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On eigenvalues of generalized shift linear vector isomorphisms

Fatemeh Ayatollah Zadeh Shirazi*
Faculty of Mathematics, Statistics and Computer Science,
College of Science, University of Tehran, Tehran, Iran.
E-mail: fatemah@khayam.ut.ac.ir

Elham Soleimani
Faculty of Mathematics, Statistics and Computer Science,
College of Science, University of Tehran, Tehran, Iran.
E-mail: elhamsolaimani73@gmail.com

Abstract Our main aim is to compute eigenvalues of generalized shift isomorphism $\sigma_\varphi : V^\Gamma \to V^\Gamma$ with $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}) = (x_{\varphi(\alpha)})_{\alpha\in\Gamma}$ $((x_\alpha)_{\alpha\in\Gamma} \in V^\Gamma)$ where V is a vector space (over field F), Γ is a nonempty arbitrary set and $\varphi : \Gamma \to \Gamma$ is an arbitrary bijection.

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1. Introduction

Let's mention that one–sided shift $\underset{(x_1,x_2,x_3,\cdots)\mapsto(x_2,x_3,x_4,\cdots)}{\{1,\ldots,k\}^{\mathbb{N}} \to \{1,\ldots,k\}^{\mathbb{N}}}$ and two–sided shift $\underset{(x_n)_{n\in\mathbb{Z}}\mapsto(x_{n+1})_{n\in\mathbb{Z}}}{\{1,\ldots,k\}^{\mathbb{Z}} \to \{1,\ldots,k\}^{\mathbb{Z}}}$ are two well-known operators in dynamical systems, ergodic theory [5], etc. For arbitrary set X with at least two elements, nonempty set Γ and $\varphi : \Gamma \to \Gamma$ generalized shift $\sigma_\varphi : \underset{(x_\alpha)_{\alpha\in\Gamma}\mapsto(x_{\varphi(\alpha)})_{\alpha\in\Gamma}}{X^\Gamma \to X^\Gamma}$ has been introduced for the first time in [2] as a generalization of one–sided and two–sided shifts. It's evident that if X has group structure, then $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is group homomorphism (see e.g., [1]) and for topological space X, $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is continuous, where X^Γ is equipped with product topology, see e.g. [3].

Convention. In the following text, suppose $V(\neq 0)$ is a linear vector space over field F, Γ is an arbitrary set with at least two elements and $\varphi : \Gamma \to \Gamma$ is an arbitrary map. Generalized shift $\sigma_\varphi : V^\Gamma \to V^\Gamma$ with $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}) = (x_{\varphi(\alpha)})_{\alpha\in\Gamma}$ is a linear vector space homomorphism since for $(x_\alpha)_{\alpha\in\Gamma}, (y_\alpha)_{\alpha\in\Gamma} \in V^\Gamma$ and $r \in F$, let $z_\alpha = x_\alpha + ry_\alpha$, now we have:

$$\begin{aligned}\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma} + r(y_\alpha)_{\alpha\in\Gamma}) &= \sigma_\varphi((z_\alpha)_{\alpha\in\Gamma}) = (z_{\varphi(\alpha)})_{\alpha\in\Gamma} \\ &= (x_{\varphi(\alpha)} + ry_{\varphi(\alpha)})_{\alpha\in\Gamma} = (x_{\varphi(\alpha)})_{\alpha\in\Gamma} + r(y_{\varphi(\alpha)})_{\alpha\in\Gamma} \\ &= \sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}) + r\sigma_\varphi((y_\alpha)_{\alpha\in\Gamma}).\end{aligned}$$

* Corresponding author.

For $\alpha, \beta \in \Gamma$, let $\alpha \sim_\varphi \beta$ or briefly $\alpha \sim \beta$ if there exist $n, m > 0$ with $\varphi^n(\alpha) = \varphi^m(\beta)$. Clearly $\sim_\varphi$ is an equivalence relation on Γ. For $\alpha \in \Gamma$ by $\frac{\alpha}{\sim_\varphi}$ we mean the equivalence class of α in $\sim_\varphi$. In other words $\frac{\alpha}{\sim_\varphi} = \bigcup\{\varphi^n(\alpha) : n \in \mathbb{Z}\}$. In particular one may consider $\varphi|_{\frac{\alpha}{\sim_\varphi}} : \frac{\alpha}{\sim_\varphi} \to \frac{\alpha}{\sim_\varphi}$.

2. Eigenvalues of generalized shifts

In this section we compute eigenvalues of homomorphism $\sigma_\varphi : V^\Gamma \to V^\Gamma$.

Lemma 2.1 (Decomposition Lemma)**.** *For $r \in F$, r is an eigenvalue of $\sigma_\varphi : V^\Gamma \to V^\Gamma$ if and only if there exists $\alpha \in \Gamma$ such that r is an eigenvalue of $\sigma_{\varphi|_{\frac{\alpha}{\sim_\varphi}}} : V^{\frac{\alpha}{\sim_\varphi}} \to V^{\frac{\alpha}{\sim_\varphi}}$. I.e.,*

$$\mathrm{Eigen}(\sigma_\varphi, V^\Gamma) = \bigcup_{\alpha\in\Gamma} \mathrm{Eigen}(\sigma_{\varphi|_{\frac{\alpha}{\sim_\varphi}}}, V^{\frac{\alpha}{\sim_\varphi}}),$$

where for vector space W and homomorphism $h : W \to W$, Eigen(h, W) *denotes the collection of all eigenvalues of $h : W \to W$.*

Proof. For $\alpha \in \Gamma$, suppose $r \in \mathrm{Eigen}(\sigma_{\varphi|_{\frac{\alpha}{\sim_\varphi}}}, V^{\frac{\alpha}{\sim_\varphi}})$. Choose $(x_\lambda)_{\lambda\in\frac{\alpha}{\sim_\varphi}} \in V^{\frac{\alpha}{\sim_\varphi}} \setminus \{0\}$ with $\varphi|_{\frac{\alpha}{\sim_\varphi}}((x_\lambda)_{\lambda\in\frac{\alpha}{\sim_\varphi}}) = r(x_\lambda)_{\lambda\in\frac{\alpha}{\sim_\varphi}}$. For $\lambda \in \Gamma \setminus \frac{\alpha}{\sim_\varphi}$ let $x_\lambda := 0$. For all $\lambda \in \Gamma \setminus \frac{\alpha}{\sim_\varphi}$ we have $\varphi(\lambda) \in \Gamma \setminus \frac{\alpha}{\sim_\varphi}$ which leads to

$$x_\lambda = x_{\varphi(\lambda)} = rx_\lambda, \quad (\lambda \in \Gamma \setminus \frac{\alpha}{\sim_\varphi}). \tag{*}$$

Using (*) and $(x_{\varphi(\lambda)})_{\lambda\in\frac{\alpha}{\sim_\varphi}} = (rx_\lambda)_{\lambda\in\frac{\alpha}{\sim_\varphi}}$ we have $\sigma_\varphi((x_\lambda)_{\lambda\in\Gamma}) = r(x_\lambda)_{\lambda\in\Gamma}$. Moreover $(x_\lambda)_{\lambda\in\Gamma} \neq 0$ since $(x_\lambda)_{\lambda\in\frac{\alpha}{\sim_\varphi}} \neq 0$. Hence r is an eigenvalue of $\sigma_\varphi : V^\Gamma \to V^\Gamma$.
Conversely, if $r \in F$ an eigenvalue of $\sigma_\varphi : V^\Gamma \to V^\Gamma$, there exist $(x_\lambda)_{\lambda\in\Gamma} \in V^\Gamma \setminus \{0\}$ such that $\sigma_\varphi((x_\lambda)_{\lambda\in\Gamma}) = r(x_\lambda)_{\lambda\in\Gamma}$. Choose $\alpha \in \Gamma$ such that $x_\alpha \neq 0$. Using $\sigma_{\varphi|_{\frac{\alpha}{\sim_\varphi}}}((x_\lambda)_{\lambda\in\frac{\alpha}{\sim_\varphi}}) = r(x_\lambda)_{\lambda\in\frac{\alpha}{\sim_\varphi}}$, we have $r \in \mathrm{Eigen}(\sigma_{\varphi|_{\frac{\alpha}{\sim_\varphi}}}, V^{\frac{\alpha}{\sim_\varphi}})$ which completes the proof. □

Remark 2.2. We have the following statements [2]:
- the mapping $\sigma_\varphi : V^\Gamma \to V^\Gamma$ is injective if and only if $\varphi : \Gamma \to \Gamma$ is surjective;
- the mapping $\sigma_\varphi : V^\Gamma \to V^\Gamma$ is bijective if and only if $\varphi : \Gamma \to \Gamma$ is injective.

Corollary 2.3. *By Remark 2.2, 0 is an eigenvalue of $\sigma_\varphi : V^\Gamma \to V^\Gamma$ if and only if $\varphi : \Gamma \to \Gamma$ is not surjective.*

Definition 2.4. For $\alpha, \beta \in \Gamma$ let:

$$m(\alpha,\beta) = \begin{cases} \min\{n > 0 : \varphi^n(\alpha) = \beta\}, & \exists n > 0(\varphi^n(\alpha) = \beta), \\ 0, & \text{otherwise}\,. \end{cases}$$

It's evident that for distinct $\alpha, \beta \in \Gamma$ we have $m(\alpha,\beta) + m(\beta,\alpha) > 0$ if and only if $\beta \in \frac{\alpha}{\sim_\varphi}$.

Definition 2.5. For $\alpha \in \Gamma$ we say:
- α is a periodic point of φ if there exists $n \geq 1$ with $\varphi^n(\alpha) = \alpha$,
- α is a quasi–periodic point of φ if there exist $n > m \geq 1$ with $\varphi^n(\alpha) = \varphi^m(\alpha)$.

By $W(\Gamma, \varphi)$ we mean the collection of all non–quasi–periodic points of $\varphi : \Gamma \to \Gamma$. Consequently $Q(\Gamma, \varphi)$ denotes the set of all quasi–periodic points of $\varphi : \Gamma \to \Gamma$ and by $P(\Gamma, \varphi)$ we mean the collection of all periodic points of $\varphi : \Gamma \to \Gamma$.

Remark 2.6. For $\alpha \in \Gamma$ we have:
- $\alpha \in Q(\Gamma, \varphi)$ if and only if $\frac{\alpha}{\sim_\varphi} \subseteq Q(\Gamma, \varphi)$.
- $\alpha \in W(\Gamma, \varphi)$ if and only if $\frac{\alpha}{\sim_\varphi} \subseteq W(\Gamma, \varphi)$.

In particular by $W(\Gamma,\varphi) \cap Q(\Gamma,\varphi) = \emptyset$ and $W(\Gamma,\varphi) \cup Q(\Gamma,\varphi) = \Gamma$, hence if $\sim_\varphi = \Gamma \times \Gamma$, then $\Gamma = W(\Gamma, \varphi)$ or $\Gamma = Q(\Gamma, \varphi)$.

Proof. Consider $\alpha, \beta \in \Gamma$ with $\beta \in \frac{\alpha}{\sim_\varphi}$, so there exist $n, m > 1$ such that $\varphi^n(\alpha) = \varphi^m(\beta)$. Now we have the following cases:
- Case 1: $\alpha \in Q(\Gamma, \varphi)$. In this case there exists $p > q \geq 1$ with $\varphi^p(\alpha) = \varphi^q(\alpha)$, thus:

$$\varphi^{p+m}(\beta) \overset{\varphi^m(\beta)=\varphi^n(\alpha)}{=} \varphi^{p+n}(\alpha) \overset{\varphi^p(\alpha)=\varphi^q(\alpha)}{=} \varphi^{p+n}(\alpha) \overset{\varphi^n(\alpha)=\varphi^m(\beta)}{=} \varphi^{p+n}(\beta)$$

and $\beta \in Q(\Gamma, \varphi)$.
- Case 2: $\alpha \in W(\Gamma, \varphi)$. If $\beta \notin W(\Gamma, \varphi) = \Gamma \setminus Q(\Gamma, \varphi)$, then $\beta \in Q(\Gamma, \varphi)$ and using Case 1 we have $\alpha \in Q(\Gamma, \varphi)$ which leads to contradiction $\alpha \in Q(\Gamma, \varphi) \cap W(\Gamma, \varphi) = \emptyset$. Hence $\beta \in W(\Gamma, \varphi)$. □

Lemma 2.7. *If $\varphi : \Gamma \to \Gamma$ is bijective and $\sim_\varphi = \Gamma \times \Gamma$ with $W(\Gamma, \varphi) \neq \emptyset$, then $F \setminus \{0\} = \mathrm{Eigen}(\sigma_\varphi, V^\Gamma)$.*

Proof. Suppose $\varphi : \Gamma \to \Gamma$ is bijective and $\sim_\varphi = \Gamma \times \Gamma$ with $W(\Gamma, \varphi) \neq \emptyset$. By Corollary 2.3 we have $0 \notin \mathrm{Eigen}(\sigma_\varphi, V^\Gamma)$ and $\mathrm{Eigen}(\sigma_\varphi, V^\Gamma) \subseteq F \setminus \{0\}$.
Choose $\theta \in W(\Gamma, \varphi) \neq \emptyset$, $x \in V \setminus \{0\}$ and $r \in F \setminus \{0\}$. Since $\sim_\varphi = \Gamma \times \Gamma$ we have $\Gamma = \frac{\theta}{\sim_\varphi}$, moreover by Remark 2.6 we have $\frac{\theta}{\sim_\varphi} \subseteq W(\Gamma, \varphi)$. As a matter of fact $\Gamma = \{\varphi^n(\theta) : n \in \mathbb{Z}\}$. Let:

$$x_\alpha = \begin{cases} r^{m(\theta,\alpha)}x, & m(\theta,\alpha) > 0\,, \\ r^{-m(\alpha,\theta)}x, & m(\alpha,\theta) > 0\,, \\ x, & \alpha = \theta\,, \end{cases}$$

and in brief words:

$$x_\alpha = r^n x \quad (\varphi^n(\theta) = \alpha, n \in \mathbb{Z})\,.$$

For $\alpha \in \Gamma$ there exists unique $n \in \mathbb{Z}$ with $\varphi^n(\theta) = \alpha$, thus $\varphi^{n+1}(\theta) = \varphi(\alpha)$ and

$$x_{\varphi(\alpha)} = x_{\varphi^{n+1}(\theta)} = r^{n+1}x = r(r^n x) = r x_{\varphi^n(\theta)} = r x_\alpha\,.$$

Hence $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}) = (x_{\varphi(\alpha)})_{\alpha\in\Gamma} = (rx_\alpha)_{\alpha\in\Gamma} = r(x_\alpha)_{\alpha\in\Gamma}$. By $(x_\alpha)_{\alpha\in\Gamma} \neq 0$ and $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}) = r(x_\alpha)_{\alpha\in\Gamma}$ we have $r \in \mathrm{Eigen}(\sigma_\varphi, V^\Gamma)$, which completes the proof. □

Lemma 2.8. *If $\varphi : \Gamma \to \Gamma$ is bijective with $W(\Gamma, \varphi) \neq \emptyset$, then $F\setminus\{0\} = \mathrm{Eigen}(\sigma_\varphi, V^\Gamma)$.*

Proof. By Corollary 2.3 we have $\mathrm{Eigen}(\sigma_\varphi, V^\Gamma) \subseteq F \setminus \{0\}$. Choose $\theta \in W(\Gamma, \varphi)$, then $\varphi|_{\frac{\theta}{\sim}} : \frac{\theta}{\sim} \to \frac{\theta}{\sim}$ is bijective and $\theta \in W(\varphi|_{\frac{\theta}{\sim}})$, thus by Lemma 2.8 we have $F\setminus\{0\} = \mathrm{Eigen}(\sigma_{\varphi|_{\frac{\theta}{\sim}}}, V^{\frac{\theta}{\sim}})$. Using Lemma 2.1 we have $F\setminus\{0\} = \mathrm{Eigen}(\sigma_{\varphi|_{\frac{\theta}{\sim}}}, V^{\frac{\theta}{\sim}}) \subseteq \mathrm{Eigen}(\sigma_\varphi, V^\Gamma)$ which completes the proof. □

Lemma 2.9. *If $\varphi : \Gamma \to \Gamma$ is bijective and $\sim_\varphi = \Gamma \times \Gamma$ with $W(\Gamma, \varphi) = \emptyset$, then (where for $r \in F \setminus \{0\}$ by $o(r)$ we mean order of r in commutative multiplying group $F \setminus \{0\}$ and for finite set A, $|A|$ denotes cardinality of A):*

1. *$P(\Gamma, \varphi) = \Gamma$ is finite,*
2. *for all $\alpha \in \Gamma$ we have $\{\varphi^n(\alpha) : 0 \leq n < |\Gamma|\} = \Gamma(= \{\varphi^n(\alpha) : n \geq 0\})$ and $\varphi^{|\Gamma|}(\alpha) = \alpha$,*
3. *$\mathrm{Eigen}(\sigma_\varphi, V^\Gamma) = \{r \in F \setminus \{0\} : o(r)$ is finite and $o(r)$ divides $|\Gamma|\}$,*

Proof. **1.** Since $W(\Gamma, \varphi) = \emptyset$, we have $Q(\Gamma, \varphi) = \Gamma$ and for $\alpha \in \Gamma = Q(\Gamma, \varphi)$ there exist $n > m \geq 1$ with $\varphi^n(\alpha) = \varphi^m(\alpha)$, thus $\varphi^{n-m}(\alpha) = \alpha$ and $\alpha \in P(\Gamma, \varphi)$.
2. Use (1).
3. Suppose $r \in \mathrm{Eigen}(\sigma_\varphi, V^\Gamma)$, by Corollary 2.3 we have $r \neq 0$. There exists nonzero $(x_\alpha)_{\alpha\in\Gamma} \in V^\Gamma$ with $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma} = r(x_\alpha)_{\alpha\in\Gamma}$. By (2) we have $\varphi^{|\Gamma|}(\alpha) = \alpha$ for all $\alpha \in \Gamma$, thus

$$\sigma_\varphi^{|\Gamma|}((x_\alpha)_{\alpha\in\Gamma}) = (x_{\varphi^{|\Gamma|}\alpha})_{\alpha\in\Gamma} = (x_\alpha)_{\alpha\in\Gamma}.$$

On the other hand $\sigma_\varphi^{|\Gamma|}((x_\alpha)_{\alpha\in\Gamma}) = r^{|\Gamma|}(x_\alpha)_{\alpha\in\Gamma} = (r^{|\Gamma|}x_\alpha)_{\alpha\in\Gamma}$. Hence:

$$\forall \alpha \in \Gamma \ (r^{|\Gamma|}x_\alpha = x_\alpha)\,.$$

Choose $\beta \in \Gamma$ with $x_\beta \neq 0$. By $r^{|\Gamma|}x_\beta = x_\beta$ we have $r^{|\Gamma|} = 1$ and $o(r)$ divides $|\Gamma|$.
Conversely, for $r \in F\setminus\{0\}$ if $o(r)$ divides $|\Gamma|$, then choose fix $x \in V\setminus\{0\}$ and $\theta \in \Gamma$. For $\alpha \in \Gamma$ let $x_\alpha = r^{m(\theta,\alpha)}x$. Using $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}) = r(x_\alpha)_{\alpha\in\Gamma}$ we have $r \in \mathrm{Eigen}(\sigma_\varphi, V^\Gamma)$ which completes the proof. □

Lemma 2.10. *For $\varphi : \Gamma \to \Gamma$ if $P(\Gamma, \varphi) = \Gamma$, then:*

- *$\varphi : \Gamma \to \Gamma$ is bijective,*
- *$\mathrm{Eigen}(\sigma_\varphi, V^\Gamma) = \{r \in F \setminus \{0\} : o(r)$ is finite and there exists $\alpha \in \Gamma$ such that $o(r)$ divides $|\frac{\alpha}{\sim}|\}$.*

Proof. Use Lemmas 2.1 and 2.9. □

Theorem 2.11 (Main Theorem). *For isomorphism* $\sigma_\varphi : V^\Gamma \to V^\Gamma$ *(so* $\varphi : \Gamma \to \Gamma$ *is bijective) we have:*

$$\mathrm{Eigen}(\sigma_\varphi, V^\Gamma) = \begin{cases} F \setminus \{0\}, & W(\varphi,\Gamma) \neq \emptyset\,, \\ \{r \in F \setminus \{0\} : \exists \theta \in P(\varphi,\Gamma)(o(r)|m(\theta,\theta))\}, & W(\varphi,\Gamma) = \emptyset\,. \end{cases}$$

In particular $\mathrm{Eigen}(\sigma_\varphi, V^\Gamma)$ *does not depend on* V.

Proof. Note that for periodic point α of φ we have $m(\alpha,\alpha) = |\{\varphi^n(\alpha) : n \geq 0\}|$. Also by bijection of φ we have $Q(\varphi,\Gamma) = P(\varphi,\Gamma)$. Use Lemmas 2.8 and 2.10 to complete the proof. □

Corollary 2.12. *For isomorphism* $\sigma_\varphi : V^\Gamma \to V^\Gamma$ *and* $r, s \in F \setminus \{0\}$ *with* $r \in \mathrm{Eigen}(\sigma_\varphi, V^\Gamma)$ *if* $o(s)|o(r)$*, then* $s \in \mathrm{Eigen}(\sigma_\varphi, V^\Gamma)$.

Proof. If $r \in \mathrm{Eigen}(\sigma_\varphi, V^\Gamma)$, then by Theorem 2.11 there exists $\alpha \in \Gamma$ with $o(r)|m(\alpha,\alpha)$ thus $o(s)|m(\alpha,\alpha)$ too, which leads to $s \in \mathrm{Eigen}(\sigma_\varphi, V^\Gamma)$. □

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