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Some notions of $(\sigma,\tau)-$amenability for unitization of Banach algebras

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Abstract Let $\mathcal{A}$ be a Banach algebra and σ and τ be continuous endomorphisms on $\mathcal{A}$. In this paper, we investigate $(\sigma,\tau)-$amenability and $(\sigma,\tau)-$weak amenability for unitization of Banach algebras, and also the relation between of them. We introduce and study the concepts $(\sigma,\tau)-$trace extention property, $(\sigma,\tau)-I-$weak amenability and $(\sigma,\tau)-$ideal amenability for $\mathcal{A}$ and its unitization, where I is a closed two-sided ideal in $\mathcal{A}$.

Keywords. Banach algebra, $(\sigma,\tau)-$ideal amenability, $(\sigma,\tau)-$weak amenability, Unitization.

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1. Introduction

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a Banach $\mathcal{A}-$bimodule. Then $\mathcal{X}^*$, the dual space of $\mathcal{X}$, by thy following module actions is a Banach $\mathcal{A}-$bimodule.

$$\langle x^* \cdot a, x\rangle = \langle x^*, a \cdot x\rangle, \qquad \langle a \cdot x^*, x\rangle = \langle x^*, x \cdot a\rangle,$$

for all $a \in \mathcal{A}$, $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$. So, it is clear that any closed two-sided ideal I of $\mathcal{A}$ and its dual I^* are Banach $\mathcal{A}-$bimodules. The linear map $d : \mathcal{A} \longrightarrow \mathcal{X}$ is derivation if $d(aa') = a \cdot d(a') + d(a) \cdot a'$ for all $a, a' \in \mathcal{A}$. Also, for $x \in \mathcal{X}$ the derivation $\mathrm{ad}_x : \mathcal{A} \longrightarrow \mathcal{X}$ defined by $\mathrm{ad}_x(a) = a \cdot x - x \cdot a$ called inner derivation and if for derivation $d : \mathcal{A} \longrightarrow \mathcal{X}$ there exists a net $(x_\alpha) \subseteq \mathcal{X}$ such that $d(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a)$ for all $a \in \mathcal{A}$, then d is an approximate inner derivation. The set of all continuous derivations and inner derivations from $\mathcal{A}$ to Banach $\mathcal{A}-$bimodule $\mathcal{X}$ are denoted by $Z^1(\mathcal{A},\mathcal{X})$ and $N^1(\mathcal{A},\mathcal{X})$, respectively. Also, the quotient space $H^1(\mathcal{A},\mathcal{X}) = Z^1(\mathcal{A},\mathcal{X})/N^1(\mathcal{A},\mathcal{X})$ is first cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{X}$. The Banach algebra $\mathcal{A}$ is called amenable if $H^1(\mathcal{A},\mathcal{X}^*) = \{0\}$ for all Banach $\mathcal{A}-$bimodule $\mathcal{X}$. The concept of amenability of Banach algebra was first introduced by Johnson [2] in 1972.

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We recall that from [6], for any two bounded continuous endomorphisms σ and τ on Banach algebra $\mathcal{A}$, the linear map $d : \mathcal{A} \longrightarrow \mathcal{X}$ is called $(\sigma, \tau)-$derivation if

$$d(aa') = \sigma(a) \cdot d(a') + d(a) \cdot \tau(a'),$$

for all $a, a' \in \mathcal{A}$. For any $x \in \mathcal{X}$, the derivation $\mathrm{ad}_x : \mathcal{A} \longrightarrow \mathcal{X}$ defined via $\mathrm{ad}_x(a) = \sigma(a) \cdot x - x \cdot \tau(a)$ is called $(\sigma, \tau)-$inner derivation for all $a \in \mathcal{A}$. Also, the derivation $d : \mathcal{A} \longrightarrow \mathcal{X}$ is $(\sigma, \tau)-$approximately inner if there exists the net $(x_\alpha) \subseteq \mathcal{X}$ such that $d(a) = \mathrm{ad}_{x_\alpha}(a) = \lim_\alpha(\sigma(a) \cdot x_\alpha - x_\alpha \cdot \tau(a))$ in the stronge topology for all $a \in \mathcal{A}$. Similarly, the set of all continuous $(\sigma, \tau)-$derivations and $(\sigma, \tau)-$ inner derivations from $\mathcal{A}$ to $\mathcal{X}$ denoted by $Z^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$ and $N^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$, respectively, and $H^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}) = Z^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})/N^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$ is the first $(\sigma, \tau)-$cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{X}$. The Banach algebra $\mathcal{A}$ is $(\sigma, \tau)-$amenable if any $(\sigma, \tau)-$derivation $d : \mathcal{A} \longrightarrow \mathcal{X}^*$ be $(\sigma, \tau)-$inner, for all Banach $\mathcal{A}-$bimodule $\mathcal{X}$. The notion of $(\sigma, \tau)-$amenability of Banach algebra introduced and studied by Moslehian [5]. Moreover, for σ as an bounded endomorphism of $\mathcal{A}$, Momeni, Yazdanpanah and Mardanbeigi [3] introduced the notios $\sigma-I-$weak amenability and $\sigma-$ideal amenability for Banach algebras. Here, we introduce and study the concepts $(\sigma, \tau) - I-$weak amenability and $(\sigma, \tau)-$ideal amenability for $\mathcal{A}$, where I is a closed two-sided ideal in $\mathcal{A}$.

In section 1, we study the notions $(\sigma, \tau)-$amenability and $(\sigma, \tau)-$weak amenability of Banach algebra $\mathcal{A}$ and ,$\mathcal{A}^\sharp$, its unitization. Also, we investigate the relationship these concepts between Banach algebra $\mathcal{A}$ and $\mathcal{A}^\sharp$. In section 2, we introduce $(\sigma, \tau)-$ideal amenability of Banach algebras and We examine the relationship between these concepts for $\mathcal{A}$ and $\mathcal{A}^\sharp$. finally, we obtain results for hereditary properties of $(\sigma, \tau)-$ideal amenability of Banach algebras.

2. $(\sigma, \tau)-$weak amenability

For two endomorphisms $\sigma, \tau : \mathcal{A} \longrightarrow \mathcal{A}$, the Banach algebra $\mathcal{A}$ is $(\sigma, \tau)-$weak amenable if every derivation $d : \mathcal{A} \longrightarrow \mathcal{A}^*$ is $(\sigma, \tau)-$inner. In the special case if σ and τ are identity homomorphisms on $\mathcal{A}$, then weak amenability of $\mathcal{A}$ is obtained (See [1] for more details). Suppose that $\mathcal{A}$ is not unital Banach algebra, $\mathcal{A}^\sharp = \mathcal{A} \oplus \mathbb{C}e$ and σ be a homomorphism on $\mathcal{A}$. Then we define $\sigma^\sharp : \mathcal{A}^\sharp \longrightarrow \mathcal{A}^\sharp$ by

$$\sigma^\sharp((a, \lambda)) = (\sigma(a), \lambda)$$

for all $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Clearly, $\sigma^\sharp$ is a homomorphism on $\mathcal{A}^\sharp$. It is clear that if $d : \mathcal{A} \longrightarrow \mathcal{A}$ be any endomorphism on Banach algebra $\mathcal{A}$, then d is a $\frac{d}{2}$-derivation. Moreover every ordinary derivation of an algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-module $\mathcal{X}$ is an $id_{\mathcal{A}}$-derivation, where $id_{\mathcal{A}}$ is the identity mapping on the algebra $\mathcal{A}$.

Example 2.1. Let $\mathcal{A}$ be a Banach algebra and suppose that $\sigma, \tau : \mathcal{A} \longrightarrow \mathcal{A}$ are two arbitrary linear mappings. Moreover, suppose that $\mathcal{X}$ is a Banach $\mathcal{A}^\sharp$-bimodule. If x is an element of $\mathcal{X}$ satisfying

$$x(\sigma(aa') - \sigma(a)\sigma(a'), 0) = (\tau(aa') - \tau(a)\tau(a'), 0)x,$$

for all $a, a' \in \mathcal{A}$. Then the mapping $d : \mathcal{A}^\sharp \longrightarrow \mathcal{X}$ defined by

$$d(a, \lambda) = x(\sigma(a), \lambda) - (\tau(a), \lambda)x$$

is a $(\sigma^\sharp, \tau^\sharp)$-derivation, for each $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$.

Note that in the Example 2.1, if σ and τ are two endomorphisms, then x can be any arbitrary element of $\mathcal{X}$. Now, we have the following Theorem.

Proposition 2.2. *Let σ and τ be two homomorphisms on Banach algebra $\mathcal{A}$. Then $\mathcal{A}$ is $(\sigma, \tau)-$amenable if and only if $\mathcal{A}^\sharp$ be $(\sigma^\sharp, \tau^\sharp)-$amenable.*

proof. First suppose that $\mathcal{A}$ is $(\sigma, \tau)-$amenable, $\mathcal{X}$ is Banach $\mathcal{A}^\sharp-$bimodule and $D^\sharp : \mathcal{A}^\sharp \longrightarrow \mathcal{X}^*$ is a $(\sigma^\sharp, \tau^\sharp)-$derivation. Then by below module actions $\mathcal{X}$ is a Banach $\mathcal{A}-$bimodule

$$x \cdot a = x(a, 0), \qquad a \cdot x = (a, 0)x,$$

for all $a \in \mathcal{A}$ and $x \in \mathcal{X}$. Also, by a rutine calculating conclude that the restriction $D^\sharp$ to $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ by $D(a) = D^\sharp(a, 0)$ is a $(\sigma, \tau)-$derivation. So, there exists $f \in \mathcal{X}^*$ such that $D(a) = f \cdot \sigma(a) - \tau(a) \cdot f$. It is clear that $D^\sharp(0, 1) = 0$, thus we have that

$$\begin{aligned} D^\sharp(a, \lambda) &= D(a) + \lambda D^\sharp(0, 1) \\ &= D(a) \\ &= f \cdot \sigma^\sharp(a, \lambda) - \tau^\sharp(a, \lambda) \cdot f \end{aligned}$$

for all $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$.

Conversely, suppose that $\mathcal{X}$ is a Banach $\mathcal{A}-$bimodule and $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ is a $(\sigma, \tau)-$derivation. Then by module actions

$$x \cdot (a, \lambda) = xa + \lambda x \qquad (a, \lambda) = ax + \lambda x,$$

$\mathcal{X}$ is $\mathcal{A}^\sharp-$bimodule for all $a \in \mathcal{A}$, $x \in \mathcal{X}$ and $\lambda \in \mathbb{C}$. Now we define $D^\sharp : \mathcal{A}^\sharp \longrightarrow \mathcal{X}^*$ by $D^\sharp(a, \lambda) = D(a)$. Clearly, $D^\sharp$ is $(\sigma^\sharp, \tau^\sharp)-$derivation. Thus, there exists $f \in \mathcal{X}^*$ such that

$$D^\sharp(a, \lambda) = f \cdot \sigma^\sharp(a, \lambda) - \tau^\sharp(a, \lambda) \cdot f.$$

Therefore, we have that

$$\begin{aligned} \langle D(a), x\rangle &= \langle D^\sharp(a, \lambda), x\rangle \\ &= \langle f \cdot \sigma^\sharp(a, \lambda) - \tau^\sharp(a, \lambda) \cdot f, x\rangle \\ &= \langle f, (\sigma(a), \lambda) \cdot x\rangle - \langle f, x \cdot (\tau(a), \lambda)\rangle \\ &= \langle f, \sigma(a) \cdot x + \lambda x - x \cdot \tau(a) - \lambda x\rangle \\ &= \langle f \cdot \sigma(a) - \tau(a) \cdot f, x\rangle, \end{aligned}$$

for all $a \in \mathcal{A}$ and $x \in \mathcal{X}$. Hence, $\mathcal{A}$ is $(\sigma, \tau)-$amenable. □

Suppose that $\mathcal{A}$ is not unital and $\mathcal{A}^\sharp = \mathcal{A} \oplus \mathbb{C}e$. Then we define $e^* \in (\mathcal{A}^\sharp)^*$ such that $\langle e^*, e\rangle = 1$ and $\langle e^*, a\rangle = 0$, for all $a \in \mathcal{A}$. Therefore, we have that

$$\mathcal{A}^{\sharp^*} = \mathcal{A}^* \oplus \mathbb{C}e^*.$$

The module actions of $\mathcal{A}^\sharp$ on $\mathcal{A}^{\sharp^*}$ are given by

$$(a + \lambda e) \cdot (f + \mu e^*) = (\lambda f + a \cdot f) + (\lambda\mu + f(a))e^*,$$

$$(f+\mu e^*)\cdot(a+\lambda e)=(\lambda f+f\cdot a)+(\lambda\mu+f(a))e^*.$$

where $a\in\mathcal{A}$, $f\in\mathcal{A}^*$ and $\lambda,\mu\in\mathbb{C}$.

Corollary 2.3. *If σ be a homomorphism with dense range on Banach algebra $\mathcal{A}$ and $\mathcal{A}$ be $\sigma-$weakly amenable, then $\overline{\mathcal{A}^2}=\mathcal{A}$.*

Proof. The proof of this result is similar to [4], Proposition 2.1 but here it is enough define $d:\mathcal{A}\longrightarrow\mathcal{A}^*$ by $\langle d(a),a'\rangle=\langle f,\sigma(a)\rangle\langle f,a'\rangle$ for all $a,a'\in\mathcal{A}$. □

Proposition 2.4. *Let $\mathcal{A}$ be a Banach algebra, σ and τ are two continuous homomorphisms on $\mathcal{A}$ such that $\sigma(a)b=a\tau(b)$ for all $a,b\in\mathcal{A}$ and $\mathcal{A}$ is $(\sigma,\tau)-$weakly amenable. Then $\mathcal{A}^\sharp$ is $(\tilde\sigma,\tilde\tau)-$weakly amenable if one of the following statements hold.*
(i) $a\tau(\sigma(b))=\sigma(\tau(a))b$, for all $a,b\in\mathcal{A}$.
(ii) $f\circ\sigma(\mathcal{A}^2)=f\circ\tau(\mathcal{A}^2)$, for all $f\in\mathcal{A}^*$

Proof. Suppose that $\tilde D:\mathcal{A}^\sharp\longrightarrow(\mathcal{A}^\sharp)^*$ is a $(\tilde\sigma,\tilde\tau)-$derivation. We show that $\tilde D$ is $(\tilde\sigma,\tilde\tau)-$inner. Clearly, $\tilde D(e)=0$. So, we can write $\tilde D:\mathcal{A}\longrightarrow(\mathcal{A}^\sharp)^*$. Thus, there are bounded linear maps such as $D:\mathcal{A}\longrightarrow\mathcal{A}^*$ and $\varphi:\mathcal{A}\longrightarrow\mathbb{C}$ such that $\tilde D(a)=D(a)+\varphi(a)e^*$ for all $a\in\mathcal{A}$. By hypothesis conclude

$$\begin{aligned}\tilde D(ab)&=\tilde\sigma(a)\cdot\tilde D(b)+\tilde D(a)\cdot\tilde\tau(b)\\&=\sigma(a)\cdot(D(b)+\varphi(b)e^*)+(D(a)+\varphi(a)e^*)\cdot\tau(b)\\&=\sigma(a)\cdot D(b)+\langle D(b),\sigma(a)\rangle e^*+D(a)\cdot\tau(b)+\langle D(a),\tau(b)\rangle e^*\end{aligned}$$

for all $a,b\in\mathcal{A}$. On the other hand, $\tilde D(ab)=D(ab)+\varphi(ab)e^*$. Therefore, for all $a,b\in\mathcal{A}$ we have that

$$D(ab)=\sigma(a)D(b)+D(a)\tau(b),\qquad \varphi(ab)=\langle D(b),\sigma(a)\rangle+\langle D(a),\tau(b)\rangle.$$

Thus, there exists $f\in\mathcal{A}^*$ such that $D(a)=\mathrm{ad}_f(a)=\sigma(a)\cdot f-f\cdot\tau(a)$ for all $a\in\mathcal{A}$. So, we conclude

$$\begin{aligned}\varphi(ab)&=\langle\sigma(b)\cdot f-f\cdot\tau(b),\sigma(a)\rangle+\langle\sigma(a)\cdot f-f\cdot\tau(a),\tau(b)\rangle\\&=\langle f,\sigma(ab)\rangle-\langle f,\tau(ab)\rangle\qquad(1)\end{aligned}$$

for all $a,b\in\mathcal{A}$. It is clear that if one of the conditions (i) or (ii) is holds, then $\varphi(ab)=0$ for all $a,b\in\mathcal{A}$. Since, $\sigma(a)b=a\tau(b)$ for all $a,b\in\mathcal{A}$. Thus, similar to [4], Proposition 2.1, $\mathcal{A}$ is essential and consequent that $\varphi(a)=0$ for all $a\in\mathcal{A}$. Therefore, we have that $\tilde D(a)=D(a)$ for all $a\in\mathcal{A}$. So, $\tilde D$ is $(\tilde\sigma,\tilde\tau)-$inner. □

Proposition 2.5. *Let $\mathcal{A}$ be a Banach algebra, σ and τ are two continuous homomorphisms on $\mathcal{A}$ and $\mathcal{A}^\sharp$ be $(\tilde\sigma,\tilde\tau)-$weakly amenable. Then $\mathcal{A}$ is $(\sigma,\tau)-$weakly amenable.*

Proof. Let $D:\mathcal{A}\longrightarrow\mathcal{A}^*$ be a $(\sigma,\tau)-$derivation. We define $\tilde D:\mathcal{A}^\sharp\longrightarrow(\mathcal{A}^\sharp)^*$ via $\tilde D(a+\lambda)=D(a)$. Since, for all $a,b\in\mathcal{A}$ conclude

$$\langle D(a),\tau(b)\rangle e^*=0,\langle D(b),\sigma(a)\rangle e^*=0.$$

Hence, we have that

$$\begin{aligned}\tilde{D}((a+\lambda e)(b+\mu e)) &= \sigma(a)D(b)+D(a)\tau(b)+\mu D(a)+\lambda D(b)\\ &= (\sigma(a)D(b)+\lambda D(b)+\langle D(b),\sigma(a)\rangle e^*)\\ &\quad +(D(a)\tau(b)+\mu D(a)+\langle D(a),\tau(b)\rangle e^*)\\ &= \tilde{\sigma}(a+\lambda e)\cdot\tilde{D}(b+\mu e)+\tilde{D}(a+\lambda e)\cdot\tilde{\tau}(b+\mu e).\end{aligned}$$

for all $a,b\in\mathcal{A}$ and $\lambda,\mu\in\mathbb{C}$. So, there exists $(f+\mu e^*)\in(\mathcal{A}^\sharp)^*$ such that $\tilde{D}(a+\lambda e)=\hat{\sigma}(a+\lambda e)\cdot(f+\mu e^*)-(f+\mu e^*)\cdot\hat{\tau}(a+\lambda e)$ for all $a\in\mathcal{A}$ and $\lambda\in\mathbb{C}$. Therefore, we have that $D(a)=\sigma(a)\cdot f+f\cdot\tau(a)$ for all $a\in\mathcal{A}$. □

3. $(\sigma,\tau)-$ideal amenability

We recall that, if we use the ideal of $\mathcal{A}$ in the definition of amenability instead of the Banach bimodule $\mathcal{X}$, then the ideal amenability concepts are obtained.

Definition 3.1. Suppose that σ and τ are two bounded endomorphism on Banach algebra $\mathcal{A}$ and I is a closed two-sided ideal in $\mathcal{A}$. Then we say that $\mathcal{A}$ is $(\sigma,\tau)-I-$weakly amenable if every $(\sigma,\tau)-$derivation from $\mathcal{A}$ into I^* is $(\sigma,\tau)-$inner. Also, $\mathcal{A}$ is called $(\sigma,\tau)-$ideally amenable if $\mathcal{A}$ be $(\sigma,\tau)-I-$weakly amenable for all closed two-sided ideal I in $\mathcal{A}$.

Next, we characterize the ideals of $\mathcal{A}^\sharp$. Note that if M is a closed two-sided ideal of $\mathcal{A}^\sharp$, then we can write M to the one of the below form.

(i) $M=\mathcal{A}\oplus Je$ such that J is a closed two-sided ideal of $\mathbb{C}$.

(ii) $M=I\oplus 0e$ such that I is a closed two-sided ideal of $\mathcal{A}$.

Theorem 3.2. *Suppose that σ and τ are two bounded endomorphisms on Banach algebra $\mathcal{A}$. Then*

i) $\mathcal{A}$ is $(\sigma,\tau)-$ideally amenable, if $\mathcal{A}^\sharp$ be $(\hat{\sigma},\hat{\tau})$-ideally amenable.

ii) $\mathcal{A}^\sharp$ is $(\tilde{\sigma},\tilde{\tau})-$ideally amenable, if $\mathcal{A}$ be (σ,τ)-ideally amenable and $f\circ\sigma=f\circ\tau$ for all $f\in\mathcal{A}^*$.

Proof. i) Let $d:\mathcal{A}\longrightarrow I^*$ be a (σ,τ)-derivation such that I is a closed two-sided ideal of Banach algebra $\mathcal{A}$. It is clear that I is a closed two-sided ideal of $\mathcal{A}^\sharp$ and $\tilde{d}:\mathcal{A}^\sharp\longrightarrow(I)^*$ via $\tilde{d}(a+\lambda)=d(a)$ is a $(\hat{\sigma},\hat{\tau})$-derivation. So, there exists $g\in I^*$ such that

$$\tilde{d}(a,\lambda)=\mathrm{ad}_g^{(\hat{\sigma},\hat{\tau})}(a,\lambda)=\hat{\sigma}(a,\lambda)\cdot g-g\cdot\hat{\tau}(a,\lambda)$$

for all $a\in\mathcal{A}$ and $\lambda\in\mathbb{C}$. Therefore, conclude that $d(a)=\sigma(a)\cdot g-g\cdot\tau(a)$ for all $a\in\mathcal{A}$.

ii) By above statement, if $\tilde{D}:\mathcal{A}^\sharp\longrightarrow M^*$ is $(\tilde{\sigma},\tilde{\tau})-$derivation, where M is a closed two-sided ideal of $\mathcal{A}^\sharp$, then it is clear that the map $D:\mathcal{A}\longrightarrow(M_{|_\mathcal{A}})^*$ via $D(a)=\tilde{D}(a,0)$ is $(\sigma,\tau)-$derivation. Thus, there exists $f\in(M_{|_\mathcal{A}})^*$ such that $D(a)=\sigma(a)\cdot f-f\cdot\tau(a)$ for all $a\in\mathcal{A}$. We know that $\tilde{D}(e)=0$. So, $\tilde{D}(a,\lambda)=D(a)$. On the other hand, let $\tilde{f}$ be the extension of f on $\mathcal{A}$. Then

$$\begin{aligned}\tilde{\sigma}(a,\lambda)\cdot\tilde{f}-\tilde{f}\cdot\tilde{\tau}(a,\lambda) &= \sigma(a)\cdot\tilde{f}+\lambda\tilde{f}+\tilde{f}(\sigma(a))e^{*}-\tilde{f}\cdot\tau(a)-\lambda\tilde{f}-\tilde{f}(\tau(a))e^{*}\\ &= \sigma(a)\cdot\tilde{f}-\tilde{f}\cdot\tau(a)\\ &= \tilde{D}(a,\lambda),\end{aligned}$$

for all $a\in\mathcal{A}$ and $\lambda\in\mathbb{C}$. □

Corollary 3.3. *Let σ be a bounded homomorphisms on Banach algebra $\mathcal{A}$ and $\mathcal{A}^{\sharp}$ be $\tilde{\sigma}-$ideally amenable. Then $\mathcal{A}$ is $\sigma-$ ideally amenable.*

Proof. If I be a closed two-sided ideal of $\mathcal{A}$ and $d:\mathcal{A}\longrightarrow I^{*}$ be a $\sigma-$derivation, then I is a closed two-sided ideal of $\mathcal{A}^{\sharp}$ and we can extend d to $\tilde{d}:\mathcal{A}^{\sharp}\longrightarrow I^{*}$ by $\tilde{d}(a,\lambda)=d(a)$ for all $a\in\mathcal{A}$ and $\lambda\in\mathbb{C}$. Moreover, I is $\mathcal{A}^{*}-$bimodule actions defined as follow

$$f\cdot(a,\lambda)=f\cdot a+\lambda f \qquad\qquad (a,\lambda)\cdot f=acdotf+\lambda f$$

for all $a\in\mathcal{A}$, $\lambda\in\mathbb{C}$ and $f\in\mathcal{A}^{*}$. Clearly, $\tilde{d}$ is $\hat{\sigma}-$derivation. Thus, there exists $g\in I^{*}$ such that $\tilde{d}(a,\lambda)=\mathrm{ad}_{g}(a,\lambda)$ for all $a\in\mathcal{A}$ and $\lambda\in\mathbb{C}$. finally, we have that

$$\begin{aligned}d(a) &= \tilde{d}(a,\lambda)\\ &= \hat{\sigma}(a,\lambda)\cdot f-f\cdot\hat{\sigma}(a,\lambda)\\ &= \sigma(a)\cdot f-f\cdot\sigma(a)\end{aligned}$$

for all $a\in\mathcal{A}$ and $\lambda\in\mathbb{C}$. Therefore, d is a $\sigma-$inner derivation and $\mathcal{A}$ is $\sigma-$ideally amenable. □

Definition 3.4. We say that the closed two-sided ideal I of Banach algebra $\mathcal{A}$ has the (σ,τ)-trace extension property if for every $m\in I^{*}$ such that $\sigma(a)\cdot m=m\cdot\tau(a)$ for each $a\in\mathcal{A}$, can be extend to an element $f\in\mathcal{A}^{*}$ such that $\sigma(a)\cdot f=f\cdot\tau(a)$ for all $a\in\mathcal{A}$.

Proposition 3.5. *Suppose that σ and τ are two bounded endomorphisms on Banach algebra $\mathcal{A}$ and $\mathcal{A}$ is $(\sigma,\tau)-$ideally amenable. Moreover, suppose that the closed two-sided ideal I of $\mathcal{A}$ has $(\sigma,\tau)-$trace extension property. Then $\frac{\mathcal{A}}{I}$ is $(\tilde{\sigma},\tilde{\tau})-$ideally amenable.*

Proof. Let $\frac{J}{I}$ be a closed two-sided ideal of $\frac{\mathcal{A}}{I}$ and $d:\frac{\mathcal{A}}{I}\longrightarrow(\frac{J}{I})^{*}$ be a $(\tilde{\sigma},\tilde{\tau})-$derivation. It is to see that J is a closed two-sided ideal of $\mathcal{A}$. Now, if $q:\mathcal{A}\longrightarrow\frac{\mathcal{A}}{I}$ and $\pi:J\longrightarrow\frac{J}{I}$ are natural quotient maps and π^{*} is a first adjoint map of π, then it is clear that $\pi^{*}\circ d\circ q:\mathcal{A}\longrightarrow J^{*}$ is a $(\sigma,\tau)-$derivation. Thus, since $\mathcal{A}$ is $(\sigma,\tau)-$ideally amenable, there exists an element $g\in J^{*}$ such that

$$(\pi^{*}\circ d\circ q)(a)=\mathrm{ad}_{g}^{(\sigma,\tau)}(a)=\sigma(a)\cdot g-g\cdot\tau(a)$$

for all $a \in \mathcal{A}$. We can restrict g to I and denote by m that $m \in I^*$. Thus, we have that

$$\begin{aligned} \langle \sigma(a)\cdot m - m\cdot\tau(a), i\rangle &= \langle g, i\sigma(a) - \tau(a)i\rangle \\ &= \langle \mathrm{ad}_g^{(\sigma,\tau)}(a), i\rangle \\ &= \langle (\pi^* \circ d \circ q)(a), i\rangle \\ &= \langle d(a+I), I\rangle \\ &= 0 \end{aligned}$$

for all $a \in \mathcal{A}$ and $i \in I$. But, by hypothesis I has $(\sigma,\tau)-$trace extension property. Hence, there exists $f \in \mathcal{A}^*$ such that $\sigma(a)\cdot f = f\cdot\tau(a)$ for all $a \in \mathcal{A}$. Let $g' = f_{|_J} : J \longrightarrow \mathbb{C}$. Then $g - g' \in J^*$, $(g-g')_{|_I} = 0$ and $g - g' \in (\frac{J}{I})^*$. Therefore, we have that

$$\begin{aligned} \langle d(a+I), j+I\rangle &= \langle (\pi^* \circ d \circ q)(a), j\rangle \\ &= \langle \sigma(a)\cdot g - g\cdot\tau(a), j\rangle \\ &= \langle \sigma(a)\cdot(g-g') - (g-g')\cdot\tau(a), j\rangle \\ &= \langle \mathrm{ad}_{(g-g')}^{(\tilde{\sigma},\tilde{\tau})}(a+I), j+I\rangle. \end{aligned}$$

for all $a \in \mathcal{A}$ and $j \in J$, as required. □

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