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Stability and numerical approximation for a spacial class of semilinear parabolic equations on the Lipschitz bounded regions: Sivashinsky equation

Mehdi Mesrizedeh
Department of mathematics,
Imam Khomeini International University, Qazvin, Iran.
E-mail: mehdi.mesrizadeh@yahoo.com

Kamal Shanazari*
Department of mathematics,
University of Kurdistan, Sanandaj, Iran.
E-mail: k.shanazari@uok.ac.ir

Abstract This paper aims to investigate the stability and numerical approximation of the Sivashinsky equations. We apply the Galerkin meshfree method based on the radial basis functions (RBFs) to discretize the spatial variables and use a group presenting scheme for the time discretization. Because the RBFs do not generally vanish on the boundary, they can not directly approximate a Dirichlet boundary problem by Galerkin method. To avoid this difficulty, an auxiliary parametrized technique is used to convert a Dirichlet boundary condition to a Robin one. In addition, we extend a stability theorem on the higher order elliptic equations such as the biharmonic equation by the eigenfunction expansion. Some experimental results will be presented to show the performance of the proposed method.

Keywords. Eigenvalue, Eigenfunction, Galerkin meshless method, Sivashinsky equation, Stability.
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1. Introduction

The approximate solution of the Galerkin method is to determine the solution of a system of equations obtained by the variational formulation of the main problem. Also the key property of the Galerkin approach is that the error is orthogonal to the chosen subspaces. In the variational formulation, it is almost always necessary to estimate some integrals. This method has also more computational cost than the collocation method, but it is a stable algorithm and well computes the approximate solution of the well-posed problems. Although it has suffered from deficiencies, special techniques can be used to alleviate the defects in the applications.
Investigated by authors of [6], a set of background cells were required to evaluate the integrals resulted from the use of the Galerkin weak-form. Global numerical integrations were needed to obtain the coefficient matrix of the equation system. Hence, a global background cell structure had to be used for these integrations, so

$*$ Corresponding author.

that the method was not truly meshless. Also, emphasis was placed on the relationship between the supports of the shape functions and the sub-domains used to integrate the discrete equations. It was shown that constructing the quadrature cells with no attention to the local supports of the shape functions may result in the considerable integration error. They employed the conventional Gauss quadrature rules in the cells for which the high order quadrature rule was needed for sufficient accuracy in the Galerkin meshfree method. Nodal integration, on the other hand, led to rank instability and significant loss of accuracy in the numerical solution. In contrast with using background meshes for integration, discussed above, a truly meshfree Galerkin computational method rely chiefly on the nodal integration maintaining the meshfree characteristics of the weak forms.
The Sivashinsky differential equation, which models a planar solid-liquid interface for a binary alloy for $a = 0$, is considered as the following nonlinear biharmonic equation [6, 4, 12]

$$\begin{aligned} u_t + \Delta^2 u - a\ \Delta u + b\ u = -\Delta f(u), \quad in \ \ \Omega, \\ u = 0, \ \ \Delta u = 0, \quad on \ \ \partial\Omega, \\ u(0) = u_\circ, \quad on \ \ \bar{\Omega}. \end{aligned} \tag{1.1}$$

where a, b are two positive real numbers, Ω is a bonded Lipschitz region with its boundary $\partial\Omega$, and $u_\circ$, and f satisfy suitable conditions. This problem was surveyed and solved by the numerical method given in [9, 10]. Dehghan [6] applied the meshless weak form techniques based on the radial point interpolation method for solving this equation and derived an error estimation of its global weak form.

In this work, we apply the Galerkin meshfree method based on the RBFs to the Sivashinsky equation and consider the stability of this equation by the eigenfunction expansion. This paper is organized as follows. In section 2, we introduce eigenfunctions of the biharmonic operators and use them to study the stability of the method. The Galerkin meshless method is presented in section 3. Some numerical results will be presented in section 4.

2. Stability

Eigenvalue problems are important in the mathematical analysis of partial differential equations (PDEs), and occur, e.g. in the modeling of vibrating membranes and other applications. In our study of time-dependent PDEs it will be important to develop functions in eigenfunction expansions, and we therefore discuss such expansions in this section.

2.1. **Solution of $\mathcal{A}u = f$ by Eigenfunction Expansion.** We first solve $\mathcal{A} = \Delta^2 - a\Delta + b\mathbb{I}$ with the Positive Parameters a and b by the use of eigenfunction expansions. We denote by $(.,.)_\Omega$ and $\|.\|_\Omega$ the inner product and the norm in $\mathbf{L}^2 = \mathbf{L}^2(\Omega)$, respectively. Also a Hilbert space of functions satisfying the double Dirichlet boundary condition $\mathcal{H} = \left\{u \in \mathbf{H}^2(\Omega)\ s.t.\ u = 0, \ \ \Delta u = 0, \ \ \text{on} \ \ \partial\Omega\right\}$ with the inner product $(.,.)_\mathcal{H} = (\Delta.\ ,\ \Delta.)_\Omega + a\ (\nabla.\ ,\ \nabla.)_\Omega + b\ (\ .,\ .)_\Omega$ that $\mathbf{H}^k(\Omega)$, $k > 0$, is the Hilbert

Sobolev space of kth order [2]. We find an eignepair $(\lambda, \psi) \in \mathbb{C} \times \mathcal{H}\backslash\{0\}$ such that for all $\nu \in \mathcal{H}$

$$(\mathcal{A}\psi, \nu)_\Omega = \lambda\ (\psi, \nu)_\Omega. \tag{2.1}$$

Using integration by parts twice, we obtain

$$(\mathcal{A}\psi, \nu)_\Omega = (\psi, \nu)_\mathcal{H}. \tag{2.2}$$

Substituting (2.2) into (2.1), we have

$$(\psi, \nu)_\mathcal{H} = \lambda\ (\psi, \nu)_\Omega. \tag{2.3}$$

By the result of Theorem 19.9 in [3] for classical regions with smooth or convex polygon boundary, we may conclude at once that the eigenfunctions are smooth, $\psi \in \mathbf{C}^\infty$, because if u is a solution of $\mathcal{A}u = f$ and $f \in \mathbf{H}^k(\Omega)$, then $u \in \mathbf{H}^{k+4}(\Omega) \cap \mathcal{H}$ being similar with elliptic regularity [2] that we call $\mathcal{A}-$regularity and

$$\|u\|_{k+4} \leq \|f\|_k, \tag{2.4}$$

where $\|.\|_k$ is the norm of $\mathbf{H}^k(\Omega)$. Since $\psi \in \mathbf{L}^2$, $\mathcal{A}-$regularity implies that $\psi \in \mathbf{H}^4(\Omega) \cap \mathcal{H}$, which in turn shows $\psi \in \mathbf{H}^8(\Omega) \cap \mathcal{H}$, and so on.

Theorem 2.1. *The eigenvalues of $\mathcal{A}$ are real, positive and tend to infinity. Two eigenfunctions corresponding to different eigenvalues are orthogonal in $\mathbf{L}^2$ and $\mathcal{H}$. Also the nth eigenvalue is calculated as*

$$\lambda_n = \inf\left\{\|\nu\|^2_\mathcal{H} \ \text{ s.t. } \ \nu \in \mathcal{H},\ \ \|\nu\|_\Omega = 1 \quad \text{and } \ (\psi_i, \nu)_\Omega = 0 \ \text{ for all } i = 1, 2, \cdots, n-1\right\},$$

where ψ_i is eigenfunction corresponding to eigenvalue λ_i. $\{\psi_n\}_{n=1}^\infty$ is an orthonormal base of $\mathbf{L}^2$ and $\mathcal{H}$.

Proof. Let λ be an eigenvalue and ψ be the corresponding eigenfunction. Then

$$\lambda\|\psi\|^2_\Omega = \|\psi\|^2_\mathcal{H},$$

which implies that $\lambda > 0$. Let λ_1 and λ_2 be two different eigenvalues and ψ_1 and ψ_2 be the corresponding eigenfunctions. Then

$$\lambda_1(\psi_1, \psi_2)_\Omega = (\psi_1, \psi_2)_\mathcal{H} = (\psi_2, \psi_1)_\mathcal{H} = \lambda_2(\psi_2, \psi_1)_\Omega,$$

so that

$$(\lambda_1 - \lambda_2)(\psi_1, \psi_2)_\Omega = 0.$$

Since $\lambda_1 \neq \lambda_2$, it follows that $(\psi_1, \psi_2)_\Omega = 0$ and $(\psi_1, \psi_2)_\mathcal{H} = 0$. The property of tending to infinity can be proved by Theorem 6.3 in [7]. It is clear that the inequality

$$\|\nu\|_\mathcal{H} \geq \|\nu\|_1, \quad \nu \in \mathcal{H}. \tag{2.5}$$

holds and obviously $\lambda_n \leq \lambda_{n+1}$. Let $\|\psi_n\|_\mathcal{H} = \lambda_n$ and $\|\psi_n\|_\Omega = 1$, we get $v \in \mathcal{H}$ such that $v = \sum_{i=1}^{n-1} v_i\psi_i$ and $\|v\|_\Omega > 1$, for an integer number n, $\mathbf{C}_n$ corresponds to the transformed of $\mathbf{B}_n = \big\{\nu \in \mathcal{H},\ \ \|\nu\|_\Omega \leq 1 \ \text{ and } \ (\psi_i, \nu)_\Omega = 0 \ \text{ for all } i = 1, 2, \cdots, n-1\big\}$ v is a weakly compact set on $\mathcal{H}$ by the Banach-Alaoglu Theorem [11]. Denote $f_n(\nu) = \frac{1}{2}\big(\|\nu\|^2_\mathcal{H} - \lambda_n\|v\|^2_\Omega - \lambda_n\big)$ on $\mathbf{C}_n$ that is the closed convex bounded set, also function f_n has an argument of minimum value on $\mathbf{C}_n$ such as $v+\psi_n$, so for any λ_n there is at least a solution $\psi_n \in \mathbf{B}_n$ from Theorem 3.1 of [5] such that

$$D_G f_n(v+\psi_n)[\nu-\psi_n] \geq 0, \quad \forall \nu \in \mathbf{B}_n,$$

where $D_G f_n$ is Ĝateaux differential f_n at ψ_n, so

$$(v+\psi_n, \nu-\psi_n)_{\mathcal{H}} = (\psi_n, \nu-\psi_n)_{\mathcal{H}} \geq 0, \quad \forall \nu \in \mathbf{B}_n,$$

from $(\psi_n, \psi_n)_{\mathcal{H}} = \lambda_n$, $v+\psi_n$ implementing minimal argument of f_n in $\mathbf{C}_n$, and simplifying above term we have

$$(\psi_n, \nu)_{\mathcal{H}} - \lambda_n(\psi_n, \nu)_{\Omega} \geq 0, \quad \forall \nu \in \mathbf{B}_n.$$

It is clear that $\mathcal{H} = span\mathbf{B}_n \oplus span\{\psi_1, \cdots, \psi_{n-1}\}$, also

$$(\psi_n, \nu)_{\mathcal{H}} - \lambda_n(\psi_n, \nu)_{\Omega} \geq 0, \quad \forall \nu \in \mathcal{H},$$

thus

$$(\psi_n, \nu)_{\mathcal{H}} = \lambda_n(\psi_n, \nu)_{\Omega}, \quad \forall \nu \in \mathcal{H}.$$

By Theorem 12.10 of [11] and that $\mathcal{A}$ has the self-adjoint linear operator, we conclude that $\{\psi_n\}_{n=1}^{\infty}$ is an orthonormal base for $\mathbf{L}^2$ and $\mathcal{H}$.

□

Theorem 2.2. *Any solution of $\mathcal{A}u = f$ for $f \in \mathbf{L}^2$ has a representation in terms of $\{\psi_n\}_{n=1}^{\infty}$ as*

$$u = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} (\psi_n, f)_{\Omega} \psi_n.$$

Proof. Multiplying both sides of $Au = f$ by ψ_n, yields

$$\big(\mathcal{A}u, \psi_n\big)_{\Omega} = \big(f, \psi_n\big)_{\Omega},$$

and using integration by parts, we have

$$\big(u, \mathcal{A}\psi_n\big)_{\Omega} = \lambda_n \big(u, \psi_n\big)_{\Omega} = \big(f, \psi_n\big)_{\Omega},$$

but we get $\hat{u}_n = \big(u, \psi_n\big)_{\Omega}$ which is a multiple of Fourier representation for $u = \sum_{n=1}^{\infty} \hat{u}_n \psi_n$, that is, $\hat{u}_n = \frac{1}{\lambda_n} \big(f, \psi_n\big)_{\Omega}$. □

We consider the time-independent solution in the next section.

2.2. **Stability of the Sivashinsky equation.** We first solve $u_t + \mathcal{A}u = 0$ by using eigenfunction expansions and seeking a solution of the form

$$u(\mathbf{x}, t) = \sum_{n=1}^{\infty} \hat{u}_n(t) \psi_n(\mathbf{x}), \tag{2.6}$$

where $\hat{u}_n : [0, T) \to \mathbb{R}$, $T \in (0, \infty]$, are coefficients to be determined. Because this is a sum of products of functions of $\mathbf{x}$ and t, this approach is called the method of separation of variables. Inserting (2.6) into the differential equation in the initial-boundary value problem $u_t+\mathcal{A}u=0$ and using eignepairs property we obtain formally

$$\sum_{n=1}^{\infty}\big(\frac{d\hat{u}_n(t)}{dt}+\lambda_n\hat{u}_n(t)\big)\psi_n(\mathbf{x})=0,\quad \text{for}\quad (t,\mathbf{x})\in(0,T]\times\Omega,$$

and, since the ψ_n's form a base, we have

$$\frac{d\hat{u}_n(t)}{dt}+\lambda_n\hat{u}_n(t)=0,\quad t\in[0,T],\quad \forall n=1,2,\cdots,$$

so that

$$\hat{u}_n(t)=\hat{u}_n(0)e^{-\lambda_n t}.$$

Moreover, from the initial condition, it follows that

$$u(\mathbf{x},0)=\sum_{n=1}^{\infty}\hat{u}_n(0)\psi_n(\mathbf{x})=u_\circ(\mathbf{x})=\sum_{n=1}^{\infty}\hat{u}_n\psi_n(\mathbf{x}),\quad \hat{u}_n=(\psi_n,u_\circ)_\Omega,\ \forall n=1,2,\cdots.$$

We thus see that, at least formally, the solution has to be

$$u(\mathbf{x},t)=\sum_{n=1}^{\infty}\hat{u}_n e^{-\lambda_n t}\psi_n(\mathbf{x}), \tag{2.7}$$

where by Parsevals relation, with $\mathbf{L}^2-$norm,

$$\|u(.,t)\|_\Omega^2=\sum_{n=1}^{\infty}|\hat{u}_n|^2e^{-2\lambda_n t}\le e^{-2\lambda_1 t}\sum_{n=1}^{\infty}|\hat{u}_n|^2=e^{-2\lambda_1 t}\|u_\circ\|_\Omega^2<\infty,$$

thus

$$\|u(.,t)\|_\Omega\le\|u_\circ\|_\Omega,\quad \forall t\in[0,T]. \tag{2.8}$$

Modifying the above inequality, yields

$$\|u(.,t)\|_\Omega\le e^{-\lambda_1 t}\|u_\circ\|_\Omega,\quad \forall t\in[0,T]. \tag{2.9}$$

Theorem 2.3. *Assume, for $u(t)\in\mathcal{H}$, for all $t>0$, that $D_t^m\mathcal{A}^k u\in\mathbf{L}^2$, D_t being time differential operator, for k and m, then*

$$\|D_t^m\mathcal{A}^k u(.,t)\|_\Omega\le C_{m,k}t^{-m-k}\|u_\circ\|_\Omega,\quad \forall t\in(0,T],$$

and

$$\|D_t^m\mathcal{A}^k u(.,t)\|_{\mathcal{H}}\le C_{m,k}t^{-m-k-\frac{1}{2}}\|u_\circ\|_\Omega,\quad \forall t\in(0,T].$$

Proof. The solution of this problem has the representation as (2.7). We first note that for any $m,\ k\ge0$ there is a constant C_k such that $t^ke^{-t}\le C_k$ for $t\ge0$. Using this with $k,\ m$,

$$\|D_t^m\mathcal{A}^k u(.,t)\|_\Omega^2=t^{-2(k+m)}\sum_{n=1}^{\infty}|\hat{u}_n|^2(\lambda_n t)^{2(k+m)}e^{-2\lambda_n t}\le C_{k+m}^2t^{-2(k+m)}e^{-2\lambda_1 t}\|u_\circ\|_\Omega^2,$$

so that

$$\|D_t^m\mathcal{A}^k u(.,t)\|_\Omega\le C_{k+m}t^{-(k+m)}\|u_\circ\|_\Omega.$$

Because $\|u(t)\|_{\mathcal{H}}^2=(\mathcal{A}u(t),u(t))_\Omega$, the other part can be proven similarly. $\square$

Acting the solution operator $E(t)$, being the defined linear operator, on the initial data $u_\circ$ is resulted in $u(t) = E(t)u_\circ$. By (2.7) this operator satisfies the stability estimate

$$\|E(t)u_\circ\|_\Omega \leq \|u_\circ\|_\Omega, \quad \forall t \in [0,T],$$

and Theorem 2.3 may be expressed as

$$\|D_t^m E(t)u_\circ\|_\Omega \leq C_m t^{-m}\|u_\circ\|_\Omega,$$

which expresses a smoothing property of the solution operator. In fact, as we shall see, the solution of $u_t + \mathcal{A}u = f$ may be expressed as

$$u(\mathbf{x},t) = E(t)u_\circ(\mathbf{x}) + \int_0^t E(t-s)f(\mathbf{x},s)ds. \tag{2.10}$$

This formula represents the solution of the inhomogeneous equation as a superposition of solutions of the homogeneous equations, so that

$$\|u(t)\|_\Omega \leq \|u_\circ\|_\Omega + \int_0^t \|f(s)\|_\Omega ds, \tag{2.11}$$

where we write $u(t)$ for $u(.,t)$ and similarly for $f(s)$. We can modify the inequality (2.11) similar to the above process such that

$$\|u(t)\|_\Omega \leq e^{-\lambda_1 t}\|u_\circ\|_\Omega + \int_0^t e^{-\lambda_1(t-s)}\|f(s)\|_\Omega ds. \tag{2.12}$$

Now we are ready to express a stability theorem for the problem (2.1).

Theorem 2.4 (Stability of Sivashinsky problem)**.** *Assume that u is the solution of equation (1.1) and there is a $L_f > 0$ for all $\nu \in \mathcal{H}$*

$$(\nabla f(\nu), \nabla\nu)_\Omega \leq L_f\|\nabla\nu\|_\Omega^2,$$

then for a $C > 0$ we have

$$\|u(t)\|_\Omega \leq C\|u_\circ\|_\Omega.$$

Proof. Multiplying both sides of (2.10) by $\nu \in \mathbf{H}_\circ^1$, $\|\nu\|_\Omega = 1$, and using integration by parts, gives

$$(u(t),\nu)_\Omega = E(t)(u_\circ,\nu)_\Omega + \int_0^t E(t-s)(\nabla f(u(s)), \nabla\nu)_\Omega ds,$$

where $\mathcal{H} \subset \mathbf{H}_\circ^1$ is the set of functions of $\mathbf{H}^1$ that vanish on the boundary of Ω. Let $\nu = u(t)$

$$(u(t),u(t))_\Omega \leq e^{-\lambda_1 t}|(u_\circ,u(t))_\Omega| + L_f\int_0^t e^{-\lambda_1(t-s)}(\nabla u(s), \nabla u(s))_\Omega ds,$$

associating to the inequality (2.5), so that

$$\|u(t)\|_\Omega^2 \leq e^{-\lambda_1 t}\|u(t)\|_\Omega\|u_\circ\|_\Omega + L_f\int_0^t e^{-\lambda_1(t-s)}\|u(s)\|_{\mathcal{H}}^2 ds,$$

also by Cauchy-Schwarz inequality and Theorem 2.3 we derive as follows

$$\begin{aligned}\|u(t)\|_\Omega^2 &\leq e^{-\lambda_1 t}\|u_\circ\|_\Omega \|u(t)\|_\Omega + L_f \int_0^t e^{-\lambda_1(t-s)} s^{-\frac{1}{2}} \|u_\circ\|_\Omega^2 ds \\ &\leq \frac{e^{-\lambda_1 t}}{2} \left(\|u(t)\|_\Omega^2 + \|u_\circ\|_\Omega^2\right) + \|u_\circ\|_\Omega^2 \big(L_f \int_0^t e^{-\lambda_1(t-s)} s^{\frac{-1}{2}} ds\big).\end{aligned}$$

Let $C^2 = \frac{\left(\frac{1}{2}+\frac{2}{\sqrt{\lambda_1}}\right)}{1-\frac{1}{2}e^{-\lambda_1 t}} \leq \left(1+\frac{4}{\sqrt{\lambda_1}}\right)$ and thus we have

$$\|u(t)\|_\Omega \leq C\|u_\circ\|_\Omega.$$

□

$f(u) = 2u - \frac{1}{2}u^2$ (see [6]), and we can give $L_f = 3$. Hence problem (1.1) is stable.

3. Galerkin meshfree method

In this section, we introduce the Galerkin meshfree method based on the RBFs. Assume that $\mathbb{X}_N = \{x_1, \cdots, x_N\}$ is a set of pairwise distinct points in the compact set Ω. An RBF is a radial function $\Pi(\mathbf{x}) = \pi(\|\mathbf{x}\|)$, where $\pi \in \mathbf{C}[0,\infty)$, that is positive definite or m-order conditionally positive definite on $\mathbb{R}^n$ with respect to the set of polynomials Π_m^n having total degree $m-1$ or less when all nonzero $a \in \mathbb{R}^n$ satisfying $\sum_{i=1}^N a_i p(\mathbf{x}_i) = 0$, for all $p \in \Pi_m^n$, we have

$$\sum_{i=1}^N \sum_{j=1}^N a_i a_j \Pi(\mathbf{x}_j - \mathbf{x}_i) > 0.$$

We use compactly-supported positive definite RBFs (CS-RBFs) in this paper. For $m = 0$, the fourth order CS-RBFs is defined by

$$\Pi(\mathbf{x}) = \begin{cases} c^2\left(\frac{r^2}{4} - \frac{5r^4}{8} + \frac{4r^5}{5} - \frac{5r^6}{12} + \frac{4r^7}{49}\right) & 0 \leq r \leq 1, \\ 0 & 1 < r, \end{cases}$$

where $\Pi \in \mathbf{C}^4$, and $c > 0$ is a parameter. For the biharmonic equation, we must apply higher-order Wendlands CS-RBFs that causes complex and costly calculations.

3.1. Galerkin meshfree method based on the radial basis functions (GRBFs).

The approximation space is used to Galerkin meshfree method as follows

$$\mathbb{V}_N = \Big\{ \sum_{i=1}^N a_i \Pi_i(\mathbf{x}) \quad \text{such that} \quad \Pi_i(\mathbf{x}) = \Pi(\mathbf{x}-\mathbf{x}_i) \;\; a_i \in \mathbb{R}, \; \forall i = 1, 2, \cdots, N \Big\}.$$

The meshless weak form techniques based on radial point interpolation has been applied to the nonlinear biharmonic equation [6]. Also, the error estimate of meshless global weak form methods has been derived for this problem. RBFs do not satisfy the double boundary conditions, directly; however, the Robin boundary conditions to approximate Dirichlet boundary conditions of Eq. (1.1) by two positive parameters α and β, can be used as follows,

$$\begin{aligned} \frac{\partial u_{\alpha,\beta}}{\partial t} + \Delta^2 u_{\alpha,\beta} - a\ \Delta u_{\alpha,\beta} + b\ u_{\alpha,\beta} = -\Delta f(u_{\alpha,\beta}), \quad in \quad \Omega, \\ u_{\alpha,\beta} + \alpha\frac{\partial u_{\alpha,\beta}}{\partial n} = 0, \ \ \Delta u_{\alpha,\beta} + \beta\frac{\partial \Delta u_{\alpha,\beta}}{\partial n} = 0, \quad on \quad \partial\Omega, \\ u_{\alpha,\beta}(0) = u_{\circ}, \quad \bar{\Omega}, \end{aligned} \tag{3.1}$$

where n is the surface normal to $\partial\Omega$. The weak form of (3.1) with twice integration by parts can be obtained when for all $v \in \mathbf{H}^2(\Omega)$ and from Theorem 2.4 there is a unique $u_{\alpha,\beta} \in \mathbf{H}^2(\Omega)$ called weak solution of Eq. (3.1) as follows

$$\begin{aligned} \frac{d}{dt}\Big(u_{\alpha,\beta}, v\Big)_{\Omega} + \Big(\Delta u_{\alpha,\beta}, \Delta v\Big)_{\Omega} + \beta\Big(\Delta u_{\alpha,\beta}, v\Big)_{\partial\Omega} + \\ a\Big(\nabla u_{\alpha,\beta}, \nabla v\Big)_{\Omega} - a\alpha\Big(u_{\alpha,\beta}, v\Big)_{\partial\Omega} + b\Big(u_{\alpha,\beta}, v\Big)_{\Omega} = -\Big(\Delta f(u_{\alpha,\beta}), v\Big)_{\Omega} \\ u_{\alpha,\beta}(0) = u_{\circ}, \quad \bar{\Omega}, \end{aligned} \tag{3.2}$$

where $\big(u_{\alpha,\beta}, v\big)_{\Omega} = \int_{\Omega} u_{\alpha,\beta}(\mathbf{x},t)v(\mathbf{x})d\mathbf{x}$ and $\big(u_{\alpha,\beta}, v\big)_{\partial\Omega} = \int_{\partial\Omega} u_{\alpha,\beta}(\mathbf{x},t)v(\mathbf{x})d\sigma$. We investigate by some examples that solution of Eq. (3.2) tend to solution of Eq. (1.1) as $\alpha \to \infty$ and $\beta \to \infty$ via typically weak convergence. The approximate solution represented in $\mathbb{V}_N$ is given as

$$\widetilde{u}_{N,\alpha,\beta}(\mathbf{x},t) = \sum_{i=1}^{N} a_i(t)\Pi_i(\mathbf{x}),$$

By substituting the approximate solution into Eq. (1.1), we have, for all $j = 1, 2, \cdots, N$,

$$\begin{aligned} \frac{d}{dt}\Big(u_{N,\alpha,\beta}, \Pi_j\Big)_{\Omega} + \Big(\Delta u_{N,\alpha,\beta}, \Delta\Pi_j\Big)_{\Omega} + \beta\Big(\Delta u_{N,\alpha,\beta}, \Pi_j\Big)_{\partial\Omega} + \\ a\Big(\nabla u_{N,\alpha,\beta}, \nabla\Pi_j\Big)_{\Omega} - a\alpha\Big(u_{N,\alpha,\beta}, \Pi_j\Big)_{\partial\Omega} + b\Big(u_{N,\alpha,\beta}, \Pi_j\Big)_{\Omega} = -\Big(\Delta f(u_{N,\alpha,\beta}), \Pi_j\Big)_{\Omega} \\ u_{N,\alpha,\beta}(0) = u_{\circ}, \quad \bar{\Omega}, \end{aligned} \tag{3.3}$$

so that

$$\begin{aligned} \sum_{i=1}^{N} \frac{da_i(t)}{dt}\Big(\Pi_i, \Pi_j\Big)_{\Omega} + \sum_{i=1}^{N} a_i(t)\Big(\Delta\Pi_i, \Delta\Pi_j\Big)_{\Omega} + \\ \beta\sum_{i=1}^{N} a_i(t)\Big(\Delta\Pi_i, \Pi_j\Big)_{\partial\Omega} + a\sum_{i=1}^{N} a_i(t)\Big(\nabla\Pi_i, \nabla\Pi_j\Big)_{\Omega} - \\ a\alpha\sum_{i=1}^{N} a_i(t)\Big(\Pi_i, \Pi_j\Big)_{\partial\Omega} + b\sum_{i=1}^{N} a_i(t)\Big(\Pi_i, \Pi_j\Big)_{\Omega} = -\sum_{i=1}^{N} f(u_{N,\alpha,\beta}(\mathbf{x}_i))\Big(\Delta\Pi_i, \Pi_j\Big)_{\Omega} \\ \sum_{i=1}^{N} a_i(0)\Big(\Pi_i, \Pi_j\Big)_{\Omega} = \Big(\Pi_i, u_{\circ}\Big)_{\Omega}, \quad \bar{\Omega}, \end{aligned} \tag{3.4}$$

which results in the following first order system of differential equations with the initial conditions

$$\begin{aligned} &\frac{d\boldsymbol{a}(t)}{dt} = \mathbf{f}(\boldsymbol{a}(t)) - \mathbb{A}_{\alpha,\beta}\boldsymbol{a}(t), \quad t \in [0,T], \\ &\boldsymbol{a}(0) = \boldsymbol{a}_{\circ}, \end{aligned} \tag{3.5}$$

where the vectors $\boldsymbol{a}_{\circ}$ can be obtained. Also $\mathbb{A}_{\alpha,\beta}$ is the constant coefficient matrix corresponding to (3.3). Let

$$\mathbf{u}(t) = \boldsymbol{a}(t),$$

so that

$$\mathbf{f}(\boldsymbol{a}(t), t) = \mathbf{f}(\boldsymbol{a}(t)) - \mathbb{A}_{\alpha,\beta}\boldsymbol{a}(t), \quad t \in [0,T],$$

with the initial condition

$$\mathbf{u}(0) = \boldsymbol{a}_{\circ}.$$

Authors of [8] presented geometric numerical integration method to solve first order differential equation such as Eq. (3.5) called group preserving scheme (GPS) as follows

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \eta_k \mathbf{f}_k, \tag{3.6}$$

where

$$\eta_k = \frac{(\alpha_k - 1)\mathbf{f}_k^T.\mathbf{u}_k + \beta_k \|\mathbf{u}_k\|\|\mathbf{f}_k\|}{\|\mathbf{f}_k\|^2},$$

$$\alpha_k = \cosh(\Delta x \frac{\|\mathbf{f}_k\|}{\|\mathbf{u}_k\|}),$$

$$\beta_k = \sinh(\Delta x \frac{\|\mathbf{f}_k\|}{\|\mathbf{u}_k\|}),$$

where $\mathbf{f}_k = \mathbf{f}(\mathbf{U}_k, x_k)$.

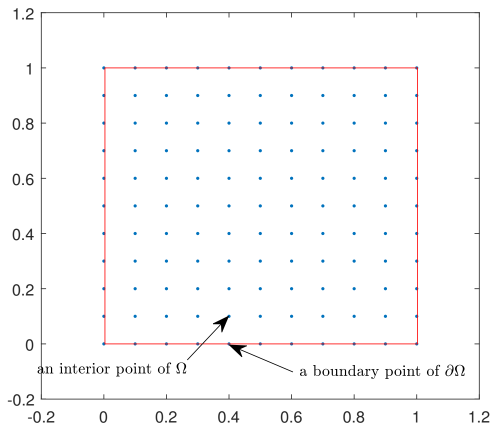
4. Numerical results

In this section, we present some numerical results by applying GRBFs to the two-dimensional Sivashinsky equation. In all examples, different values of parameters of Eq. (1.1) with some regularly distributed nodes are used. As mentioned before, to approximate the time variable, the GPS is employed and the relative least square error (r.l.s.e) is used to measure the error as follows

$$r.l.s.e = \sqrt{\frac{\sum_{i=1}^{NT} \Big(u_{h,\delta t,\beta}(x_i, y_i, T) - u(x_i, y_i, T) \Big)^2}{\sum_{i=1}^{NT} \Big(\sum_{k=0}^{4} \binom{4}{k} \frac{\partial^4}{\partial x^k \partial y^{4-k}} u(x_i, y_i, T) \Big)^2}},$$

where NT is the number of test points and $u_{h,\delta t}$ denotes the numerical solution for the parameters h and δt. So an iterative scheme is used to achieve the final time $T = 1$, in each problem. The results are obtained in the case of $\delta t = 0.01$ and 0.001.

FIGURE 1. Considered domain Ω of Example 4.1.



Example 4.1. Consider Eq. (1.1) with two sub-examples corresponding to sets of different parameters of $a = 1,\ b = 1,\ c = 0.7$ and $\alpha,\ \beta \in \{10^n \ \ s.t.\ \ n = 3, 4, 5, 6\}$ via various mesh spatial and time steps $h = 0.2,\ 0.1,\ 0.05$ and $\delta t = 0.01,\ 0.001$ on the domain as Figure 1. The corresponding exact solution of the collective parameters is given by

$$u_1(\mathbf{x}, t) = e^{-t} \sin(\pi \mathbf{x}_1) \sin(\pi \mathbf{x}_2),$$

where $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$, the functions $f(u) = 2u - \frac{1}{2}u$ and $u_\circ$ is obtained using the exact solutions. Table 1,2 presents $-\log_{10}(r.l.s.e)$ comparing $h = 0.2,\ 0.1,\ 0.05$ for different values $\log_{10}\alpha$ and $\log_{10}\beta$ for $\delta t = 0.01$ and 0.001, respectively. In Figure 3, the red and blue points respectively show the dispersion to different values of α and β. Planes A and B are the regression plane based on the spatial common parameter $h = 0.05$ and the different time parameters $\delta t = 0.01$ and $\delta t = 0.001$, respectively, which demonstrate growth of the accuracy can be deduced from the growth of the accuracy when increasing auxiliary parameters α and β.

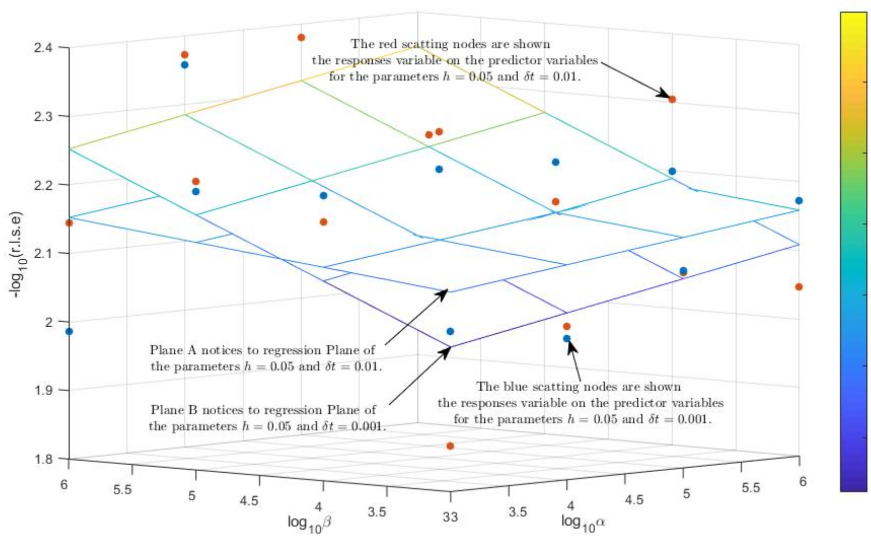
TABLE 1. Comparing $-\log_{10}(r.l.s.e)$ errors for $\delta t = 0.01$ of Example 4.1 by different parameters α, β and h for discretization GPS.

	$h = 0.2$				$h = 0.1$				$h = 0.05$			
	$\log_{10}\alpha = 3$	4	5	6	$\log_{10}\alpha = 3$	4	5	6	$\log_{10}\alpha = 3$	4	5	6
$\log_{10}(\beta) = 3$	1.0068	1.1849	1.1016	1.1610	1.6915	1.7457	1.7275	1.5822	2.0350	2.0066	2.0884	2.1731
4	0.9591	1.0870	0.9259	1.1138	1.5353	1.5533	1.5498	1.6105	2.2161	2.2371	2.2305	2.1992
5	1.0091	0.9777	0.9948	0.9502	1.5764	1.6492	1.7772	1.5590	2.2061	2.0808	2.1387	2.2270
6	1.0810	1.1370	0.9840	0.9888	1.4690	1.5681	1.5836	1.6941	1.9866	2.3576	2.1330	2.1721

TABLE 2. Comparing $-\log_{10}(r.l.s.e)$ errors for $\delta t = 0.001$ of Example 4.1 by different parameters α, β and h for discretization GPS.

	$h = 0.2$				$h = 0.1$				$h = 0.05$			
	$\log_{10}\alpha = 3$	4	5	6	$\log_{10}\alpha = 3$	4	5	6	$\log_{10}\alpha = 3$	4	5	6
$\log_{10}(\beta) = 3$	0.9216	0.9716	0.9025	1.0377	1.5478	1.6951	1.5257	1.5210	1.8673	2.0243	2.0847	2.0473
4	1.0688	1.1887	0.9037	1.0140	1.5052	1.5842	1.7244	1.7792	2.1786	2.2916	2.1720	2.3041
5	1.0522	1.1094	1.1734	1.0364	1.5290	1.5145	1.5334	1.5860	2.2218	2.0915	2.2546	2.1918
6	1.1349	0.9043	1.1242	0.9907	1.5667	1.5021	1.7506	1.7786	2.1441	2.3717	2.3798	2.2610

FIGURE 2. The above and below planes are estimated by multi-linear regressions of the responses in $-\log_{10}(r.l.s.e)$ on the predictors in $\left(\log_{10}(\alpha), \log_{10}(\beta)\right)$ for the parameters $h = 0.05$, $\delta t = 0.001$ and $h = 0.1$, $\delta t = 0.01$, respectively.



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