.

Theorem 2.4. *Let $f:[0,\infty)\to\mathbb{R}$ be a $(h-m)-$convex function with $m\in(0,1]$, also let $f\in L_1[a,b]$, $a,b\in[0,\infty)$. Then we will have*

$$\begin{aligned}&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{a+}f(b)+I^{\alpha,k}_{b-}f(a)\right]\qquad(2.3)\\&\leq[f(a)+f(b)]\int_0^1t^{\frac{\alpha}{k}-1}h(t)dt+m\left[f\left(\frac{b}{m}\right)+f\left(\frac{a}{m}\right)\right]\int_0^1t^{\frac{\alpha}{k}-1}h(1-t)dt\\&\leq\frac{1}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left(\int_0^1(h(t))^qdt\right)^{\frac{1}{q}}\left[f(a)+f(b)+m\left(f\left(\frac{b}{m}\right)+f\left(\frac{a}{m}\right)\right)\right],\end{aligned}$$

where $p^{-1}+q^{-1}=1$ *and* $p>1$.

Proof. Since f is $(h-m)-$convex on $[a,b]$, then for $m\in(0,1]$ and $t\in[0,1]$, we have

$$\begin{aligned}&f(ta+(1-t)b)+f((1-t)a+tb)\\&\leq h(t)[f(a)+f(b)]+mh(1-t)\left[f\left(\frac{b}{m}\right)+f\left(\frac{a}{m}\right)\right].\end{aligned}$$

from which multiplying both sides with $t^{\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, we will have

$$\begin{aligned}&\int_0^1 t^{\frac{\alpha}{k}-1}[f(ta+(1-t)b)+f((1-t)a+tb)]dt\\&\leq [f(a)+f(b)]\int_0^1 t^{\frac{\alpha}{k}-1}h(t)dt+m\left[f\left(\frac{b}{m}\right)+f\left(\frac{a}{m}\right)\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t)dt.\end{aligned}$$

By changing variables we will have

$$\begin{aligned}&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(b)+I_{b^-}^{\alpha,k}f(a)]\\&\leq [f(a)+f(b)]\int_0^1 t^{\frac{\alpha}{k}-1}h(t)dt+m\left[f\left(\frac{b}{m}\right)+f\left(\frac{a}{m}\right)\right]\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t)dt\end{aligned}\tag{2.4}$$

Which completes the proof of first inequality in (2.3). The second inequality in (2.3) follows by using the Hölder's inequality

$$\int_0^1 t^{\frac{\alpha}{k}-1}(h(t))dt\leq\frac{1}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left(\int_0^1(h(t))^q dt\right)^{\frac{1}{q}}.$$

Thus from (2.4) we will get (2.3). □

from this theorem, if we will put $k=1$ and $m=1$, then we get [27, Theorem 2.1], which is stated in the following corollary.

Corollary 2.5. *Let* $f:[0,\infty)\to\mathbb{R}$ *be* $h-$*convex function and* $f\in L_1[a,b]$. *Then for RL fractional integrals, we have*

$$\begin{aligned}&\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}[I_{a^+}^{\alpha}f(b)+I_{b^-}^{\alpha}f(a)]\\&\leq [f(a)+f(b)]\int_0^1 t^{\alpha-1}[h(t)+h(1-t)]dt\leq\frac{2\,[f(a)+f(b)]}{(\alpha p-p+1)^{\frac{1}{p}}}\left(\int_0^1(h(t))^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Theorem 2.6. *Let* $f:[0,\infty)\to\mathbb{R}$ *be a* $(h-m)-$*convex function with* $m\in(0,1]$, h *be superaddtive and* $f\in L_1[a,b], a,b\in[0,\infty)$. *Then for RL* $k-$*fractional integrals we have*

$$\begin{aligned}&\frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(b)+I_{b^-}^{\alpha,k}f(a)]\\&\leq h(1)\left[\frac{f(a)+f(b)}{2}+m\left(\frac{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)}{2}\right)\right].\end{aligned}\tag{2.5}$$

Proof. Since f is $(h-m)-$convex on $[a,b]$, then for $t\in[0,1]$, we get

$$f(ta+(1-t)b)+f((1-t)a+tb) \leq [h(t)+h(1-t)]\left[\frac{f(a)+f(b)}{2}+m\left(\frac{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)}{2}\right)\right].$$

Since h is superadditive, therefore

$$f(ta+(1-t)b)+f((1-t)a+tb) \leq h(1)\left[\frac{f(a)+f(b)}{2}+m\left(\frac{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)}{2}\right)\right].$$

By multiplying both sides of above inequality with $t^{\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, we will have

$$\int_0^1 t^{\frac{\alpha}{k}-1}[f(ta+(1-t)b)+f((1-t)a+tb)]dt \leq h(1)\left[\frac{f(a)+f(b)}{2}+m\left(\frac{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)}{2}\right)\right]\int_0^1 t^{\frac{\alpha}{k}-1}dt.$$

By substituting $w=ta+(1-t)b$ in left side of above inequality leads to (2.5). □

Corollary 2.7. *In Theorem 2.6, if we take $k=1$, then for $(h-m)-$convex function via RL fractional integrals we get*

$$\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}[I^{\alpha}_{a^+}f(b)+I^{\alpha}_{b^-}f(a)] \leq h(1)\left[\frac{f(a)+f(b)}{2}+m\left(\frac{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)}{2}\right)\right].$$

3. $k-$fractional integral inequality of Hadamard type for functions whose derivatives absolute values are $(h-m)-$convex

In the following $k-$fractional integral inequalities of Hadamard type for $(h-m)-$convex function in terms of the first derivatives have been obtained. For next result we use the following lemma.

Lemma 3.1. [9] *Let function $f:[a,b]\to\mathbb{R}$ be differentiable on interval (a,b). If $f'\in L[a,b]$, then we will have*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}(I^{\alpha,k}_{a^+}f(b)+I^{\alpha,k}_{b^-}f(a)) = \frac{b-a}{2}\int_0^1[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}]f'(ta+(1-t)b)dt.$$

Theorem 3.2. *Let $f:[0,\infty)\to\mathbb{R}$ be a function such that $[a,b]\subset[0,\infty)$ and $f\in L_1[a,b]$. If $|f'|$ is an $(h-m)-$convex with $m\in(0,1]$ and $h^q\in[0,1], q>1$. Then for RL $k-$fractional integrals we will have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(b)+I_{b^-}^{\alpha,k}f(a)]\right| \tag{3.1}$$
$$\leq \frac{(b-a)\left[|f'(a)|+m\left|f'\left(\frac{b}{m}\right)\right|\right]}{2}\left[\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right]$$
$$\times\left[\left[\int_0^{\frac{1}{2}}(h(t))^q dt\right]^{\frac{1}{q}}+\left[\int_{\frac{1}{2}}^1(h(t))^q dt\right]^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using $(h-m)-$convexity of $|f'|$ and Lemma 3.1, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(b)+I_{b^-}^{\alpha,k}f(a)]\right|$$
$$\leq \frac{b-a}{2}\int_0^1\left|(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right||f'(ta+(1-t)b)|\,dt.$$

Furthermore, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(b)+I_{b^-}^{\alpha,k}f(a)]\right| \tag{3.2}$$
$$\leq \frac{b-a}{2}\int_0^{\frac{1}{2}}\left[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right]\left[h(t)\left|f'(a)\right|+mh(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right]dt$$
$$+\int_{\frac{1}{2}}^1\left[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right]\left[h(t)\left|f'(a)\right|+mh(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right]dt$$
$$=\frac{b-a}{2}\left\{\left|f'(a)\right|\int_0^{\frac{1}{2}}(1-t)^{\frac{\alpha}{k}}h(t)dt-\left|f'(a)\right|\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}}h(t)dt\right.$$
$$+m\left|f'\left(\frac{b}{m}\right)\right|\int_0^{\frac{1}{2}}(1-t)^{\frac{\alpha}{k}}h(1-t)dt-m\left|f'\left(\frac{b}{m}\right)\right|\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}}h(1-t)dt$$
$$+\left|f'(a)\right|\int_{\frac{1}{2}}^1 t^{\frac{\alpha}{k}}h(t)dt-\left|f'(a)\right|\int_{\frac{1}{2}}^1(1-t)^{\frac{\alpha}{k}}h(t)dt$$
$$\left.+m\left|f'\left(\frac{b}{m}\right)\right|\int_{\frac{1}{2}}^1 t^{\frac{\alpha}{k}}h(1-t)dt-m\left|f'\left(\frac{b}{m}\right)\right|\int_{\frac{1}{2}}^1(1-t)^{\frac{\alpha}{k}}h(1-t)dt\right\}.$$

Now, by using the Hölder's inequality, we have

$$\int_0^{\frac{1}{2}}(1-t)^{\frac{\alpha}{k}}h(t)dt=\int_{\frac{1}{2}}^1 t^{\frac{\alpha}{k}}h(1-t)dt$$
$$\leq\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\int_0^{\frac{1}{2}}[h(t)]^q dt\right]^{\frac{1}{q}},$$

$$\begin{aligned}\int_0^{\frac{1}{2}}(1-t)^{\frac{\alpha}{k}}h(1-t)dt &= \int_{\frac{1}{2}}^1 t^{\frac{\alpha}{k}}h(t)dt\\ &\leq \left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\alpha p+1)}\right]^{\frac{1}{p}}\left[\int_{\frac{1}{2}}^1[h(t)]^q dt\right]^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}}h(1-t)dt &= \int_{\frac{1}{2}}^1 (1-t)^{\frac{\alpha}{k}}h(t)dt\\ &\leq \left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\int_{\frac{1}{2}}^1[h(t)]^q dt\right]^{\frac{1}{q}},\end{aligned}$$

and

$$\begin{aligned}\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}}h(t)dt &= \int_{\frac{1}{2}}^1 (1-t)^{\frac{\alpha}{k}}h(1-t)dt\\ &\leq \left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\int_0^{\frac{1}{2}}[h(t)]^q dt\right]^{\frac{1}{q}}.\end{aligned}$$

By using the above inequalities in the right hand side of (3.2), we get

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[I_{a+}^{\alpha,k}f(b)+I_{b-}^{\alpha,k}f(a)]\right|\\ &\leq \frac{b-a}{2}\left\{|f'(a)|\left[\left(\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\left(\int_0^{\frac{1}{2}}[h(t)]^q dt\right)^{\frac{1}{q}}\right.\right.\\ &+\left.\left(\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\left(\int_{\frac{1}{2}}^1[h(t)]^q dt\right)^{\frac{1}{q}}\right]\\ &+m\left|f'\left(\frac{b}{m}\right)\right|\left[\left(\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\left(\int_{\frac{1}{2}}^1[h(t)]^q dt\right)^{\frac{1}{q}}\right.\\ &+\left.\left.\left(\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\left(\int_0^{\frac{1}{2}}[h(t)]^q dt\right)^{\frac{1}{q}}\right]\right\}.\end{aligned}$$

After a little computation one can get inequality (3.1). □

Corollary 3.3. *Theorem 3.2 with $k = 1$ gives the result for (h-m)-convex functions via RL fractional integrals, we will have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[I^{\alpha}_{a+}f(b)+I^{\alpha}_{b-}f(a)]\right|$$
$$\leq \frac{(b-a)\left[|f'(a)|+m\left|f'\left(\frac{b}{m}\right)\right|\right]}{2}\left[\left[\frac{2^{\alpha p+1}-1}{2^{\alpha p+1}(\alpha p+1)}\right]^{\frac{1}{p}} - \left[\frac{1}{2^{\alpha p+1}(\alpha p+1)}\right]^{\frac{1}{p}}\right]\times$$
$$\left[\left[\int_0^{\frac{1}{2}}(h(t))^q dt\right]^{\frac{1}{q}} + \left[\int_{\frac{1}{2}}^{1}(h(t))^q dt\right]^{\frac{1}{q}}\right].$$

Corollary 3.4. *If we put $m = 1$ and $k = 1$ in Theorem 3.2, then we have the following inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha)}{2(b-a)^{\alpha}}[I^{\alpha}_{a+}f(b)+I^{\alpha}_{b-}f(a)]\right|$$
$$\leq \frac{(b-a)\left[|f'(a)|+|f'(b)|\right]}{2}\left[\left[\frac{2^{\alpha p+1}-1}{2^{\alpha p+1}(\alpha p+1)}\right]^{\frac{1}{p}} - \left[\frac{1}{2^{\alpha p+1}(\alpha p+1)}\right]^{\frac{1}{p}}\right]$$
$$\times\left[\left(\int_0^{\frac{1}{2}}[h(t)]^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^{1}[h(t)]^q dt\right)^{\frac{1}{q}}\right].$$

which is given in [27, Theorem 2.6]. Before, we prove next theorem, we need the following lemma.

Lemma 3.5. *Let $f : [a, mb] \to \mathbb{R}$ be a differentiable mapping on interval (a, mb) with $a < mb$. If $f' \in L_1[a, mb]$, then for $k-$fractional integrals we will have*

$$\frac{f(mb)+f(a)}{2} - \frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(mb-a)^{\frac{\alpha}{k}}}(I^{\alpha,k}_{a+}f(mb)+I^{\alpha,k}_{mb-}f(a))$$
$$= \frac{mb-a}{2}\int_0^1[(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}]f'(m(1-t)b+ta)dt.$$

Proof. Consider the right hand side

$$\frac{mb-a}{2}\int_0^1[(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}]f'(m(1-t)b+ta)$$
$$= \frac{mb-a}{2}\left[\int_0^1(1-t)^{\frac{\alpha}{k}}f'(m(1-t)b+ta)dt\right.$$
$$\left.-\int_0^1 t^{\frac{\alpha}{k}}f'(m(1-t)b+ta)dt\right].$$

One has

$$\frac{mb-a}{2}\int_0^1(1-t)^{\frac{\alpha}{k}}f'(m(1-t)b+ta)dt$$
$$= \frac{f(mb)}{2} - \frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(mb-a)^{\frac{\alpha}{k}}}I^{\alpha,k}_{mb-}f(a),$$

and

$$\frac{mb-a}{2}\int_0^1 (t)^{\frac{\alpha}{k}} f'(m(1-t)b+ta)dt$$
$$=\frac{f(b)}{2}-\frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(mb-a)^{\frac{\alpha}{k}}} I_{a^+}^{\alpha,k} f(mb).$$

Hence the required equality can be established. □

Theorem 3.6. *Let $f:[a,mb]\to\mathbb{R}$ be a function such that $[a,mb]\subseteq[0,\infty)$ and $f\in L_1[a,mb]$. If $|f'|$ is an $(h-m)-$convex with $m\in(0,1]$ and $h^q\in[0,1], q>1$. Then for RL $k-$fractional integrals we will have*

$$\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(mb-a)^{\frac{\alpha}{k}}}(I_{a^+}^{\alpha,k} f(mb)+I_{mb^-}^{\alpha,k} f(a))\right| \tag{3.3}$$
$$\leq \frac{(mb-a)\left[|f'(a)|+m\left|f'\left(b\right)\right|\right]}{2}\left[\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right]$$
$$\times\left[\left(\int_0^{\frac{1}{2}}(h(t))^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1 (h(t))^q dt\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using the property of modulus from Lemma 3.5, we will get

$$\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(mb-a)^{\frac{\alpha}{k}}}(I_{b^+}^{\alpha,k} f(mb)+I_{mb^-}^{\alpha,k} f(a))\right|$$
$$\leq\frac{mb-a}{2}\int_0^1\left|(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right|\left|f'(m(1-t)b+ta)\right|dt.$$

By $(h-m)-$convexity of $|f'|$, we will have

$$\begin{aligned}
&\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(mb-a)^{\frac{\alpha}{k}}}(I_{a^+}^{\alpha,k}f(mb)+I_{mb^-}^{\alpha,k}f(a))\right| \qquad (3.4)\\
&\leq \frac{mb-a}{2}\int_0^{\frac{1}{2}}\left[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right][mh(1-t)\,|f'(b)|+h(t)\,|f'\,(a)|]\,dt\\
&+\int_{\frac{1}{2}}^{1}\left[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right][mh(1-t)\,|f'(b)|+h(t)\,|f'\,(a)|]\,dt\\
&=\frac{mb-a}{2}\left\{|f'(a)|\left[\int_0^{\frac{1}{2}}(1-t)^{\frac{\alpha}{k}}h(1-t)dt-\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}}h(1-t)dt\right]\right.\\
&+m\,|f'\,(b)|\left[\int_0^{\frac{1}{2}}(1-t)^{\frac{\alpha}{k}}h(t)dt-\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}}h(t)dt\right]\\
&+|f'(a)|\left[\int_{\frac{1}{2}}^{1}t^{\frac{\alpha}{k}}h(1-t)dt-\int_{\frac{1}{2}}^{1}(1-t)^{\frac{\alpha}{k}}h(1-t)dt\right]\\
&\left.+m\,|f'\,(b)|\left[\int_{\frac{1}{2}}^{1}t^{\frac{\alpha}{k}}h(1-t)dt-\int_{\frac{1}{2}}^{1}(1-t)^{\frac{\alpha}{k}}h(1-t)dt\right]\right\}.
\end{aligned}$$

Now, by using the Hölder's inequality in the right hand side of (3.4), we will get

$$\begin{aligned}
&\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(mb-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(mb)+I_{mb^-}^{\alpha,k}f(a)]\right|\\
&\leq\frac{mb-a}{2}\left\{|f'(a)|\left[\left(\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\left(\int_0^{\frac{1}{2}}[h(t)]^q dt\right)^{\frac{1}{q}}\right.\right.\\
&\left.+\left(\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\left(\int_{\frac{1}{2}}^{1}[h(t)]^q dt\right)^{\frac{1}{q}}\right]\\
&+m\,|f'\,(b)|\left[\left(\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\left(\int_{\frac{1}{2}}^{1}[h(t)]^q dt\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\left(\int_0^{\frac{1}{2}}[h(t)]^q dt\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}$$

By some manipulation one can get inequality (3.3). □

Corollary 3.7. *In Theorem 3.6 if we take $k = 1$, we get the following inequality for $(h-m)-$convex function via RL fractional integrals*

$$\left|\frac{f(mb)+f(a)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}(I^{\alpha}_{a+}f(mb) + I^{\alpha}_{mb-}f(a))\right|$$
$$\leq \frac{(mb-a)\left[|f'(a)| + m\,|f'(b)|\right]}{2}\left[\left[\frac{2^{\alpha p+1}-1}{2^{\alpha p+1}(\alpha p+1)}\right]^{\frac{1}{p}} - \left[\frac{1}{2^{\alpha p+1}(\alpha p+1)}\right]^{\frac{1}{p}}\right]$$
$$\times\left[\left(\int_0^{\frac{1}{2}}[h(t)]^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 [h(t)]^q dt\right)^{\frac{1}{q}}\right],$$

4. RL k-fractional integral inequalities of Hadamard type for product of two $(h-m)-$convex functions

Now, we obtain some Hadamard type inequalities for products of two $(h-m)-$convex functions via RL $k-$fractional integrals.

Theorem 4.1. *Let $f, g : [0,\infty) \to \mathbb{R}$ be functions such that $fg \in L_1[a,b]$, $a, b \in [0,\infty)$, $a < b$. If function f is $(h_1 - m)-$convex and function g is $(h_2 - m)-$convex on $[0,\infty)$ with $m \in (0,1]$, then for RL $k-$fractional integrals we have*

$$\frac{k}{\alpha h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \tag{4.1}$$
$$-\frac{\Gamma_k(\alpha)}{(mb-a)^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{a+}f(mb)g(mb) + m^{\alpha+1}I^{\alpha,k}_{b-}f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\right]$$
$$\leq m\left[f(a)g\left(\frac{a}{m^2}\right) + f(b)\,g(b)\right]\int_0^1 t^{\frac{\alpha}{k}-1}h_1(t)h_2(1-t)dt$$
$$+ m^2\left[f(b)\,g\left(\frac{a}{m^2}\right) + f\left(\frac{a}{m^2}\right)g(b)\right]\int_0^1 t^{\frac{\alpha}{k}-1}h_1(1-t)h_2(1-t)dt$$
$$+\left[f(b)\,g(a) + f(a)g(b)\right]\int_0^1 t^{\frac{\alpha}{k}-1}h_1(t)h_2(t)dt$$
$$+ m\left[f(b)\,g(b) + f\left(\frac{a}{m^2}\right)g(a)\right]\int_0^1 t^{\frac{\alpha}{k}-1}h_2(t)h_1(1-t)dt.$$

Proof. We can write

$$f\left(\frac{a+mb}{2}\right)g\left(\frac{a+mb}{2}\right)$$
$$= f\left(\frac{at+m(1-t)b}{2} + \frac{(1-t)a+mtb}{2}\right)$$
$$\times \quad g\left(\frac{at+m(1-t)b}{2} + \frac{(1-t)a+mtb}{2}\right).$$

From $(h_1 - m)-$convexity of f and $(h_2 - m)-$convexity of g, we will have

$$
\begin{aligned}
& f\left(\frac{a+mb}{2}\right) g\left(\frac{a+mb}{2}\right) \\
& \leq h_1\left(\frac{1}{2}\right)\left[f\left(at+m(1-t)b\right)+mf\left((1-t)\frac{a}{m}+tb\right)\right] \\
\times & h_2\left(\frac{1}{2}\right)\left[g\left(at+m(1-t)b\right)+mg\left((1-t)\frac{a}{m}+tb\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
= & h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right)\{f\left(at+m(1-t)b\right) g\left(at+m(1-t)b\right) \\
& + m^2 f\left((1-t)\frac{a}{m}+tb\right) g\left((1-t)\frac{a}{m}+tb\right) \\
& + mf\left(at+m(1-t)b\right) g\left((1-t)\frac{a}{m}+tb\right) \\
& + mf\left((1-t)\frac{a}{m}+tb\right) g\left(at+m(1-t)b\right)\Big\}.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
& f\left(\frac{a+mb}{2}\right) g\left(\frac{a+mb}{2}\right) \\
& \leq h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right)\{f\left(at+m(1-t)b\right) g\left(at+m(1-t)b\right) \\
& + m^2 f\left((1-t)\frac{a}{m}+tb\right) g\left((1-t)\frac{a}{m}+tb\right) \\
& + m^2 h_1(t) h_2(1-t)\left[f(a) g\left(\frac{a}{m^2}\right)+f\left(\frac{b}{m}\right) g\left(\frac{b}{m}\right)\right] \\
& + m^2 h_1(1-t) h_2(1-t)\left[f\left(\frac{b}{m}\right) g\left(\frac{a}{m^2}\right)+f\left(\frac{a}{m^2}\right) g\left(\frac{a}{m}\right)\right] \\
& + h_1(t) h_2(t)\left[f\left(\frac{b}{m}\right) g(a)+f(a) g\left(\frac{b}{m}\right)\right] \\
& + mh_2(t) h_1(1-t)\left[f\left(\frac{b}{m}\right) g\left(\frac{b}{m}\right)+f\left(\frac{a}{m^2}\right) g(a)\right]\Big\}.
\end{aligned}
$$

By multiplying both sides of above inequality with $t^{\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, we have

$$\begin{aligned}
&\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}dt\\
&-\frac{\Gamma_k(\alpha)}{(mb-a)^{\frac{\alpha}{k}}}\left[I_{a^+}^{\alpha,k}f(mb)g(mb)+m^{\alpha+1}I_{b^-}^{\alpha,k}f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\right]\\
&\leq m\left[f(a)g\left(\frac{a}{m^2}\right)+f\left(b\right)g\left(b\right)\right]\int_0^1 t^{\frac{\alpha}{k}-1}h_1(t)h_2(1-t)dt\\
&+m^2\left[f\left(b\right)g\left(\frac{a}{m^2}\right)+f\left(\frac{a}{m^2}\right)g\left(b\right)\right]\int_0^1 t^{\frac{\alpha}{k}-1}h_1(1-t)h_2(1-t)dt\\
&+\left[f\left(b\right)g(a)+f(a)g\left(b\right)\right]\int_0^1 t^{\frac{\alpha}{k}-1}h_1(t)h_2(t)dt\\
&+m\left[f\left(b\right)g\left(b\right)+f\left(\frac{a}{m^2}\right)g(a)\right]\int_0^1 t^{\frac{\alpha}{k}-1}h_2(t)h_1(1-t)dt.
\end{aligned}$$

After a little computation one can have the required result. □

Corollary 4.2. *For $k=1$, theorem 4.1 gives the following inequality for $(h-m)-$convex function via RL fractional integrals*

$$\begin{aligned}
&\frac{1}{\alpha h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\
&-\frac{\Gamma(\alpha)}{(mb-a)^{\alpha}}\left[I_{a^+}^{\alpha}f(mb)g(mb)+m^{\alpha+1}I_{b^-}^{\alpha}f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\right]\\
&\leq m\left[f(a)g\left(\frac{a}{m^2}\right)+f\left(b\right)g\left(b\right)\right]\int_0^1 t^{\alpha-1}h_1(t)h_2(1-t)dt\\
&+m^2\left[f\left(b\right)g\left(\frac{a}{m^2}\right)+f\left(\frac{a}{m^2}\right)g\left(b\right)\right]\int_0^1 t^{\alpha-1}h_1(1-t)h_2(1-t)dt\\
&+\left[f\left(b\right)g(a)+f(a)g\left(b\right)\right]\int_0^1 t^{\alpha-1}h_1(t)h_2(t)dt\\
&+m\left[f\left(b\right)g\left(b\right)+f\left(\frac{a}{m^2}\right)g(a)\right]\int_0^1 t^{\alpha-1}h_2(t)h_1(1-t)dt.
\end{aligned}$$

Theorem 4.3. *Let $f,g:[0,\infty)\to\mathbb{R}$ be functions such that $fg\in L_1[a,b]$, $a,b\in[0,\infty)$, $a<b$. If function f is $(h-m_1)-$convex and function g is $(h-m_2)-$convex on $[0,\infty)$ with $m_1,m_2\in(0,1]$, then the following inequalities hold for RL $k-$fractional integrals*

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}\left[I_{a^+}^{\alpha,k}f(b)g(b)+I_{b^-}^{\alpha,k}f(a)g(a)\right] \\
&\leq [f(a)g(a)+f(b)g(b)]\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t)dt \\
&+m_1m_2\left[f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)+f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]\int_0^1 t^{\frac{\alpha}{k}-1}h^2(1-t)dt \\
&+\left\{m_2\left[f(a)g\left(\frac{b}{m_2}\right)+f(b)g\left(\frac{a}{m_2}\right)\right]+m_1\left[g(a)f\left(\frac{b}{m_1}\right)g(b)f\left(\frac{a}{m_1}\right)\right]\right\} \\
&\times\int_0^1 t^{\frac{\alpha}{k}-1}h(t)h(1-t)dt \\
&\leq \frac{1}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left\{\left(\int_0^1 h^{2q}(t)dt\right)^{\frac{1}{q}}\left[f(a)g(a)+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right.\right. \\
&\left.+f(b)g(b)+m_1m_2f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]+\left(\int_0^1(h(t)h(1-t))^qdt\right)^{\frac{1}{q}}\times \\
&\left.\left[m_2f(a)g\left(\frac{b}{m_2}\right)+m_1g(a)f\left(\frac{b}{m_1}\right)+m_2f(b)g\left(\frac{a}{m_2}\right)+m_1g(b)f\left(\frac{a}{m_1}\right)\right]\right\},
\end{aligned} \tag{4.2}$$

where $p>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since function f is $(h-m_1)-$convex and function g is $(h-m_2)-$convex, then for $t\in[0,1]$, we have

$$\begin{aligned}
&f(ta+(1-t)b)g(ta+(1-t)b) \\
&\leq h^2(t)f(a)g(a)+m_2f(a)g\left(\frac{b}{m_2}\right)h(t)h(1-t) \\
&+m_1g(a)f\left(\frac{b}{m_1}\right)h(t)h(1-t)+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)h^2(1-t).
\end{aligned}$$

By multiplying both sides of above inequality with $t^{\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}-1}f(ta+(1-t)b)g(ta+(1-t)b)dt \\
&\leq f(a)g(a)\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t)dt+m_2f(a)g\left(\frac{b}{m_2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h(t)h(1-t)dt \\
&+m_1g(a)f\left(\frac{b}{m_1}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h(t)h(1-t)dt \\
&+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h^2(1-t)dt.
\end{aligned}$$

By substituting $z = ta + (1-t)b$ in left side of above inequality, we get

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}} I_{a^+}^{\alpha,k} f(b)g(b) \\
&= f(a)g(a)\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t)dt + m_2 f(a)g\left(\frac{b}{m_2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h(t)h(1-t)dt \\
&+ m_1 g(a)f\left(\frac{b}{m_1}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h(t)h(1-t)dt \\
&+ m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h^2(1-t)dt.
\end{aligned}$$

By using the Hölder's inequality, we will have

$$\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t)dt \leq \frac{1}{(\frac{\alpha}{k}p - p + 1)^{\frac{1}{p}}}\left(\int_0^1 h^{2q}(t)dt\right)^{\frac{1}{q}},$$

and similarly

$$\int_0^1 t^{\frac{\alpha}{k}-1}h(t)h(1-t)dt = \frac{1}{(\frac{\alpha}{k}p - p + 1)^{\frac{1}{p}}}\left(\int_0^1 (h(t)h(1-t))^q dt\right)^{\frac{1}{q}}.$$

Thus we will get

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}} I_{a^+}^{\alpha,k} f(b)g(b) \qquad (4.3)\\
&\leq f(a)g(a)\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t)dt + m_2 f(a)g\left(\frac{b}{m_2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h(t)h(1-t)dt \\
&+ m_1 g(a)f\left(\frac{b}{m_1}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h(t)h(1-t)dt \\
&+ m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h^2(1-t)dt \\
&\leq \frac{1}{(\frac{\alpha}{k}p - p + 1)^{\frac{1}{p}}}\left\{\left(\int_0^1 h^{2q}(t)dt\right)^{\frac{1}{q}}\left[f(a)g(a) + m_1 m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\right]\right. \\
&\left.+ \left(\int_0^1 (h(t)h(1-t))^q dt\right)^{\frac{1}{q}}\left[m_2 f(a)g\left(\frac{b}{m_2}\right) + m_1 g(a)f\left(\frac{b}{m_1}\right)\right]\right\}.
\end{aligned}$$

Similarly, by changing the roles of a and b, after a little computation one can get

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}} I_{a^+}^{\alpha,k} f(a)g(a) \qquad (4.4)\\
&\leq f(b)g(b)\int_0^1 t^{\frac{\alpha}{k}-1}h^2(t)dt + m_2 f(b) g\left(\frac{a}{m_2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h(t)h(1-t)dt\\
&+ m_1 g(b) f\left(\frac{a}{m_1}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h(t)h(1-t)dt\\
&+ m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h^2(1-t)dt\\
&\leq \frac{1}{(\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left\{\left(\int_0^1 h^{2q}(t)dt\right)^{\frac{1}{q}}\left[f(b)g(b)+m_1m_2 f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]\right.\\
&\left.+\left(\int_0^1 (h(t)h(1-t))^q dt\right)^{\frac{1}{q}}\left[m_2 f(b) g\left(\frac{a}{m_2}\right)+m_1 g(b) f\left(\frac{a}{m_1}\right)\right]\right\}.
\end{aligned}$$

Adding (4.3) and(4.4), we get the required result. □

For $k = 1$, $h(t) = t$ this theorem gives [25, Theorem 8], which is stated in the following corollary.

Corollary 4.4. *Let $f, g : [0,\infty) \to [0,\infty)$, $[a, b] \subseteq [0,\infty)$ be functions such that $fg \in L_1[a, b]$. If f is m_1-convex and g is m_2-convex on $[a, b]$ with $m_1, m_2 \in (0, 1]$, then one has*

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} I_{a^+}^{\alpha} f(b)g(b) \leq \frac{f(a)g(a)}{\alpha+2} + \frac{m_2}{(\alpha+1)(\alpha+2)} f(a) g\left(\frac{b}{m_2}\right) \qquad (4.5)\\
&+ \frac{m_1}{(\alpha+1)(\alpha+2)} g(a) f\left(\frac{b}{m_1}\right) + \frac{2m_1m_2}{\alpha(\alpha+1)(\alpha+2)} f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} I_{b^-}^{\alpha} f(a)g(a) \leq \frac{f(b)g(b)}{\alpha+2} + \frac{m_2}{(\alpha+1)(\alpha+2)} f(b) g\left(\frac{a}{m_2}\right) \qquad (4.6)\\
&+ \frac{m_1}{(\alpha+1)(\alpha+2)} g(b) f\left(\frac{a}{m_1}\right) + \frac{2m_1m_2}{\alpha(\alpha+1)(\alpha+2)} f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right).
\end{aligned}$$

Proof. Taking (4.3) for $k = 1$ and $h(t) = t$, we get (4.5). Similarly using $k = 1$ and $h(t) = t$ in (4.4), we get (4.6). □

In the following we give Hadamard type inequality for the product of two $(\alpha, h-m)-$convex via RL $k-$fractional integral.The following definition is needed

Definition 4.5. Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. We say that $f : [0, b] \to \mathbb{R}$ is a $(\alpha, h-m)-$convex function, if f is non-negative and for all $x, y \in [0, b], (\alpha, m) \in [0, 1]^2$ and $t \in (0, 1)$, one has

$$f(tx + m(1-t)y) \leq h(t^{\alpha})f(x) + mh(1-t^{\alpha})f(y).$$

For suitable choice of α, $h(t)$ and $m = 1$, class of $(\alpha, h-m)-$convex functions reduces to the different known classes of convex functions.

Theorem 4.6. *Let $f, g : [0,\infty) \to \mathbb{R}$ be functions such that $fg \in L_1[a,b]$, $a, b \in [0,\infty)$. If f is $(\alpha_1, h-m_1)-$convex and function g is $(\alpha_2, h-m_2)-$convex on $[0,\infty)$ with (α_1, m_1) and $(\alpha_2, m_2) \in (0,1]^2$, then the following inequalities hold for RL $k-$fractional integrals*

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}\left[I_{a^+}^{\alpha,k}f(b)g(b)+I_{b^-}^{\alpha,k}f(a)g(a)\right] \qquad (4.7)\\
&\leq [f(a)g(a)+f(b)g(b)]\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1})h(t^{\alpha_2})dt\\
&+m_1m_2\left[f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)+f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\right]\\
&\times\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t^{\alpha_1})h(1-t^{\alpha_2})dt+\left\{m_2\left[f(a)g\left(\frac{b}{m_2}\right)+f(b)g\left(\frac{a}{m_2}\right)\right]\right.\\
&\left.+m_1\left[g(a)f\left(\frac{b}{m_1}\right)+g(b)f\left(\frac{a}{m_1}\right)\right]\right\}\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_2})h(1-t^{\alpha_1})dt.
\end{aligned}$$

Proof. Since function f is $(\alpha_1, h-m_1)-$convex and function g is $(\alpha_2, h-m_2)-$convex, therefore for $t \in [0,1]$, we have

$$\begin{aligned}
&f(ta+(1-t)b)g(ta+(1-t)b)\\
&\leq h(t^{\alpha_1})h(t^{\alpha_2})f(a)g(a)+m_2f(a)g\left(\frac{b}{m_2}\right)h(t^{\alpha_1})h(1-t^{\alpha_2})\\
&+m_1g(a)f\left(\frac{b}{m_1}\right)h(t^{\alpha_2})h(1-t^{\alpha_1})\\
&+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)h(1-t^{\alpha_1})h(1-t^{\alpha_2}).
\end{aligned}$$

By multiplying both sides of the above inequality with $t^{\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, we will have

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}-1}f(ta+(1-t)b)g(ta+(1-t)b)dt\\
&\leq f(a)g(a)\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1})h(t^{\alpha_2})dt+m_2f(a)g\left(\frac{b}{m_2}\right)\\
&\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1})h(1-t^{\alpha_2})dt+m_1g(a)f\left(\frac{b}{m_1}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_2})h(1-t^{\alpha_1})dt\\
&+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t^{\alpha_1})h(1-t^{\alpha_2})dt.
\end{aligned}$$

By substituting $g = ta + (1-t)b$ in left side of above inequality we will get

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}} I_{a^+}^{\alpha,k} f(b)g(b) \qquad (4.8)\\
&\leq f(a)g(a)\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1})h(t^{\alpha_2})dt\\
&+ m_2 f(a) g\left(\frac{b}{m_2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1})h(1-t^{\alpha_2})dt\\
&+ m_1 g(a) f\left(\frac{b}{m_1}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_2})h(1-t^{\alpha_1})dt\\
&+ m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t^{\alpha_1})h(1-t^{\alpha_2})dt,
\end{aligned}$$

and similarly, changing the roles of a and b, after a little computation one can get

$$\begin{aligned}
&\frac{k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}} I_{a^+}^{\alpha,k} f(a)g(a) \qquad (4.9)\\
&\leq f(b)g(b)\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1})h(t^{\alpha_2})dt\\
&+ m_2 f(b) g\left(\frac{a}{m_2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_1})h(1-t^{\alpha_2})dt\\
&+ m_1 g(b) f\left(\frac{a}{m_1}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h(t^{\alpha_2})h(1-t^{\alpha_1})dt\\
&+ m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}h(1-t^{\alpha_1})h(1-t^{\alpha_2})dt.
\end{aligned}$$

Adding (4.8) and(4.9), we get the required result. □

If we put $k = 1$, $h(t) = t$ this theorem gives [25, Theorem 12], which is stated in the following corollary.

Corollary 4.7. *Let $f, g : [0,\infty) \to [0,\infty)$, be functions such that $fg \in L_1[a,b], a, b \in [0,\infty)$, $a < b$. If f is $(\alpha_1, m_1)-$convex and g is $(\alpha_2, m_2) -$ convex on $[a,b]$ with*

$(\alpha_1, m_1), (\alpha_2, m_2) \in (0,1]^2$ *respectively, then we will have*

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} I^{\alpha}_{a+} f(b)g(b) \qquad (4.10)\\
&\le \frac{1}{\alpha_1+\alpha_2+\alpha} f(a)g(a) + \frac{\alpha_1}{(\alpha+\alpha_1)(\alpha_1+\alpha_2+\alpha)} m_2 f(a) g\left(\frac{b}{m_2}\right)\\
&+ \frac{\alpha_1}{(\alpha+\alpha_1)(\alpha_1+\alpha_2+\alpha)} m_1 g(a) f\left(\frac{b}{m_1}\right)\\
&+ \left(\frac{1}{\alpha} - \frac{1}{\alpha+\alpha_1} - \frac{1}{\alpha+\alpha_2} + \frac{1}{\alpha_1+\alpha_2+\alpha}\right) m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} I^{\alpha}_{b-} f(a)g(a) \qquad (4.11)\\
&\le \frac{1}{\alpha_1+\alpha_2+\alpha} f(b)g(b) + \frac{\alpha_1}{(\alpha+\alpha_1)(\alpha_1+\alpha_2+\alpha)} m_2 f(b) g\left(\frac{b}{m_2}\right)\\
&+ \frac{\alpha_1}{(\alpha+\alpha_1)(\alpha_1+\alpha_2+\alpha)} m_1 g(b) f\left(\frac{b}{m_1}\right)\\
&+ \left(\frac{1}{\alpha} - \frac{1}{\alpha+\alpha_1} - \frac{1}{\alpha+\alpha_2} + \frac{1}{\alpha_1+\alpha_2+\alpha}\right) m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right).
\end{aligned}$$

Proof. From (4.8) for $k=1$ and $h(t)=t$, we get (4.10). Similarly, using $k=1$ and $h(t)=t$ in (4.9), we get (4.11). □

Conclusion

In this study some of the general versions of Hadamard inequality are analyzed in fractional calculus. A generalization of convex functions; namely $(h-m)-$convex function is used to establish these results. Some identities have been established which are further utilized in the formation of Hadamard type inequalities. Furthermore, Hadamard type inequalities for product of two $(h-m)-$convex functions have been studied and connection with already published results is investigated.

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