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Comparing present and accumulated value of annuities with different interest rates

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Abstract Assume we have k immediate (due)-annuities with different interest rates. Let $\mathbf{i} = (i_1, i_2, ..., i_k)$ and $\mathbf{i}^* = (i_1^*, i_2^*, ..., i_k^*)$ be two vectors of interest rates such that $\mathbf{i}^*$ is majorized by $\mathbf{i}$. It's shown that sum of present and accumulated value of annuities-immediate with interest rate $\mathbf{i}$ is grater than sum of present value of annuities-immediate with interest rate $\mathbf{i}^*$. We also prove the similar results for annuities-due.

Keywords. Arithmetic mean, Majorization, Schur-convex function.

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1. Introduction

Interest rate theory is a basic concept in financial and actuarial mathematics. Actuaries and business professionals use relevant mathematical relations for modeling and calculating present and accumulated value of cash flows in specific times. An annuity is a series of payments made at equal intervals and if the payments are level it is called level annuity which in this paper, we only consider this kind of annuity. Examples of annuities are regular deposit to a savings account, payments for life insurance, pension payments, regular payments for investing in funds, loans, etc. In annuities, it is usual to calculate the value of payment, considering interest rate, at the time of beginning contract, known as present value and at the end of contract, known as accumulated value.

Let X be the amount of money which is invested for example in an invested fund with interest rate or inner internal return rate i, then accumulated value at time t (after t years) will be $AV_t = X(1+i)^t$ and if X invested at time $t' < t$, the present value of X at time 0 will be $PV_t' = X(1+i)^{-t'}$.

Annuity-immediate and annuity-due are two common annuities that appear in many kinds of financial and actuarial contracts. In the following, these annuities are defined.

Definition 1.1. Annuity-immediate is a kind of annuity in which payments of 1 are made at the end of every year for n years.

Present value of annuity-immediate with interest rate i is denoted by $a_{\bar{n}|i}$ and calculated as

$$a_{\bar{n}|i} = \frac{1-(1+i)^{-n}}{i}.$$

Accumulated value of annuity-immediate is denoted by $s_{\bar{n}|i}$ and calculated as

$$s_{\bar{n}|i} = \frac{(1+i)^n - 1}{i}.$$

Definition 1.2. Annuity-due is a kind of annuity in which payments of 1 are made at the beginning of every year for n years.

Present value of annuity-due with interest rate i is denoted by $\ddot{a}_{\bar{n}|i}$ and calculated as

$$\ddot{a}_{\bar{n}|i} = \frac{1 - (1+i)^{-n}}{d},$$

where $d = \frac{i}{1+i}$, known as discount rate. Accumulated value of annuity-due is denoted by $\ddot{s}_{\bar{n}|i}$ and calculated as

$$\ddot{s}_{\bar{n}|i} = \frac{(1+i)^n - 1}{d}.$$

Obviously, we have that

$$\ddot{a}_{\overline{n-1}|i} = 1 + a_{\bar{n}|i}, \qquad s_{\bar{n}|i} = 1 + \ddot{s}_{\overline{n-1}|i}. \tag{1.1}$$

For more details on theory of interest and annuities one may refer to [1, 2, 4].
Let for a specified vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, denote the ith smallest of x_i's by $x_{(i)}$ and any permutation of $\mathbf{x}$ by $(\mathbf{x}\Pi)$. Next we introduce the notion of majorization which is one of the basic tools in establishing various inequalities in mathematics and statistics. In [3], extensive and compressive details on the theory of majorization and its applications have been provided.

Definition 1.3. A vector $\mathbf{x} \in \mathbb{R}^\mathbf{n}$ is said to be majorized by another vector $\mathbf{y} \in \mathbb{R}^\mathbf{n}$ denoted by $\mathbf{x} \leq_\mathbf{m} \mathbf{y}$, if $\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

When the relation of majorization is hold between two vectors, they are ordered according to variation. The bigger vector by the sense of majorization is more scattered than the smaller one. For example, every vector majorizes the vector of its mean, i.e.,

$$(x_1, ..., x_n) \overset{m}{\geq} (\bar{x}, ..., \bar{x}), \tag{1.2}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

Definition 1.4. A real valued function $\phi(.)$ defined on set $\mathbb{A} \subseteq \mathbb{R}^n$ is said to be Schur-convex on $\mathbb{A}$, if $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{A}$ that $\mathbf{x} \leq_\mathbf{m} \mathbf{y}$.

Definition 1.5. A set $\mathbb{A} \subseteq \mathbb{R}^n$ is said to be symmetric, if $\mathbf{x} \in \mathbb{A}$ implies $(\mathbf{x}\Pi) \in \mathbb{A}$.

Definition 1.6. A real valued function $\phi(.)$ defined on symmetric set $\mathbb{A} \subseteq \mathbb{R}^n$ is said to be symmetric on $\mathbb{A}$, if $\phi(\mathbf{x}) = \phi(\mathbf{x}\boldsymbol{\Pi})$ for any $\mathbf{x} \in \mathbb{A}$.

To prove the main results, we use the following lemma in [3].

Lemma 1.7. *Let $\mathbb{A}$ be the set with property*

$$\mathbf{y} \in \mathbb{A} \text{ and } \mathbf{x} \leq_m \mathbf{y} \quad \text{implies } \mathbf{x} \in \mathbb{A}.$$

A continuous function ϕ defined on $\mathbb{A}$ is Schur-convex on $\mathbb{A}$ if and only if ϕ is symmetric and

$$\phi(x_1, s - x_1, x_3, ..., x_n) \qquad \text{is decreasing in } x_1 \leq \frac{s}{2},$$

for each fixed $s, x_3, ..., x_n$.

2. Main results

In this section, sums of present and accumulated values of immediate (due)-annuities will be considered. Let we have k annuities in n years with different interest rate $i_1, ..., i_k$, so the present and accumulated values of these annuities are

$$a_{\bar{n}|i_1,\ldots,i_k} = \sum_{j=1}^{k} a_{\bar{n}|i_j} \quad \text{and} \tag{2.1}$$

$$s_{\bar{n}|i_1,\ldots,i_k} = \sum_{j=1}^{k} s_{\bar{n}|i_j}, \tag{2.2}$$

respectively. Two following theorems prove the main results of this paper for annuity-immediate.

Theorem 2.1. *Let $\mathbb{I} = (0, 1)$, then $a_{\bar{n}|i_1,\ldots,i_k}$ is a Schur-convex function on I^k.*

Proof. First, we prove the theorem for case $k = 2$. Without loss of generality, assume that $i_1 = x, i_2 = c - x$ where $0 < x \leq \frac{c}{2} < 1$, so we have that

$$\begin{aligned} a_{\bar{n}|x,c-x} &= a_{\bar{n}|x} + a_{\bar{n}|c-x} \\ &= \frac{1 - (1 + x)^{-n}}{x} + \frac{1 - (1 + c - x)^{-n}}{c - x} \\ &= \sum_{l=1}^{n} (1 + x)^{-l} + \sum_{l=1}^{n} (1 + c - x)^{-l} \end{aligned} \tag{2.3}$$

Using Lemma 1.7, it's enough to show that $a_{\bar{n}|x,c-x}$ is decreasing in $0 < x \leq \frac{c}{2} < 1$. From (2.3) we have that

$$\begin{aligned} \frac{da_{\bar{n}|x,c-x}}{dx} &= \sum_{l=1}^{k} -l(1 + x)^{-(l-1)} + \sum_{l=1}^{k} l(1 + c - x)^{-(l-1)} \\ &= \sum_{i=1}^{k} l[(1 + c - x)^{-l-1} - (1 + x)^{-l-1}] \leq 0. \end{aligned}$$

This completes the proof for case $k = 2$. Now for general case, using Lemma 1.7, it is enough to show that $a_{\bar{n}|x,c-x,x_3,\ldots,x_3}$ is decreasing in $0 < x \leq \frac{c}{2} < 1$, for any arbitrary $0 < x_3 < 1$. Since $a_{\bar{n}|x,c-x,x_3,\ldots,x_3} = a_{\bar{n}|x,c-x} + a_{\bar{n}|x_3,\ldots,x_3}$, $a_{\bar{n}|x,c-x,x_3,\ldots,x_3}$ is decreasing in $0 < x \leq \frac{c}{2} < 1$, for any arbitrary $0 < x_3 < 1$. This completes the proof. □

Theorem 2.2. *Let* $\mathbb{I} = (0,1)$*, then* $s_{\bar{n}|i_1,\ldots,i_k}$ *is a Schur-convex function on* I^k*.*

Proof. We only prove the theorem for case $k = 2$. Extending the result for general case k will be similar to the proof of Theorem 2.1. Without loss of generality, assume that $i_1 = x, i_2 = c - x$ where $0 < x \leq \frac{c}{2} < 1$, so we have that

$$\begin{aligned} s_{\bar{n}|x,c-x} &= s_{\bar{n}|x} + s_{\bar{n}|c-x} \\ &= \frac{(1+x)^n - 1}{x} + \frac{(1+c-x)^n - 1}{c-x} \\ &= \sum_{l=1}^{n}(1+x)^l + \sum_{l=1}^{n}(1+c-x)^l. \end{aligned} \tag{2.4}$$

Using Lemma 1.7, it's enough to show that $s_{\bar{n}|x,c-x}$ is decreasing in $0 < x \leq \frac{c}{2} < 1$. From (2.4) we have that

$$\begin{aligned} \frac{ds_{\bar{n}|x,c-x}}{dx} &= \sum_{l=1}^{k} = l(1+x)^{l-1} + \sum_{l=1}^{k} -l(1+c-x)^{l-1} \\ &= \sum_{i=1}^{k} -l[(1+c-x)^{l-1} - (1+x)^{l-1}] \leq 0, \end{aligned}$$

and this completes the proof for case $k = 2$. □

Next theorem considers annuity-due.

Theorem 2.3. *Let* $\mathbb{I} = (0,1)$*, then* $\ddot{a}_{\bar{n}|i_1,\ldots,i_k}$ *and* $\ddot{s}_{\bar{n}|i_1,\ldots,i_k}$ *are Schur-convex functions on* I^k*.*

Proof. By combining (1.1) and (1.2), proof will be obvious. □

The following corollary can be obtained from above mentioned theorems and (1.2).

Corollary 2.4. *Let* $i_1, \ldots, i_k$ *be* k *different interest rate with arithmetic mean* $\bar{i}$*. Then,*

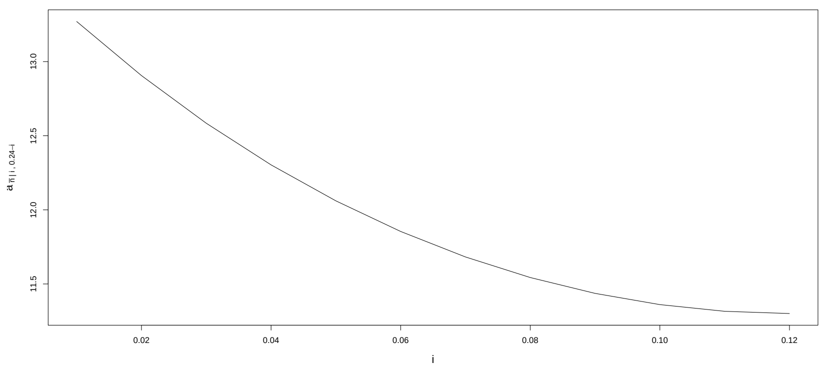
$$\begin{aligned} a_{\bar{n}|i_1,\ldots,i_k} &\geq n a_{\bar{n}|\bar{i}}, \\ s_{\bar{n}|i_1,\ldots,i_k} &\geq n s_{\bar{n}|\bar{i}}, \\ \ddot{a}_{\bar{n}|i_1,\ldots,i_k} &\geq n \ddot{a}_{\bar{n}|\bar{i}}, \\ \ddot{s}_{\bar{n}|i_1,\ldots,i_k} &\geq n \ddot{s}_{\bar{n}|\bar{i}}. \end{aligned}$$

By using the obtained inequalities in Corollary 2.4, we can compare present and accumulated value of k annuities with different interest rates with k annuities that have the same interest rate $\bar{i}$. In order to justify the result of Theorem 2.1, in Figure 1, we plot $a_{\bar{n}|i_1,i_2}$ for $\{(i_1, i_2)|i_1 + i_2 = 0.24\}$. As we expected, $a_{\bar{n}|i_1,i_2}$ is decreasing in $0 \leq i \leq 0.12$ and takes its minimum value for $(i_1 = i_2 = 0.12)$.

3. Conclusion

Given k annuities-immediate with present values $a_{\bar{n}|i_1}, \ldots, a_{\bar{n}|i_k}$ and accumulated values $s_{\bar{n}|i_1}, \ldots, s_{\bar{n}|i_k}$, we have proven that the sum of these present and accumulated values are greater than $na_{\bar{n}|\bar{i}}$ and $ns_{\bar{n}|\bar{i}}$, respectively. The simillar results also have been proven for annuities-due.

Figure 1. Sum of two level annuities with interest rates i and $0.24 - i$ for $0 \leq i \leq 0.12$ and $n = 10$.



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