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Automatic continuity of almost conjugate Jordan homomorphism on Fréchet Q-algebras

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Abstract In this paper, the notation of almost conjugate Jordan homomorphism between Fréchet algebras is introduced. It is proven that, under special hypotheses, every almost conjugate Jordan homomorphism on Fréchet algebras is an (weakly) almost homomorphism. Also, the automatic continuity of them is generalized.

Keywords. Almost conjugate Jordan homomorphism, Almost Jordan homomorphism, Almost homomorphism , Fréchet Q-algebras.

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1. Introduction

A topological algebra is called a *Fréchet algebra* if it is a complete metrizable topological algebra which has a neighbourhood basis (V_n) of zero such that V_n is absolutely convex and $V_nV_n \subseteq V_n$ for all $n \in \mathbb{N}$. The topology of a Fréchet algebra A can be generated by a sequence (p_n) of separating submultiplicative seminorms, i.e., $p_n(xy) \leq p_n(x)p_n(y)$ for all $n \in \mathbb{N}$ and $x, y \in A$, such that $p_n(x) \leq p_{n+1}(x)$, whenever $n \in \mathbb{N}$ and $x \in A$. If A is unital then p_n can be chosen such that $p_n(1) = 1$ for all $n \in \mathbb{N}$. The Fréchet algebra A with the above generating sequence of seminorms (p_n) is denoted by $(A,(p_n))$. Note that a sequence (x_k) in the Fréchet algebra $(A,(p_n))$ converges to $x \in A$ if and only if $p_n(x_k - x) \to 0$ for each $n \in \mathbb{N}$, as $k \to \infty$. Banach algebras are important examples of Fréchet algebras.

Let A be a unital algebra. The set of all invertible elements of A is denoted by $InvA$. A topological algebra A is called a *Q-algebra* if $InvA$ is open set, or equivalently, $InvA$ has an interior point in A [9, Lemma E2].

The *Jacobson radical* of an algebra A, denoted by $radA$, is the intersection of all maximal left (right) ideals in A. The algebra A is called *semisimple* if $radA = \{0\}$.

If A is a commutative Fréchet algebra, then

$$radA = \bigcap\{ker\varphi : \quad \varphi \in M(A)\},$$

where $M(A)$ is the continuous character space of A, i.e. the space of all continuous non-zero multiplicative linear functionals on A. See, for example, [4, Proposition 8.1.2].

Banach algebras are important examples of Fréchet Q-algebras, and Fréchet Q-algebras is the well-known class of Fréchet algebras. A Fréchet algebra A is a Q-algebra if and only if the spectral radius $r_A(x)$ is finite for all $x \in A$. See, for example, [3, Theorem 6.18], [8, III. Proposition 6.2], or [9, Theorem 13.6].

Homomorphisms and their automatic continuity between different classes of topological algebras, including Fréchet algebras, Q-algebras and Banach algebras, have been widely studied by many authors. One may refer to the monographs of H. G. Dales [2], M. Fragoulopoulou [3], T. G. Honary [5], K. Jarosz [7], A. Mallios [8] and the interesting article of E. A. Michael [9].

In 1952, E. A. Michael posed the question as whether each multiplicative linear functional on a commutative Frechet algebra is automatically continuous [9].

In this paper, the relationship between almost conjugate Jordan homomorphism and almost Jordan homomorphism is investigated. In particular, it is proven that, under special hypotheses, every almost conjugate Jordan homomorphism on Fréchet algebras is an (weakly) almost homomorphism. Also, the automatic continuity of them is generalized.

The following result is due to Zelazko, about relationship between Jordan homomorphism and homomorphism on Banach algebras.

Theorem 1.1. [12, Theorem 1] *Suppose that A is a Banach algebra, which need not be commutative, and B is a semisimple commutative Banach algebra. Then each Jordan homomorphism $T : A \longrightarrow B$ is a homomorphism.*

2. Preliminaries

In this section we first introduce the notations, definitions and quote some auxiliary results.

Definition 2.1. Let $(A, (p_n))$ and $(B, (q_n))$ be Fréchet algebras. A linear map $T : (A, (p_n)) \longrightarrow (B, (q_n))$ is called *almost homomorphism* with respect to (p_n) and (q_n), if there exists $\varepsilon \geq 0$ such that

$$q_n(Tab - TaTb) \leq \varepsilon p_n(a)p_n(b),$$

for all $n \in \mathbb{N}$ and every $a, b \in A$, and it is called weakly almost homomorphism, if for every $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that

$$q_k(Tab - TaTb) \leq \varepsilon p_{n(k)}(a)p_{n(k)}(b),$$

for every $a, b \in A$.

Definition 2.2. Let $(A, (p_n))$ and $(B, (q_n))$ be Fréchet algebras. A linear map $T : (A, (p_n)) \longrightarrow (B, (q_n))$ is called *almost Jordan homomorphism* with respect to (p_n) and (q_n), if there exists $\varepsilon \geq 0$ such that

$$q_n(Ta^2 - (Ta)^2) \leq \varepsilon p_n(a)^2,$$

for all $n \in \mathbb{N}$ and every $a \in A$, and it is called *weakly almost Jordan homomorphism*, if for every $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that

$$q_k(Ta^2 - (Ta)^2) \leq \varepsilon p_{n(k)}(a)^2,$$

for every $a \in A$. Moreover, T is called Jordan homomorphism if

$$Ta^2 = (Ta)^2, \quad a \in A$$

and T is called multiplicative if

$$Tab = TaTb, \quad a, b \in A.$$

Definition 2.3. Let $(A, (p_n))$ and $(B, (q_n))$ be Fréchet algebras. A linear map $T : (A, (p_n)) \longrightarrow (B, (q_n))$ is called *almost Mixed Jordan homomorphism* with respect to (p_n) and (q_n), if there exists $\varepsilon \geq 0$ such that

$$q_n(T(a^2b) - (Ta)^2T(b)) \leq \varepsilon p_n(a)^2 p_n(b),$$

for all $n \in \mathbb{N}$ and every $a, b \in A$, and it is called *weakly almost Mixed Jordan homomorphism*, if for every $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that

$$q_k(T(a^2b) - (Ta)^2T(b)) \leq \varepsilon p_{n(k)}(a)^2 p_{n(k)}(b), \quad a, b \in A.$$

Moreover, T is called Mixed Jordan homomorphism if,

$$T(a^2b) = (Ta)^2T(b), \quad a, b \in A.$$

Definition 2.4. Let $(A, (p_n))$ and $(B, (q_n))$ be Fréchet algebras. A linear map $T : A \longrightarrow B$ is called *almost conjugate Jordan homomorphism* with respect to (q_n), if there exists $\varepsilon \geq 0$ such that

$$q_n(T(a^2b) - (Ta)^2T(b)) \leq \varepsilon,$$

for all $n \in \mathbb{N}$ and every $a, b \in A$.

The following example proves that, the almost conjugate Jordan homomorphisms are different from the almost Mixed Jordan homomorphisms.

Example 2.5. Let $A = C(X)$ for some compact Hausdorff space X, μ be a regular Borel measure on X such that $\mu(X) = 1$ and T be the linear functional on A represented by μ, that is, $T(f) = \int_X f d\mu$ for all $f \in A$. Then, for all $f, g \in A$ we have,

$$\begin{aligned}
|T(f^2g) - T(f)^2T(g)| &= |\int_X f^2 g d\mu - (\int_X f d\mu)^2 \int_X g d\mu| \\
&\leq \|f^2 g\|_X \mu(X) + \|f\|_X^2 \mu(X)^2 \|g\|_X \mu(X) \\
&\leq 2\|f\|_X^2 \|g\|_X.
\end{aligned}$$

It is clear that T is not almost conjugate Jordan homomorphism, but it is almost Mixed Jordan homomorphism. In Proposition (3.1), it is proven that every almost conjugate Jordan homomorphism between Fréchet algebras is an (weakly) almost Mixed Jordan homomorphism.

Clearly, every almost (Mixed) Jordan homomorphism is weakly almost (Mixed) Jordan homomorphism, and since (q_n) is a separating sequence of seminorms on B, almost (Mixed) Jordan homomorphism and weakly almost (Mixed) Jordan homomorphism turn out to be (Mixed) Jordan homomorphism, whenever $\varepsilon = 0$. Moreover, any Jordan homomorphism is almost Jordan homomorphism for every $\varepsilon \geq 0$.

Definition 2.6. A Fréchet algebra $(A,(p_n))$ is a uniform Fréchet algebra if

$$p_n(a^2) = (p_n(a))^2,$$

for each $n \in \mathbb{N}$ and for all $a \in A$.

Remark 2.7. [4, Page 73] Let A and B be Fréchet algebras with generating sequences of seminorms (p_n) and (q_n), respectively. If $\varphi : A \longrightarrow B$ is a linear operator, then φ is continuous if and only if for each $k \in \mathbb{N}$, there exist $n(k) \in \mathbb{N}$ and a constant $c_k > 0$ such that

$$q_k(\varphi(a)) \leq c_k p_{n(k)}(a),$$

for every $a \in A$. In the case that $(B,(q_n))$ is a uniform Fréchet algebra and $\varphi : A \longrightarrow B$ is a continuous homomorphism, we may choose $c_k = 1$ for all $k \in \mathbb{N}$.

Now, we recall the following result, which concerns an interesting property of Q-algebras.

Theorem 2.8. [3, Theorem 6.18] *Let $(A,(p_k))$ be a Fréchet algebra, then the following statements are equivalent:*

(i) *$(A,(p_k))$ is a Q-algebra.*
(ii) *There is $k_0 \in \mathbb{N}$ such that $r_A(x) \leq p_{k_0}(x)$, for every $x \in A$.*
(iii) *$r_A(x) = lim_{n\to\infty} p_{k_0}(x^n)^{\frac{1}{n}}$, for every $x \in A$ and p_{k_0} as in (ii).*

If k_0 is the smallest natural number such that p_{k_0} satisfies in the above theorem, we say that p_{k_0} is original seminorm for Fréchet $Q-$algebra $(A,(p_k))$. In the sequel, we use this fixed p_{k_0} as original seminorm for every Fréchet Q-algebra.

We first bring the following result for easy reference.

Lemma 2.9. [6, Lemma 2.6] *Let $(A,(p_n))$ be a Fréchet algebra and $T : (A,(p_n)) \to \mathbb{C}$ be an (weakly) almost multiplicative linear functional. Then, at least one of the following holds:*

(i) *T is multiplicative,*
(ii) *for every $a \in A$, if $p_m(a) = 0$ then $Ta = 0$.*

3. Main Results

In this section, some interest results about almost conjugate Jordan homomorphism between Fréchet algebras are proven.

Proposition 3.1. *Let $T : (A,(p_n)) \longrightarrow (B,(q_n))$ be an almost conjugate Jordan homomorphism between Fréchet algebras. Then T is a weakly almost Mixed Jordan homomorphism.*

Proof. Let T be an ε-conjugate Jordan homomorphism for some $\varepsilon \geq 0$. Then

$$q_k(T(a^2b) - (Ta)^2T(b)) \leq \varepsilon,$$

for all $k \in \mathbb{N}$ and every $a, b \in A$. Let k be fixed and $a, b \neq 0$. It is clear that, there exists $n_k \in \mathbb{N}$ such that $p_{n_k}(a) \neq 0$ and $p_{n_k}(b) \neq 0$. Then

$$q_k(T((\frac{a}{p_{n_k}(a)})^2 \frac{b}{p_{n_k}(b)}) - (T\frac{a}{p_{n_k}(a)})^2 T(\frac{b}{p_{n_k}(b)})) \leq \varepsilon,$$

therefore

$$q_k(T(a^2b) - (Ta)^2T(b)) \leq \varepsilon p_{n_k}(a)^2 p_{n_k}(b). \tag{3.1}$$

If $a = 0$ or $b = 0$, the inequality (3.1) is correct, and this completes the proof. □

Now, we apply the above proposition and obtain the following interesting result.

Theorem 3.2. *Let $(A,(p_n))$ be a Fréchet algebra and $T : (A,(p_n)) \longrightarrow \mathbb{C}$ be an almost conjugate Jordan homomorphism, e_A be unit of A and $\varphi : A \longrightarrow \mathbb{C}$ is defined by $\varphi(x) = T(x)T(e_A)$. Then φ is an almost Jordan homomorphism. Moreover if A is commutative, then φ is an almost homomorphism.*

Proof. By applying Proposition (3.1), there exists some $\varepsilon \geq 0$ and $m \in \mathbb{N}$ such that

$$|T(a^2b) - (Ta)^2T(b)| \leq \varepsilon p_m(a)^2 p_m(b),$$

for every $a, b \in A$. Hence

$$\begin{aligned}
|\varphi(x^2) - \varphi(x)^2| &= |T(x^2)T(e_A) - (Tx)^2T(e_A)^2| \\
&= |T(e_A)||T(x^2e_A) - (Tx)^2T(e_A)| \\
&\leq \varepsilon|T(e_A)|p_m(x)^2p_m(e_A) \\
&= \varepsilon_1 p_m(x)^2,
\end{aligned}$$

where $\varepsilon_1 = \varepsilon|T(e_A)|$. Thus φ is an almost Jordan homomorphism.

Now, suppose that $a, b \in A$ and $p_m(a) = p_m(b) = 1$. Since A is commutative, then

$$\begin{aligned}
4|\varphi(ab) - \varphi(a)\varphi(b)| &= |\varphi((a+b)^2) - (\varphi(a+b))^2 - \varphi((a-b)^2) + (\varphi(a-b))^2| \\
&\leq \varepsilon_1((p_m(a+b))^2 + (p_m(a-b))^2) \\
&\leq \varepsilon_1((p_m(a) + p_m(b))^2 + (p_m(a) + p_m(b))^2) \\
&= 8\varepsilon_1,
\end{aligned}$$

therefore

$$|\varphi(ab) - \varphi(a)\varphi(b)| \leq 2\varepsilon_1.$$

Let $a, b \in A$ and $a, b \neq 0$. It is clear that, there exists $n \geq m$ such that $p_n(a) \neq 0$ and $p_n(b) \neq 0$. We may assume, without loss of generality, that $n = m$. Then

$$|\varphi(\frac{a}{p_m(a)}\frac{b}{p_m(b)}) - \varphi(\frac{a}{p_m(a)})\varphi(\frac{b}{p_m(b)})| \leq 2\varepsilon_1,$$

hence

$$|\varphi(ab) - \varphi(a)\varphi(b)| \leq 2\varepsilon_1 p_m(a)p_m(b) \tag{3.2}$$

If $a = 0$ or $b = 0$, the inequality (3.2) is correct. This means that φ is an (weakly)almost homomorphism. □

Corollary 3.3. *Let $(A,(p_n))$ be a unital Fréchet algebra and $T : (A,(p_n)) \longrightarrow \mathbb{C}$ be an almost conjugate Jordan homomorphism. Then T is an almost Jordan homomorphism. Moreover if A is commutative, then T is an almost homomorphism.*

Theorem 3.4. *Let $(A,(p_k))$ be a unital commutative Fréchet Q-algebra with p_{k_0} as original seminorm and $T:(A,(p_n))\longrightarrow\mathbb{C}$ be a (weakly) almost homomorphism. Define*

$$\|T\|_p=\sup\{|T(a)|:p_{k_0}(a)=1\}.$$

Then $\|.\|_p$ is a norm on $M_{alm}(A)$, the set of all almost homomorphism from A to $\mathbb{C}$.

Proof. If $p_{k_0}(a)\neq 0$, then $p_{k_0}(\frac{a}{p_{k_0}(a)})=1$ and $|T(\frac{a}{p_{k_0}(a)})|\leq\|T\|_p$. Therefore

$$|T(a)|\leq p_{k_0}(a)\|T\|_p. \tag{3.3}$$

Suppose that $\|T\|_p=0$, then $T(a)=0$ for each $a\in A$, where $p_{k_0}(a)=1$. Let $T\neq 0$, then there exists $b\in A$ such that $T(b)\neq 0$. If $p_{k_0}(b)=0$, by applying Lemma(2.9), T is a multiplicative and hence $T(e_A)=1$, which is contradiction by (3.3). Therefore $p_{k_0}(b)\neq 0$. Set $a=\dfrac{b}{p_{k_0}(b)}$, then we get $p_{k_0}(a)=1$ and $T(a)\neq 0$. But it is not possible, by (3.3). Thus, we conclude that $T=0$. This means that $\|\cdot\|_p$ is a norm on $M_{alm}(A)$. □

Next, the following result is obtained, by applying Theorem (3.2) and Theorem (3.4).

Corollary 3.5. *Let $(A,(p_k))$ be a unital commutative Fréchet Q-algebra. Then $\|\cdot\|_p$ is a norm on $M_{alm.J}(A)$, the set of all almost Jordan homomorphism from A to $\mathbb{C}$.*

A proof of the following theorem can be found in [10]. We apply this result to get a sufficient condition for an almost conjugate Jordan homomorphism to be continuous.

Theorem 3.6. [10, Theorem 3.2] *Let $(A,(p_n))$ be a unital commutative Fréchet Q-algebra and $T:A\longrightarrow\mathbb{C}$ be an ε- Jordan homomorphism. Then $\|T\|_p<1+\varepsilon$ and T is continuous.*

Now, by using Theorem (3.2) and Theorem (3.6), We conclude that:

Theorem 3.7. *Let $(A,(p_n))$ be a unital commutative Fréchet Q-algebra and $T:(A,(p_n))\longrightarrow\mathbb{C}$ be an almost conjugate Jordan homomorphism. Then T is continuous.*

4. Conclusion

We conclude that, every almost conjugate Jordan homomorphism between Fréchet algebras is almost Mixed Jordan homomorphism. Also, if A be a unital commutative Banach algebra and B a semisimple Banach algebra, then every almost conjugate Jordan homomorphism $T:A\longrightarrow B$ is continuous.

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