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A shifted Chebyshev-Tau method for finding a time-dependent heat source in heat equation

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Abstract This paper investigates the inverse problem of determining the time-dependent heat source and the temperature for the heat equation with Dirichlet boundary conditions and an integral over-determination conditions. The numerical method is presented for solving the Inverse problem. Shifted Chebyshev polynomial is used to approximate the solution of the equation as a base of the tau method which is based on the Chebyshev operational matrices. The main advantage of this method is based upon reducing the partial differential equation into a system of algebraic equations of the solution. Numerical results are presented and discussed.

Keywords. Inverse source problem, Heat equation, Shifted Chebyshev polynomial, Operational matrix, Shifted Chebyshev-Tau method .

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1. Introduction

The inverse problems for heat equation arise in many physical and engineering problems, and they can be roughly divided into three principal classes.

(1) Backward or reversed-time problem: the initial condition is to be found [16, 26].
(2) Coefficient inverse problem: this is a classical parameter problem where a multiplier in the governing equation is to be found [1, 5, 8, 11, 17, 22, 24, 25, 27].
(3) Boundary inverse problem: some missing information at the boundary of the domain is to be found [10, 18, 19, 23].

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In this paper, we consider the following heat equation:

$$u_t(x,t) = u_{xx}(x,t) + p(t)u(x,t) + q(x,t), \quad 0 < x < L, \quad 0 < t \leq \tau, \tag{1.1}$$

with initial condition

$$u(x,0) = f(x), \quad 0 < x < L, \tag{1.2}$$

and the Dirichlet boundary conditions

$$u(0,t) = g_1(t), \quad 0 < t \leq \tau, \tag{1.3}$$

$$u(L,t) = g_2(t), \quad 0 < t \leq \tau, \tag{1.4}$$

subject to the integral over-specification of the function $k(x)\,u(x,t)$ over the spatial domain (energy over-specification)

$$\int_0^1 k(x)\,u(x,t)\,dx = E(t), \quad 0 \leq t \leq \tau \tag{1.5}$$

where $q(x,t), f(x), g_1(t), g_2(t), k(x), E(t) \neq 0$ are given functions. Also, it is assumed that for constant $\rho > 0$ the kernel $k(x)$ satisfies

$$\int_0^1 |k(x)|\,dx \leq \rho.$$

If the function $p(t)$ is known, the problem of finding $u(x,t)$ from (1.1)-(1.4) is called the direct problem. However, the problem here is that the source parameter $p(t)$ is unknown, which needs to be determined by energy condition (1.5). This problem (1.1)–(1.5) is called the inverse problem.

There is a fundamental difference between the direct and inverse problems. It is known that an inverse problem is not well posed in general while the direct problem is well posed. The existence and uniqueness of this inverse problem are discussed in [4, 6, 13, 15, 17, 20]. In [6, 7, 11, 17, 21] the solution of this problem and similar problems are investigated. Some numerical methods are presented in [11, 17] for solving this problem. In [17], the author used the high order scheme for the solution of inverse problem (1.1)–(1.5). Also, the numerical methods suggested in [11] are based on the optimal homotopy analysis method (OHAM) is designed.

The aim of this research is presenting a numerical method for solving Equations (1.1)–(1.5) by using shifted Chebyshev Tau method. The main idea of the current work is to apply the shifted Chebyshev polynomials, the operational matrix of derivative and integration together with the tau method to get a linear system of algebraic equations thus greatly simplifying the problem. Moreover, we have applied the proposed algorithm to the numerical example in order to confirm the accuracy of this algorithm. The current article is organized as follows: In the next section we will introduce some necessary definitions, and give some relevant properties of shifted Chebyshev polynomials. Section 3 summarizes the application of the shifted Chebyshev-tau method to offer solution for the problems (1.1)–(1.5). As a result, a system of algebraic equations is formed and the solution of the considered problem is introduced. In Section 4, numerical results are given to clarify the method. Finally, a conclusion is given in Section 5.

2. Properties of shifted Chebyshev polynomials

The well-known shifted Chebyshev polynomials are defined on the interval $[0, 1]$ and can be determined with the aid of the following recurrence formulae:

$$T_{L,0}(x) = 1, \quad T_{L,1}(x) = \frac{2x}{L} - 1,$$
$$T_{L,j}(x) = 2\left(\frac{2x}{L} - 1\right) T_{L,j-1}(x) - T_{L,j-2}(x), \quad j = 2, 3, ..., n. \tag{2.1}$$

The following formula for the jth degree of $T_{L,j}(x)$

$$T_{L,j}(x) = j \sum_{k=0}^{j} (-1)^{j-k} \frac{(j+k-1)!2^{2k}}{(j-k)!\,(2k)!L^k} x^k, j = 1, 2, 3, ..., n, \tag{2.2}$$

where $T_{L,j}(0) = (-1)^j$ and $T_{L,j}(L) = 1$.

The orthogonality condition is

$$\int_0^L T_{L,j}(x)\, T_{L,k}(x)\, w_L(x)\, dx = h_j, \tag{2.3}$$

where

$$w_L(x) = \frac{1}{\sqrt{Lx - x^2}}, \tag{2.4}$$

and

$$h_j = \begin{cases} \frac{\varepsilon_j}{2}\pi & , k = j, \\ 0 & , k \neq j, \end{cases} \quad \varepsilon_0 = 2, \varepsilon_j = 1; \quad j \geq 1. \tag{2.5}$$

A function $u(x, t)$ of two independent variables defined for $0 < x < L, 0 < t \leq \tau$ may be expanded into the shifted Chebyshev polynomials as:

$$u(x, t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} T_{\tau,i}(t)\, T_{L,j}(x). \tag{2.6}$$

If the infinite series in (2.6) is truncated, then it can be written as:

$$u_{m,n}(x, t) \simeq \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} T_{\tau,i}(t)\, T_{L,j}(x) = \psi^T(t)\, A\phi(x), \tag{2.7}$$

where the shifted Chebyshev vectors $\psi(t)$ and $\phi(x)$ and the shifted Chebyshev coefficient matrix A are given as:

$$\psi(t) = [T_{\tau,0}(t), T_{\tau,1}(t), ..., T_{\tau,m}(t)]^T,$$
$$\phi(x) = [T_{L,0}(x), T_{L,1}(x), ..., T_{L,n}(x)]^T, \tag{2.8}$$
$$A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m0} & a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

where

$$a_{ij} = \frac{1}{h_i h_j} \int_0^{\tau} \int_0^{L} u(x,t)\, T_{\tau,i}(t)\, T_{L,j}(x)\, w_{\tau}(t)\, w_L(x)\, dx dt, \tag{2.9}$$
$$i = 0,1,...,m, \qquad j = 0,1,...,n.$$

Theorem 2.1. *The first derivative of the shifted Chebyshev vector $\phi(x)$ may be expressed by* [2, 3, 9, 14]

$$\frac{d\phi(x)}{dx} = D^{(1)} \phi(x), \tag{2.10}$$

where $D^{(1)}$ is the $(n+1) \times (n+1)$ operational matrix of derivative given by

$$D^{(1)} = d_{ij} = \begin{cases} \frac{4i}{\varepsilon_j L} & j = i-k, \begin{cases} k = 1,3,...,n & \text{if } (n) \text{ is odd} \\ k = 1,3,...,n-1 & \text{if } (n) \text{ is even} \end{cases} \\ 0 & \text{otherwise} \end{cases} \tag{2.11}$$

where $\varepsilon_0 = 2, \quad \varepsilon_j = 1, j \geq 1$.

For example, for odd n given as:

$$D = \frac{2}{L} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 4 & 0 & \cdots & 0 & 0 & 0 \\ 3 & 0 & 6 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 2(n-1) & 0 & \cdots & 2(n-1) & 0 & 0 \\ n & 0 & 2n & \cdots & 0 & 2n & 0 \end{bmatrix}$$

and for even n given as:

$$D = \frac{2}{L} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 4 & 0 & \cdots & 0 & 0 & 0 \\ 3 & 0 & 6 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n-1 & 0 & 2(n-1) & \cdots & 2(n-1) & 0 & 0 \\ 0 & 2n & 0 & \cdots & 0 & 2n & 0 \end{bmatrix}$$

Remark 2.2. The operational matrix for the nth derivative can be derived as [7, 9]

$$\frac{d^n \phi(x)}{dx^n} = \left(D^{(1)}\right)^n \phi(x), \tag{2.12}$$

where $n \in N$ and the superscript in $D^{(1)}$, denotes matrix powers. Thus

$$D^n = \left(D^{(1)}\right)^n, n = 1,2,... \tag{2.13}$$

Theorem 2.3. *The integration of $\psi_{\tau,m}(t)$ may be written as [23, 25]*

$$\int_0^t \psi(t')\,dt' \simeq P\psi(t), \tag{2.14}$$

where P is the $(m+1)\times(m+1)$ shifted Chebyshev operational matrix of integration and is given by

$$p = \begin{bmatrix} w_0 & \delta_0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ w_1 & 0 & \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ w_2 & \delta_2 & 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ w_3 & 0 & \delta_3 & 0 & \lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ w_{m-2} & 0 & 0 & 0 & \ddots & \ddots & \lambda_{m-2} & 0 \\ w_{m-1} & 0 & 0 & 0 & 0 & \ddots & 0 & \lambda_{m-1} \\ w_m & 0 & 0 & 0 & 0 & \cdots & \delta_m & 0 \end{bmatrix}, \tag{2.15}$$

where

$$w_k = \begin{cases} \frac{\tau}{2} & ,k=0 \\ \frac{-\tau}{8} & ,k=1 \\ \frac{(-1)^{k+1}\tau}{2(k-1)(k+1)} & ,k=2,3,... \end{cases}, \delta_k = \begin{cases} \frac{\tau}{2} & ,k=0 \\ 0 & ,k=1 \\ \frac{-\tau}{4(k-1)} & ,k=2,3,... \end{cases},$$
$$\lambda_k = \begin{cases} 0 & ,k=0 \\ \frac{\tau}{8} & ,k=1 \\ \frac{\tau}{4(k+1)} & ,k=2,3,... \end{cases} \tag{2.16}$$

Obviously similar to (2.14) we have

$$\int_0^x \phi(x')\,dx' \simeq G\phi(x), \tag{2.17}$$

where G is the $(n+1)\times(n+1)$ shifted Chebyshev operational matrix of integration and is defined similar to (2.15).

3. The numerical scheme

In this section, we will use the tau approximation together with the shifted Chebyshev operational matrix for solving inverse parabolic problems (1.1)-(1.5). We approximate $u(x,t)$, $q(x,t)$ and $f(x)$ by using the shifted Chebyshev operational matrix as:

$$u_{m,n}(x,t) = \psi^T(t) A\phi(x), \tag{3.1}$$

$$q_{m,n}(x,t) \simeq \sum_{i=0}^{m}\sum_{j=0}^{n} q_{ij} T_{\tau,i}(t) T_{L,j}(x) = \psi^T(t) Q\phi(x),$$

$$f(x) \simeq \sum_{j=0}^{n} f_j T_{L,j}(x) = \psi^T(t) F\phi(x),$$

where A is an unknown $(m+1)\times(n+1)$ matrix, Q and F are known $(m+1)\times(n+1)$ matrices, as

$$Q = \begin{bmatrix} q_{00} & q_{01} & \cdots & q_{0n} \\ q_{01} & q_{11} & \cdots & q_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ q_{m0} & q_{m1} & \cdots & q_{mn} \end{bmatrix}, \quad F = \begin{bmatrix} f_0 & f_1 & \cdots & f_{n-1} & f_n \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{3.2}$$

where

$$q_{ij} = \frac{1}{h_i h_j}\int_0^\tau \int_0^L q(x,t) T_{\tau,i}(t) T_{L,j}(x) w_\tau(t) w_L(x)\, dx dt, \tag{3.3}$$
$$i = 0,1,...,m, \quad j = 0,1,...,n$$

and

$$f_j = \frac{1}{h_j}\int_0^L f(x) T_{L,j}(x) w_L(x)\, dx, \quad j = 0,1,...,n. \tag{3.4}$$

Integrating equation (1.1) from 0 to t and using equation (1.2) (see [7, 9]), we have

$$u(x,t) - f(x) = \int_o^t u_{xx}(x,t')\, dt' + \int_o^t p(t') u(x,t')\, dt' + \int_o^t q(x,t')\, dt'. \tag{3.5}$$

Using equations (2.7), (2.12) and (2.14) we get

$$\int_o^t u_{xx}(x,t')\, dt' = \left(\int_o^t \psi^T(t')\, dt'\right) A \left(\frac{d^2\phi(x)}{dx^2}\right)$$
$$= \psi^T(t) P^T A D^2 \phi(x). \tag{3.6}$$

The function $p(t)$ may be expanded in terms of $m+1$ shifted Chebyshev series as

$$p(t) = \sum_{k=0}^{m} b_k T_{\tau,k}(t) = B^T \psi(t), \tag{3.7}$$

where $B = [b_0, b_1, ..., b_m]^T$ is an unknown vector.
Now, using equations (2.5), (2.12) and (3.7) we have

$$\int_o^t p(t') u(x,t')\, dt' = \left(\int_o^t B^T \psi(t') \psi^T(t')\, dt'\right) A\phi(x). \tag{3.8}$$

Let

$$B^T \psi(t) \psi^T(t) = \psi^T(t) H, \tag{3.9}$$

where H is an $(m+1)\times(m+1)$ matrix. To find H, we rewrite equation (3.9) (see[23]) in the form

$$\sum_{k=0}^{m} b_k T_{\tau,k}(t)\, T_{\tau,j}(t) = \sum_{k=0}^{m} H_{kj} T_{\tau,k}(t)\,, j = 0,1,...,m. \tag{3.10}$$

Multiplying the both sides of (3.10) by $T_{\tau,i}(t)\, w_\tau(t)$, $i = 0,1,...,m$ and integrating from 0 to τ yield

$$\begin{aligned}&\sum_{k=0}^{m} b_k \int_0^\tau T_{\tau,i}(t)\, T_{\tau,k}(t)\, T_{\tau,j}(t)\, w_\tau(t)\, dt\\ &\quad= \sum_{k=0}^{m} H_{kj} \int_0^\tau T_{\tau,k}(t)\, T_{\tau,i}(t)\, w_\tau(t)\, dt, i,j = 0,1,...,m.\end{aligned} \tag{3.11}$$

Using equation (3.11) and employing the orthogonality relation (2.3) give

$$\sum_{k=0}^{m} b_k \int_0^\tau T_{\tau,i}(t)\, T_{\tau,k}(t)\, T_{\tau,j}(t)\, w_\tau(t)\, dt = H_{ij} h_i,$$

or equivalently

$$H_{ij} = \frac{1}{h_i} \sum_{k=0}^{m} b_k \int_0^\tau T_{\tau,i}(t)\, T_{\tau,k}(t)\, T_{\tau,j}(t)\, w_\tau(t)\, dt, \quad i,j = 0,1,...,m. \tag{3.12}$$

Employing equations (2.14), (3.8) and equation (3.9) can be written as

$$\int_o^t p(t')\, u(x,t')\, dt' = \psi^T(t)\, P^T H A \phi(x)\,. \tag{3.13}$$

Also by using equations (2.7), (2.14) and (3.1) (see [23]), we get

$$\int_o^t q(x,t')\, dt' = \left(\int_0^t \psi^T(t')\, dt'\right) Q\phi(x) = \psi^T(t)\, P^T Q \phi(x)\,. \tag{3.14}$$

Applying equations (2.7), (3.1), (3.6), (3.13) and (3.14) the residual $R_{m,n}(x,t)$ for equation (3.5) can be written as

$$R_{m,n}(x,t) = \psi^T(t)\left[A - F - P^T H A - P^T A D^2 - P^T Q\right]\phi(x) = 0.$$

Let

$$Z = \left[A - F - P^T H A - P^T A D^2 - P^T Q\right],$$

then we have

$$\psi^T(t)\, Z \phi(x) = 0. \tag{3.15}$$

As in a typical Tau method, we generate $(m+1)\times(n-1)$ linear algebraic equations using the following algebraic equations

$$Z_{ij} = 0,\ i = 0,1,...,m,\ j = 0,1,...,n-2. \tag{3.16}$$

Also, by substituting equations (3.1) and (3.7) in equations (1.3)-(1.4) we get

$$\psi^T(t)\, A\phi(0) = g_1(t), \tag{3.17}$$

$$\psi^T(t)\, A\phi(L) = g_2(t). \tag{3.18}$$

And applying (3.1) in equation (1.5), we have

$$\int_0^1 k(x)\, u(x,t)\, dx = \psi^T(t)\, A\left(\int_0^1 k(x)\, \phi(x)\, dx\right) = E(t). \tag{3.19}$$

Let's assume that

$$I = \int_0^1 k(x)\, \phi(x)\, dx, \tag{3.20}$$

and

$$I = [I_0, I_1, I_2, ..., I_n]^T, i = 0, 1, 2, ..., n \tag{3.21}$$

with

$$I_i = \int_0^1 k(x)\, \phi_i(x)\, dx, \tag{3.22}$$

that

$$k(x) \simeq \sum_{j=0}^{n} k_j T_{L,j}(x) = \psi^T(t)\, K\phi(x), \tag{3.23}$$

and K is known $(m+1)\times(n+1)$ matrix below

$$K = \begin{bmatrix} k_0 & k_1 & \cdots & k_{n-1} & k_n \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

that is

$$k_j = \frac{1}{h_j}\int_0^L k(x)\, T_{L,j}(x)\, w_L(x)\, dx. \tag{3.24}$$

Now applying equations (3.20)-(3.24) in equation (3.19), we have

$$\psi^T(t)\, AI = E(t). \tag{3.25}$$

Equations (3.17), (3.18) and (3.25) are collocated at $m+1$ points. For suitable collocation points, we use the shifted Chebyshev roots t_i , $i = 1, 2, ..., m+1$ of $T_{\tau,m+1}(t)$. The number of the unknown coefficients $a_{ij}, i = 0, 1, ..., m, j = 0, 1, ..., n$ and b_k, $k = 0, 1, ..., m$ is equal to $(m+1)(n+1) + (m+1)$ and can be obtained from equations (3.16)-(3.18) and (3.25). Consequently $u(x,t)$ given in equation (2.7) and $p(t)$ given in equation (3.7) can be calculated.

FIGURE 1. Plot of error for $|u_{m,n}(x,1) - u(x,1)|$ with $m = n = 4, 8$.

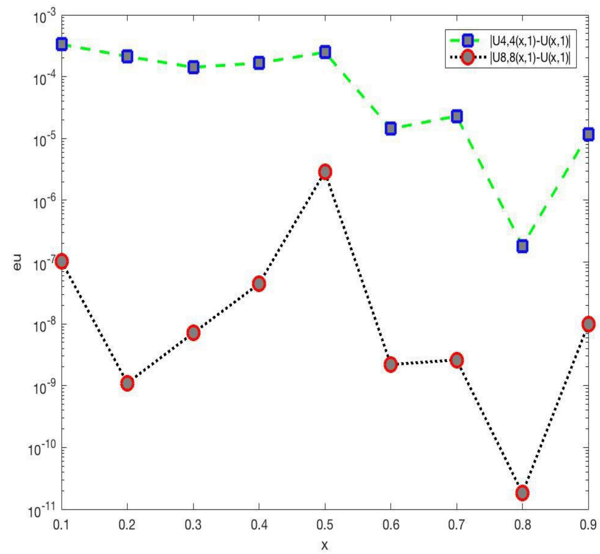
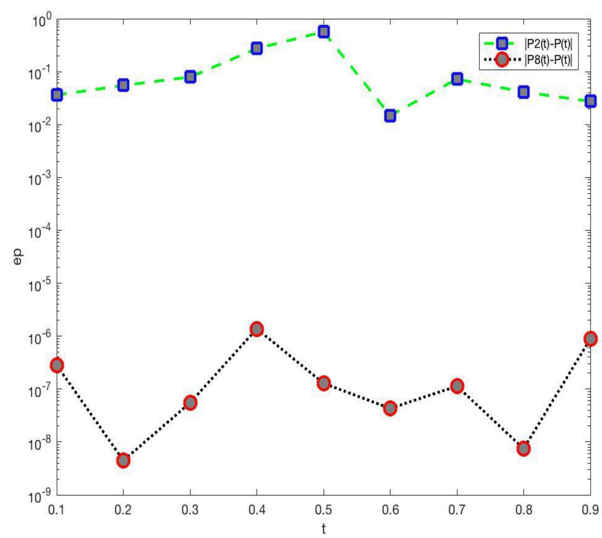


FIGURE 2. Plot of error for $|p_m(t) - p(t)|$ with $m = n = 2, 8$.



4. Numerical results

In this section, we illustrate the use of our algorithm by displaying the results obtained from its application to a test problem. In this case the exact solution $u(x,t)$ and $p(t)$ to the problem are known, we will report the accuracy and efficiency of the

FIGURE 3. Plot of error $|u_{m,n}(x,1) - u(x,1)|$ for $x = 0.1$ and $x = 0.9$ with various value of $m = n$.

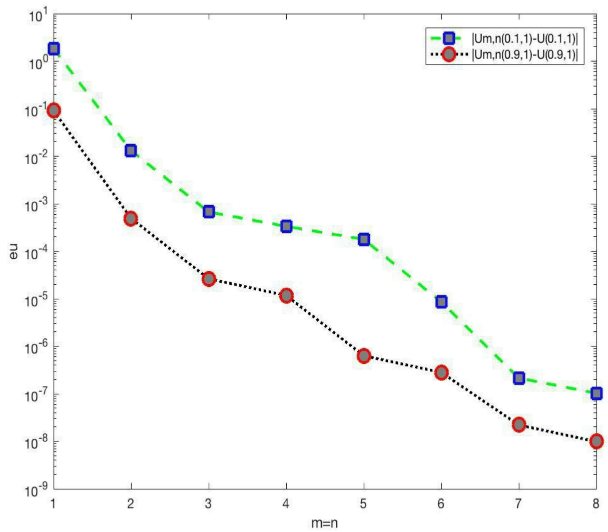
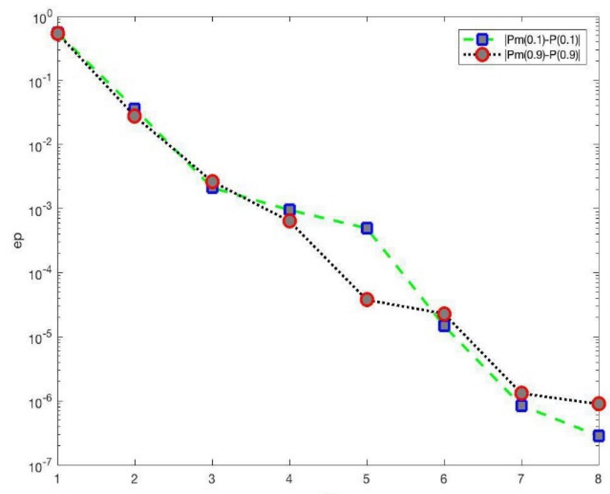


FIGURE 4. Plot of error $|p_m(t) - p(t)|$ for $t = 0.1$ and $t = 0.9$ with various value of m.



shifted chebyshev-Tau method based on absolute errors e_u and e_p defined as:

$$e_u = |u_{m,n}(x,t) - u(x,t)|, e_p = |p_m(t) - p(t)|$$

.

Example 4.1. The inverse problem (1.1)–(1.5) with the following conditions:

$$\begin{aligned}
&\tau = 1, L = 1,\\
&q(x,t) = \exp(t)\left(x + cos(\pi x) + \pi^2 cos(\pi x)\right) - \exp(t)\left(1+t^2\right)(x + cos(\pi x)),\\
&f(x) = x + cos(\pi x),\\
&g_1(t) = \exp(t),\\
&g_2(t) = 0,\\
&E(t) = \exp\left(\tfrac{3}{4} - \tfrac{2}{\pi^2}\right),\\
&k(x) = 1 + x^2
\end{aligned}$$

The exact solution of the problem is $u(x,t) = \exp(t)(x + \cos \pi x)$ and $p(t) = 1 + t^2$, see [17, 11].

We solved the problem by applying the method described in Section 3. We report the absolute error of $|u_{m,n}(x,1) - u(x,1)|$ and $|p_m(t) - p(t)|$ for $m = n = 4, 6, 8$ in Tables 1 and 2, respectively. Figure 1 shows the absolute error function $|u_{m,n}(x,1) - u(x,1)|$ at different space for $m = n = 4, 8$. In addition, Figure 2 shows the absolute error function $|p_m(t) - p(t)|$ at different time for $m = n = 2, 8$. Also, Figure 3 and Figure 4 illustrate the numerical results of the error function $|u_{m,n}(x,1) - u(x,1)|$ and $|p_m(t) - p(t)|$ by increasing of m and n in 0.1 and 0.9.

The obtained results showed that this approach can solve the problem effectively. The described computational method produces very accurate results even when employing a small number of collocation points.

TABLE 1. Results for $u(x,1)$ and the absolute error $|u_{m,n}(x,1) - u(x,1)|$ form Example.

x	exact		error	
	$u(x,1)$	$m=n=4$	$m=n=6$	$m=n=8$
0.1	2.8571e+00	3.3395e-04	8.5616e-06	1.0265e-07
0.2	2.7428e+00	2.1271e-04	3.3107e-08	1.1036e-09
0.3	2.4133e+00	1.4184e-04	1.7740e-06	7.0921e-09
0.4	1.9273e+00	1.6643e-04	3.7018e-06	4.4484e-08
0.5	1.3591e+00	2.4769e-04	1.7070e-05	2.8897e-06
0.6	7.9097e-01	1.4388e-05	2.4996e-07	2.2013e-09
0.7	3.0503e-01	2.2960e-05	2.4139e-07	2.6079e-09
0.8	-2.4511e-02	1.8292e-07	5.9385e-10	1.8227e-11
0.9	-1.3879e-01	1.1727e-05	2.8145e-07	9.8509e-09

5. Conclusion

In this article, the inverse problem of finding the time-dependent heat source and the temperature for the heat equation, under the Dirichlet boundary condition and the integral over-specification of the function $k(x)u(x,t)$ have been investigated. An efficient direct solver method is developed for solving such problems using the shifted Cheyshev-Tau method. The construction of the proposed algorithm is based on the Tau approximation in addition to the shifted Chebyshev operational matrix.

TABLE 2. Results for $p(t)$ and absolute error $|p_m(t) - p(t)|$ form Example.

t	exact $p(t)$	error $m=4$	error $m=6$	error $m=8$
0.1	1.0100e+00	9.5029e-04	1.4864e-05	2.8458e-07
0.2	1.0400e+00	6.8131e-04	1.3393e-07	4.4644e-09
0.3	1.0900e+00	1.1442e-03	1.3743e-05	5.4910e-08
0.4	1.1600e+00	3.2483e-03	4.0245e-04	1.3798e-06
0.5	1.2500e+00	1.5113e-02	7.5872e-04	1.2867e-07
0.6	1.3600e+00	1.1035e-03	2.1054e-05	4.3196e-08
0.7	1.4900e+00	1.0017e-03	8.1872e-06	1.1354e-07
0.8	1.6400e+00	7.4802e-05	2.4329e-07	7.4621e-09
0.9	1.8100e+00	6.3590e-04	2.2886e-05	8.9241e-07

Illustrative numerical example with satisfactory approximate solutions is achieved to demonstrate the accuracy of method. The numerical results in Section 4 demonstrate the good accuracy of the described method. Moreover, only a small number of shifted Chebyshev polynomials is needed to obtain a satisfactory solution.

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