Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 7, No. 3, 2019, pp. 396-417

A novel hybrid method for solving combined functional neutral differential equations with several delays and investigation of convergence rate via residual function

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Abstract In this study, we introduce a novel hybrid method based on a simple graph along with operational matrix to solve the combined functional neutral differential equations with several delays. The matrix relations of the matching polynomials of complete and path graphs are employed in the matrix-collocation method. We improve the obtained solutions via an error analysis technique. The oscillation of them on time interval is also estimated by coupling the method with Laplace-Padé technique. We develop a general computer program and so we can efficiently monitor the precision of the method. We investigate a convergence rate of the method by constructing a formula based on the residual function. Eventually, an algorithm is described to show the easiness of the method.

Keywords. Collocation points, Graph theory, Laplace-Padé method, Matching polynomial, Vulnerability.
2010 Mathematics Subject Classification. 34K40, 65L60, 68R10, 05C31.

1. Introduction

In this study, we consider the combined functional neutral differential equation with several delays (CFNDEs)

$$\sum_{k=0}^{m} P_k(t)\, y^{(k)}(h_k(t)) + \left(Q(t)\, y^{(n)}(t) + \sum_{j=0}^{n} \binom{n}{j} R^{(j)}(t)\, y^{(n-j)}(\tau_j(t)) \right) = f(t),\ m \geq n \quad (1.1)$$

Received: 16 August 2018 ; Accepted: 10 November 2018.
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subject to the mixed conditions

$$\sum_{k=0}^{m-1} a_{ik} y^{(k)}(a) + b_{ik} y^{(k)}(b) = \tau_i,\ i = 0, 1, \ldots, m-1. \tag{1.2}$$

Here, $P_k(t)$, $Q(t)$, $R(t)$, $f(t)$, and two functional delays $h_k(t)$ and $\tau_j(t)$ are analytic functions on $a \le t \le b$.

Our first aim in this study is to efficiently solve Eq. (1.1) and our second aim is to observe which of complete (K_n) and path (P_n) graphs provides more proper matrix structure for the method. By applying our method to solve Eq. (1.1), we seek a matching polynomial solution of Eq. (1.1) as

$$y(t) \cong y_N(t) = \sum_{n=0}^{N} y_n M_n(G_n, t), \tag{1.3}$$

where y_n's are unknown coefficients to be determined, and G_n is a simple graph with n vertices and m edges. The standard (SCPs) and Chebyshev-Lobatto (CLCPs) collocation points, which are used in the method, are defined respectively by

$$t_i = a + \left(\frac{b-a}{N}\right) i \text{ and } t_i = \frac{a+b}{2} + \frac{a-b}{2} \cos\left(\frac{\pi i}{N}\right), \tag{1.4}$$

where $i = 0, 1, \ldots, N$ and $a = t_0 < t_1 < \ldots < t_N = b$.

In recent years, many remarkable efforts have focused mainly on differential equations and their types, which are neutral, retarded, functional, and delay ones. These equations are encountered while investigating the real life problems arising in mathematics, physics, biology, and engineering [12, 13, 20, 28, 31, 50, 40, 47]. For example, pantograph equations encountered in electrodynamics are one of the types of the neutral differential equations with proportional delays [40].

We draw attention to the fact that there is a stiff problem turns out be exist, while analytically solving the mentioned equations. Thus, numerical methods are employed for eliminating this hardiness. So far, Dickson [32, 33, 34, 35], Taylor [45], Chebyshev matrix-collocation [19], homotopy perturbation [25], variational iteration [26] methods have been successively employed. Furthermore, Lu and Ge [38] have studied on the existence periodic solutions of neutral functional differential equations. Liu et al. [37] have investigated the existence of nonoscillatory solutions of higher-order neutral differential equations with distributed coefficients and delays. Zhao [51] has discussed the stability of neutral differential equations with variable delays. Fabiano and Payne [14] have established the spline approximation to treat the linear neutral delay-differential equations. Raza and Khan [44] have obtained the Haar wavelet series solutions of neutral delay differential equations. The readers are also encouraged to see theoretical and numerical studies in [11, 30, 41].

We become Eq. (1.1) a generalized and combined form of FNDEs, in order to extend this equation for other well-known problems. A novel hybrid method that aimed to solve Eq. (1.1), consists of a graph polynomial and operational matrix along with collocation points. This polynomial is the matching polynomial, which was denominated by Farrell [15] in 1979. The detailed information about the matching polynomial is given in section 1.1.1.

1.1. **A brief introduction to graph theory.**

A finite simple graph consists of vertex set $V(G)$ and edge set $E(G)$. Edge(s) in a graph connects vertices. Some fundamental graphs, such as complete, complete bipartite, cycle, wheel, path, tree and star graphs, form a basis of graph theory. A detailed terminology and notations for graph theory can be found in [18, 24].

Graph theory provides several fields of study, such as vulnerability, the modeling of real life and networks, transportation, algorithms, and mathematical chemistry [3, 4, 6, 21, 18, 29, 24]. Some K_n and P_n are illustrated in Figure 1 and it is evident that K_1 contains only one vertex. When K_n and P_n are exposed to an intervention, they have different reactions in terms of vulnerability, which investigates the reaction of a graph or network after encountering a failure or an intervention [3, **?**]. Connectivity, which was introduced by Harary [24], is one of the vulnerability parameters. In order to obtain a disconnected form of a simple graph G, connectivity $\kappa(G)$ measures the minimum number(s) of vertices to be deleted from G, so

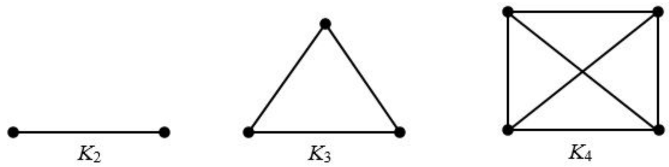
$$\kappa(G) = \min\{|S| : S \subseteq V(G)\},$$

where $V(G)$ is vertex set of G and $|S|$ is a vertex cut set [24]. Similarly, edge connectivity was introduced in [24]. It can be noticed from Figure 1 that $\kappa(P_n) = 1$ and $\kappa(K_n) = n - 1$. Since the value of $\kappa(K_n)$ is far bigger than that of $\kappa(P_n)$, complete graph represents a durable and dependable structure for connectivity. The reason is that its all vertices are incident to each other. When it encounters a damage or an interruption, the connection is strictly sustainable. On the other hand, some vulnerability parameters, such as the edge scattering number [3] and edge neighbor rupture degree [4] prove this result. It is worth specifying that if any vertex or edges in path graph is removed, the structure of path graph will be disrupted immediately. The reader can refer to [18, 24] for more information about vulnerability and graph theory.

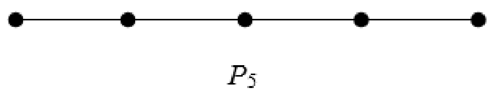
1.1.1. *Some fundamental properties of matching polynomial of a graph.*

In 1972, a polynomial was introduced by Heilman and Lieb [27] for the theory of monomer-dimer systems, but they determined no specific name for this polynomial. In 1979, Farrell [15] denominate it as the matching polynomial. In addition, the matching polynomial possesses the different names, such as acyclic [22] and reference

Figure 1. Some fundamental graphs.



(A) Complete graphs K_2, K_3 and K_4.



(B) A path graph P_5.

[1]. Let G_n be a graph with n vertices and m edges. It is generally defined to be (see [15, 27, 22, 23, 48])

$$M_n\left(G_n,t\right)=\sum_{k=0}^{m}\left(-1\right)^k p\left(G_n,k\right)t^{n-2k}, \tag{1.5}$$

where $p\left(G_n,k\right)$ is the matching number to be obtained by k-matching numbers, and also $p\left(G,0\right)=1$ is designated for k=0 and any graph G.

In 1988, Hosoya [29] gave mathematical properties and physico-chemical interpretations of the matching and some other polynomials in chemistry. Aihara [1] used the reference (matching) polynomial for monocyclic conjugated system. Godsil and Gutman [17] constructed the properties of the matching polynomial and the relation between the matching and characteristic polynomials of a graph. Yan and Yeh [49] performed a study on the matching polynomial of subdivision graphs. The matching polynomial was generalized by Araujo et al. [2] to find the relation between the generalized matching polynomial and hypergeometric functions. A general method was introduced by Babic et al. [5] for obtaining the matching polynomial of a polygraph. Bian et al. [9] obtained the recurrences of the matching polynomial of ortho-chains and meta-chains. Some fundamental properties of this polynomial are as follows [23]:

- $p\left(G,1\right)=m$,
- $p\left(G,2\right)=\frac{m(m+1)}{2}-\frac{1}{2}\left(d_1^2+d_2^2+\ldots+d_n^2\right)$, where d_i is the degree of vertex v_i in G.
- $k>n/2\Rightarrow p\left(G,k\right)=0$,
- $p\left(G,k\right)=0\Rightarrow p\left(G,k+1\right)=0$.

There is a relation between the matching and the independence polynomials. The independence polynomial of a graph G equals to the matching polynomial of the line graph of G [36]. In addition, the other relation between matching and chromatic polynomials can be found in [16].

Some well-known polynomials can be obtained by using the matching polynomial for some specific graphs. For example, the matching polynomial of complete graph K_n is obtained by Gutman [22, 23] as

$$M_0\left(K_0,t\right)=1,\ M_1\left(K_1,t\right)=t,$$

$$\begin{aligned} M_2\left(K_2,t\right) &= \sum_{k=0}^{1}(-1)^k p\left(K_2,k\right)t^{2-2k}\begin{cases} k=0\Rightarrow p\left(K_2,0\right)=1 \\ k=1\Rightarrow p\left(K_2,1\right)=m=1 \end{cases} \\ &= (-1)^0.1.t^{2-0}+(-1)^1.1.t^{2-2}=t^2-1, \end{aligned}$$

$$\begin{aligned} M_3\left(K_3,t\right) &= \sum_{k=0}^{3}(-1)^k p\left(K_3,k\right)t^{3-2k}\begin{cases} k=0\Rightarrow p\left(K_3,0\right)=1 \\ k=1\Rightarrow p\left(K_3,1\right)=3 \end{cases} \\ &= (-1)^0.1.t^{3-0}+(-1)^1.3.t^{3-2}=t^3-3t, \end{aligned}$$

$$\begin{aligned} M_4\left(K_4,t\right) &= \sum_{k=0}^{6}(-1)^k p\left(K_4,k\right)t^{4-2k}\begin{cases} k=0\Rightarrow p\left(K_4,0\right)=1 \\ k=1\Rightarrow p\left(K_4,1\right)=6 \\ k=2\Rightarrow p\left(K_4,2\right)=3 \end{cases} \\ &= (-1)^0.1.t^{4-0}+(-1)^1.6.t^{4-2}+(-1)^2.3.t^{4-4}=t^4-6t^2+3, \end{aligned}$$

where $p\left(K_4,2\right)=\frac{6.7}{2}-\frac{1}{2}\left(3^2+3^2+3^2+3^2\right)=3$.
Then, its explicit formula is of the form [22, 23]

$$M_n\left(K_n,t\right)=\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{(2k)!}{2^k k!}\binom{n}{2k}t^{n-2k}$$

or the recurrence relation

$$M_n\left(K_n,t\right)=tM_n\left(K_{n-1},t\right)-(n-1)M_n\left(K_{n-2},t\right),$$

which are equivalent to the modified Hermite polynomials (see [22, 23, 48]). Similarly, the matching polynomial of path graph P_n is obtained as follows [48]:

$$M_n\left(P_n,t\right)=t^{n-2}\left(t^2-n+1\right)$$

or the recurrence relation

$$M_n\left(P_n,t\right)=tM_{n-1}\left(P_{n-1},t\right)-M_{n-2}\left(P_{n-2},t\right),$$

where $M_0\left(P_0,t\right)=1$, $M_1\left(P_1,t\right)=t$, $M_2\left(P_2,t\right)=t^2-1$, and $M_3\left(P_3,t\right)=t^3-2t$.

2. Basic matrix relations and method of solution

In this section, we construct basic matrix relations for the method. First, we give the matrix relation of the matching polynomial of complete graph K_n. Similar matrix relation can be obtained for the matching polynomial of path graph. By considering complete graph K_n, the matrix relation of the matching polynomial solution (1.3) of Eq. (1.1) yields

$$y(t) \cong y_N(t) = \boldsymbol{M}_n(K_n, t)\,\boldsymbol{Y} \Rightarrow y(t) = \boldsymbol{X}(t)\,\boldsymbol{K}\,\boldsymbol{Y},$$

and

$$y^{(k)}(t) = \boldsymbol{X}^{(k)}(t)\,\boldsymbol{K}\,\boldsymbol{Y} \Rightarrow y^{(k)}(t) = \boldsymbol{X}(t)\,\boldsymbol{B}^k\boldsymbol{K}\,\boldsymbol{Y},\ k = 0, 1, 2, \ldots, \tag{2.1}$$

where $\boldsymbol{M}_n(K_n, t) = \boldsymbol{X}(t)\,\boldsymbol{K}$, $\boldsymbol{X}^{(k)}(t) = \boldsymbol{X}(t)\,\boldsymbol{B}^k$, $\boldsymbol{K}^T$ is a lower triangular coefficient matrix between $\boldsymbol{M}_n(K_n, t)$ and $\boldsymbol{X}(t)$,

$$\boldsymbol{X}(t) = \begin{bmatrix} 1 & t & t^2 & \cdots & t^N \end{bmatrix},\ \boldsymbol{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & N \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\boldsymbol{Y} = \begin{bmatrix} y_0 & y_1 & \cdots & y_N \end{bmatrix}^T.$$

Let us now construct the matrix form of Eq. (1.1). If $t \to h_k(t)$ is inserted into the relation (2.1), then we can write

$$y^{(k)}(h_k(t)) = \boldsymbol{X}(h_k(t))\,\boldsymbol{B}^k\boldsymbol{K}\,\boldsymbol{Y}, \tag{2.2}$$

where

$$\boldsymbol{X}(h_k(t)) = \begin{bmatrix} 1 & h_k(t) & h_k^2(t) & \cdots & h_k^N(t) \end{bmatrix}.$$

Similarly,

$$y^{(n-j)}(\tau_j(t)) = \boldsymbol{X}(\tau_j(t))\,\boldsymbol{B}^{n-j}\boldsymbol{K}\,\boldsymbol{Y}. \tag{2.3}$$

By substituting the collocation points (1.4) along with the relations (2.2) and (2.3) into Eq. (1.1), we have a basic matrix equation for Eq. (1.1)

$$\underbrace{\left\{\boldsymbol{P}_k\boldsymbol{X}_k\boldsymbol{B}^k + \boldsymbol{Q}\,\boldsymbol{X}\,\boldsymbol{B}^n + \boldsymbol{R}^{(j)}\boldsymbol{X}_j\boldsymbol{B}^{n-j}\right\}\boldsymbol{K}}_{\boldsymbol{W}}\ \boldsymbol{Y} = \boldsymbol{F}. \tag{2.4}$$

where

$$\boldsymbol{W}=\left[\begin{array}{c} W(t_0)\\ W(t_1)\\ \vdots \\ W(t_N)\end{array}\right],\ \boldsymbol{P}_k=\left[\begin{array}{cccc} P_k(t_0) & 0 & \cdots & 0\\ 0 & P_k(t_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & P_k(t_N)\end{array}\right],$$

$$\boldsymbol{Q}=\left[\begin{array}{cccc} Q(t_0) & 0 & \cdots & 0\\ 0 & Q(t_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & Q(t_N)\end{array}\right],$$

$$\boldsymbol{R}^{(j)}=\left[\begin{array}{cccc} R^{(j)}(t_0) & 0 & \cdots & 0\\ 0 & R^{(j)}(t_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & R^{(j)}(t_N)\end{array}\right],$$

$$\boldsymbol{X}_k=\left[\begin{array}{c} \boldsymbol{X}(h_k(t_0))\\ \boldsymbol{X}(h_k(t_1))\\ \vdots \\ \boldsymbol{X}(h_k(t_N))\end{array}\right]=\left[\begin{array}{cccc} 1 & h_k(t_0) & \cdots & h_k^N(t_0)\\ 1 & h_k(t_1) & \cdots & h_k^N(t_1)\\ \vdots & \vdots & \ddots & \vdots\\ 1 & h_k(t_N) & \cdots & h_k^N(t_N)\end{array}\right],$$

$$\boldsymbol{X}_j=\left[\begin{array}{c} \boldsymbol{X}(\tau_j(t_0))\\ \boldsymbol{X}(\tau_j(t_1))\\ \vdots \\ \boldsymbol{X}(\tau_j(t_N))\end{array}\right]=\left[\begin{array}{cccc} 1 & \tau_j(t_0) & \cdots & \tau_j^N(t_0)\\ 1 & \tau_j(t_1) & \cdots & \tau_j^N(t_1)\\ \vdots & \vdots & \ddots & \vdots\\ 1 & \tau_j(t_N) & \cdots & \tau_j^N(t_N)\end{array}\right],$$

$$\boldsymbol{F}=\left[\begin{array}{cccc} f(t_0) & f(t_1) & \cdots & f(t_N)\end{array}\right]^T.$$

Then, we can write the basic matrix equation (2.4) as

$$\boldsymbol{W}\,\boldsymbol{Y}=\boldsymbol{F}\ \text{or}\ [\boldsymbol{W}\ ;\ \boldsymbol{F}]. \tag{2.5}$$

The matrix relation of the conditions (1.2) is

$$\boldsymbol{U}_i\boldsymbol{Y}=\tau_i\Rightarrow[\boldsymbol{U}_i\,;\tau_i],\ i=0,1,\ldots,m-1, \tag{2.6}$$

where

$$\boldsymbol{U}_i=\left[\begin{array}{cccc} u_{i0} & u_{i1} & \ldots & u_{iN}\end{array}\right]=\sum_{k=0}^{m-1}\left[a_{ik}\boldsymbol{X}(a)+b_{ik}\boldsymbol{X}(b)\right]\boldsymbol{B}^k\boldsymbol{K}\,\boldsymbol{Y}.$$

We insert the row(s) of the condition matrix (2.6) into the matrix equation (2.5), after removing m row(s) of the matrix equation (2.5). Then, we have the augmented matrix system

$$\left[\tilde{\boldsymbol{W}}\ ;\ \tilde{\boldsymbol{F}}\right]=\left[\begin{array}{cccccc} w_{00} & w_{01} & \cdots & w_{0N} & ; & f(t_0)\\ w_{10} & w_{11} & \cdots & w_{1N} & ; & f(t_1)\\ \vdots & \vdots & \vdots & \vdots & \vdots; & \vdots\\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N} & ; & f(t_{N-m})\\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \tau_0\\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \tau_1\\ \vdots & \vdots & \vdots & \vdots & \vdots; & \vdots\\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & ; & \tau_{m-1} \end{array}\right],$$

where rank $\tilde{\boldsymbol{W}}=$ rank $\left[\tilde{\boldsymbol{W}}\ ;\ \tilde{\boldsymbol{F}}\right]=N+1$. Thus, the coefficients matrix $\boldsymbol{Y}$ can be obtained as

$$\boldsymbol{Y}=\left(\tilde{\boldsymbol{W}}\right)^{-1}\tilde{\boldsymbol{F}}.$$

By substituting these coefficients into the solution (1.3), we obtain the matching polynomial solution of Eq. (1.1).

3. Error analysis and oscillation of solution via Laplace-Padé method

In this section, our aim is to improve the obtained approximate solutions and estimate the oscillation of solution on time interval. The theory of residual error analysis has also been discussed in [10, 34, 42]. The residual error analysis is based mainly on the residual function, which can be obtained as

$$R_N(t)=\sum_{k=0}^{m}P_k(t)\,y_N^{(k)}(h_k(t))+(Q(t)\,y_N^{(n)}(t)+\sum_{j=0}^{n}\binom{n}{j}R^{(j)}(t)\,y_N^{(n-j)}(\tau_j(t)))-f(t). \tag{3.1}$$

By using Eq. (3.1), the error equation is of the form

$$L[e_N(t)]=L[y(t)]-L[y_N(t)]=-R_N(t), \tag{3.2}$$

subject to the conditions

$$\sum_{k=0}^{m-1}\left[a_{jk}e_N^{(k)}(a)+b_{jk}e_N^{(k)}(b)\right]=0, \tag{3.3}$$

where

$$e_N(t)=y(t)-y_N(t).$$

When the error problem (3.2)-(3.3) is solved using the procedure described in Section 2, we obtain a solution of the error problem

$$e_{N,M}(t)=\sum_{n=0}^{M} y_n^* M_n(G_n,t)\ , \tag{3.4}$$

By the form (3.4), the corrected matching polynomial solution is obtained as

$$y_{N,M}(t)=y_N(t)+e_{N,M}(t)\,.$$

Thereby, we improve a matching polynomial solution. The corrected absolute error is also of the form

$$E_{N,M}(t)=y(t)-y_{N,M}(t).$$

On the other hand, in recent paper, Pezza and Pitolli [43] have made use of a convergence rate including both the exact and approximate solutions to investigate the consistency of their method. We want to show here that whether a problem possesses an exact solution or not, the residual function provides a better way to observe the consistency of the method. By following this motivation, we now give a formula of the convergence rate based on the residual function (3.1) as a corollary.

Corollary 3.1. *Let $R_N(t)$ be an analytic function on [a,b], then the convergence rate corresponding to N is calculated by*

$$Cr_N=\log\left(\frac{\max\limits_{a\le x\le b}\{|R_N(x)|\}}{\max\limits_{a\le x\le b}\{|R_{N+1}(x)|\}}\right)\frac{1}{\log(2)}>0,$$

such that if $Cr_N<0$, then the convergence consistency is ruined. We can thereby investigate the convergence rate of the method with respect to the computation limit N.

We know that it is hardly possible to entirely estimate the oscillation of the matching polynomial solution (1.3) on time interval, we therefore merge our method with Laplace-Padé method [39, 46] based on Padé approximant [7] and Laplace transform. In addition, Laplace-Padé method has been applied by some authors for different problems in [39, 46].

Padé approximant [7] is basically constituted by taking Taylor expansion of a function $g(t)$ and then the outcome is written as a rational form and its polynomial degree of numerator and denominator is K and L, respectively. As K and L are increased, the approximation to $g(t)$ is improved. Padé approximant is also easily performed on computer programs, such as Mathematica, Matlab and Maple.

In accordance with the algorithm described in [46, 39]. By using the matching polynomial solution (1.3), the algorithm of Laplace-Padé method can be constituted as follows:

Step 1: $G(s) \leftarrow L\{y_N(t)\} = \int\limits_0^\infty y_N(t)\, e^{-st} dt$, ($L\{\bullet\}$: Laplace transform);

Step 2: $G\left(\frac{1}{t}\right) \underset{s\to 1/t}{\leftarrow} G(s)$;

Step 3: $H(s) \underset{t\to 1/s}{\leftarrow} H\left(\frac{1}{t}\right) \leftarrow P\left[G\left(\frac{1}{t}\right)\right]$, where K and L are the degrees of numerator and denominator (K, $L<M$), ($P[\bullet]$: Padé approximant);

Step 4: $y_{P,N}(t) \leftarrow L^{-1}\{H(s)\}$, where $y_{P,N}(t)$ is a Padé-matching polynomial solution and $L^{-1}\{\bullet\}$ is inverse Laplace transform.

Thus, we can scrutinize the oscillatory behavior of new solution $y_{P,N}(t)$ on time interval. Furthermore, we can easily get information about the problems consisting of an oscillatory solution on time interval.

4. Numerical applications

Let us apply illustrative problems to observe the efficiency of the novel method. The numerical results are compared and interpreted in tables. The behaviors of the matching polynomial solutions are demonstrated in figures. A useful computer program module, which is developed by the authors, is used on Mathematica software, where our PC contains 2 GB RAM and 2.80 GHz CPU. Hence, we can determine the precision of the solutions. Here, $\left|e_N^K(t)\right|$ and $\left|e_N^P(t)\right|$ represent the absolute error obtained with K_n and P_n, respectively. The other similar representation is for $y_N^K(t)$ and $y_N^P(t)$.

Application 4.1. [14] Consider the neutral delay differential equation

$$y'(t) + \frac{1}{4}y'\left(t-\frac{1}{2}\right) = -y(t) + \frac{1}{4}y(t-1) + \frac{1}{4}y\left(t-\frac{3}{2}\right),\ 0 \le t \le \frac{3}{2}$$

where the non-smooth exact solutions on piecewise intervals are of the forms

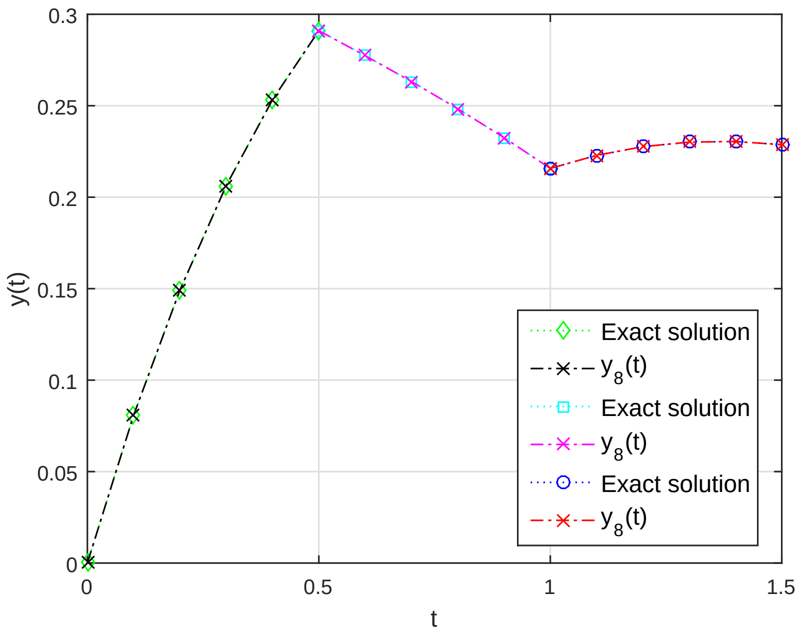
$$y(t) = -\frac{11}{8}e^{-t} - \frac{1}{2}t + \frac{11}{8},\ 0 \le t \le \frac{1}{2};$$

$$y(t) = \frac{5}{4} - \frac{t}{2} - \frac{11t}{32}e^{-\left(t-\frac{1}{2}\right)} - \frac{11}{8}e^{-t} + \frac{19}{64}e^{-\left(t-\frac{1}{2}\right)},\ \frac{1}{2} \le t \le 1;$$

$$\begin{aligned} y(t) \;=\; & \frac{43}{32} - \frac{3t}{8} - \frac{11t^2}{256}e^{-(t-1)} - \frac{11t}{32}e^{-\left(t-\frac{1}{2}\right)} - \frac{9t}{64}e^{-(t-1)} + \frac{19}{64}e^{-\left(t-\frac{1}{2}\right)} \\ & -\frac{9}{256}e^{-(t-1)} - \frac{11}{8}e^{-t},\ 1 \le t \le \frac{3}{2} \end{aligned}$$

and their initial conditions are $y(0) = 0$, $y(0.5) = 0.2910203$, and $y(1) = 0.2157346$, respectively. We solve this problem using the complete graph-based method with $N = 8$ and CLCPs. We demonstrate the obtained solutions along with the exact solutions in Figure 2 to observe the precision of the method. In [14], the authors have plotted the approximate and exact solutions of this problem in a figure. As seen from there, it can be easily observed that our method provides better approximation than the spline method [14].

FIGURE 2. The displacements of the matching polynomial and exact solutions on piecewise intervals for Application 4.1.



Application 4.2. [13, 28] Consider the functional pantograph equation used in electrodynamics

$$y'(t) - y\left(\frac{t}{3}\right) + 2\cos\left(\frac{t}{3}\right) y\left(\frac{t}{6}\right) - 2\sin\left(\frac{t}{3}\right) y\left(\frac{t}{6}\right) = f(t), 0 \le t \le T$$

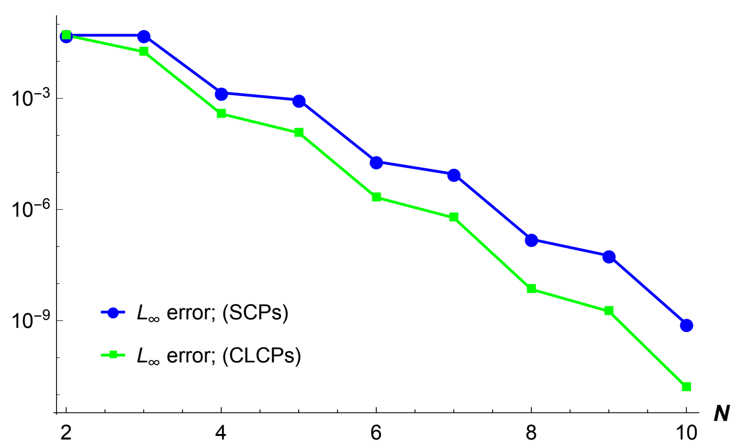
subject to the initial condition $y(0) = 0.5$. Here, $T = \{1, 5, 14\}$, the oscillatory exact solution is $y(t) = 0.5(\cos(2t) + \sin(2t))$ and

$$f(t) = \cos(2t) - \sin(2t) - \frac{1}{2}\sin\left(\frac{2t}{3}\right) + \frac{1}{2}\cos\left(\frac{2t}{3}\right).$$

We solve this problem by using SCPs and CLCPs in the method from N=2 to 10. We compare the present results with those obtained by Adomian [13] and direct [28] methods in Table 1. Due to the residual error analysis technique, we improve the approximate solutions as seen in Table 1. Table 2 shows CPU time and L_∞ error with respect to different N and graph-based method. It is readily seen from Table 2 that the results of complete graph-based method are better those of path graph-based method. At CLCPs, the absolute errors of the present method based on K_n and P_n graphs are compared in Table 3. It is easily observed that the present results are remarkable in comparison to the existing results. Figure 3 shows the logarithmic scaled plot consisting of L_∞ errors at CLCPs and SCPs. As seen from there, CLCPs give rise to obtain better results as N is increased. In addition, the Padé-matching polynomial solution on time interval [0,14] is plotted along with the exact solution in Figure 4. The phase plane of the matching polynomial solution on time interval [0,5] is also illustrated in Figure 5. The convergence consistency of the method based on complete graph is prescribed as

$$\{Cr_7, Cr_8, Cr_9, Cr_{10}, Cr_{11}, Cr_{12}\} = \{6.00,\ 1.80,\ 6.36,\ 2.12,\ 6.65,\ 2.39\} > 0. \quad (4.1)$$

FIGURE 3. The logarithmic scaled plot of L_∞ errors in terms of different collocation points for Application 4.2.



Application 4.3. [8] Consider the differential equation with state dependent delay

$$y'(t) + y(y(t)) - y(t) = \frac{1}{t} - \ln(t) + \ln(1 + \ln(t)), 1 \le t \le 5,$$

subject to the initial condition $y(1) = 1$. The exact solution of this problem is $y(t) = \ln(t) + 1$. Bellen and Zennaro [8] have derived a convergence result from a theorem for state dependent delay differential equations and they have shown that this problem satisfies the convergence result of the considered theorem. By taking $N = 12$, we apply different graph-based method to solve this problem. The results at CLCPs are presented in Table 4. It is clearly seen that the numerical results are changed via the method including different graphs. The behavior of the solution is compared in Figure 6. Besides, the convergence consistency of the method based on

FIGURE 4. The oscillation of the Padé-matching polynomial and exact solutions on [0,14] for Application 4.2.

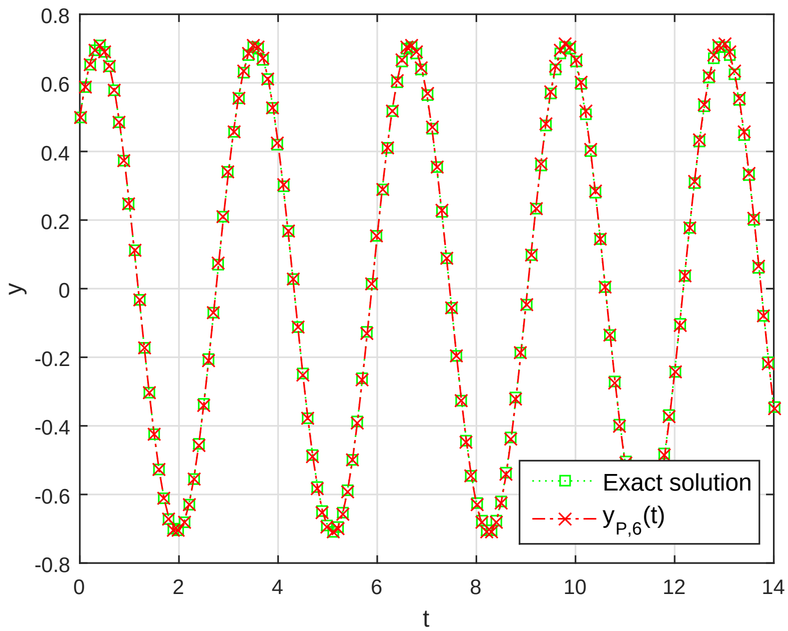


FIGURE 5. The phase plane of the matching polynomial solution $y_{12}(t)$ on [0,5] for Application 4.2.

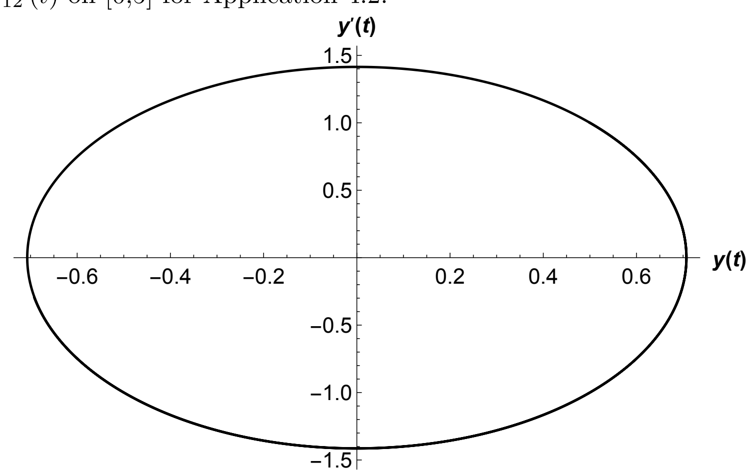


TABLE 1. Comparison of the absolute errors in terms of different collocation points for Application 4.2.

t_i	$\lvert e_6^K(t_i)\rvert$ CLCPs	$\lvert e_{10}^K(t_i)\rvert$ CLCPs	$\lvert E_{10,12}^K(t_i)\rvert$ CLCPs	$\lvert e_{10}^K(t_i)\rvert$ SCPs	Adomian meth. [13]	Direct meth. [28]
0.2	$4.68e-08$	$9.72e-13$	$9.99e-16$	$3.02e-12$	$7.47e-11$	$2.98e-12$
0.4	$1.82e-07$	$7.71e-13$	$3.55e-15$	$3.02e-12$	$2.44e-09$	$9.65e-13$
0.6	$1.51e-08$	$3.38e-12$	$8.88e-15$	$2.34e-12$	$1.88e-08$	$3.17e-12$
0.8	$7.23e-07$	$1.07e-11$	$1.22e-15$	$5.67e-14$	$8.07e-08$	$6.27e-13$
1.0	$2.12e-06$	$1.60e-11$	$2.54e-14$	$8.16e-10$	$2.50e-07$	$3.09e-12$

K_n yields

$$\{Cr_{10}, Cr_{11}, Cr_{12}, Cr_{13}, Cr_{14}\} = \{1.22,\ 1.23,\ 1.25,\ 1.26,\ 1.27\} > 0. \quad (4.2)$$

TABLE 2. CPU time and L_∞ error in terms of N for Application 4.2.

$y_N^{K,P}(t)$	$y_{50}^K(t)$	$y_{50}^P(t)$
Time (sec.)	0.109	5.672
L_∞ error	$1.55e-15$	$7.21e-13$
	$y_{200}^K(t)$	$y_{200}^P(t)$
Time (sec.)	2.594	208.13
L_∞ error	$7.43e-14$	$1.66e-01$

TABLE 3. Comparison of the absolute errors at CLCPs for Application 4.2.

t_i	$\lvert e_{10}^K(t_i)\rvert$	$\lvert e_{10}^P(t_i)\rvert$
0.095492	$3.21632e-13$	$3.21965e-13$
0.345492	$2.43350e-12$	$2.43350e-12$
0.654509	$7.13041e-12$	$7.13063e-12$
0.904509	$1.35744e-11$	$1.30747e-11$
1.000000	$1.60165e-11$	$1.60166e-11$

TABLE 4. Comparison of the absolute errors of different graph-based method at CLCPs for Application 4.3.

t_i	$\lvert e_{12}^K(t_i)\rvert$	$\lvert e_{12}^P(t_i)\rvert$
1.267949	$8.76517e-07$	$1.75461e-05$
2.000000	$2.37452e-06$	$1.64821e-05$
3.000000	$1.84892e-06$	$1.65046e-05$
4.000000	$5.11436e-07$	$1.67478e-05$
4.732051	$2.19583e-06$	$1.89133e-05$
5.000000	$3.16548e-06$	$1.53346e-04$

Application 4.4. Consider third order combined functional neutral differential equation

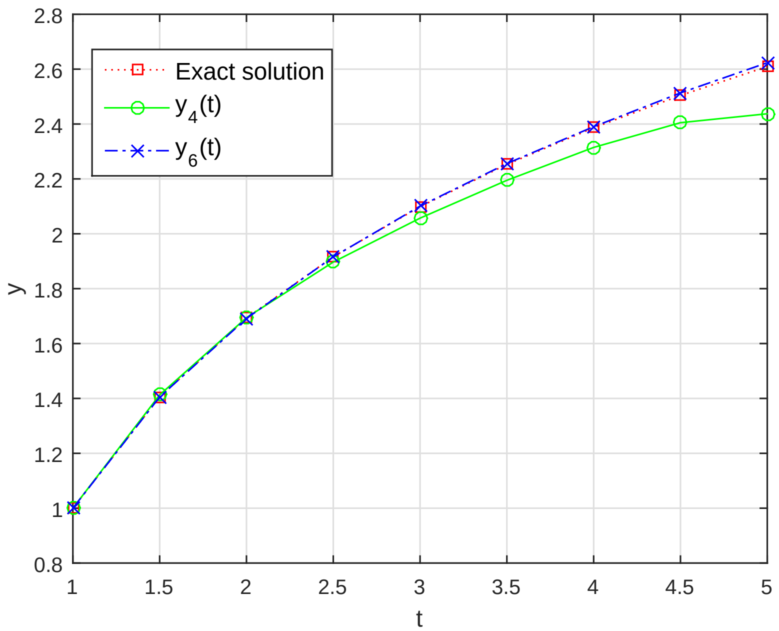
$$y'''\left(t^2\right)+ty''(t)-e^t y''\left(t^3-1\right)-(t+1)\,y'(t-1)+y(0.5t)=f(t),$$

subject to the initial conditions $y(0)=1$, $y'(0)=0$ and $y''(0)=-4$. Here, $0\leq t\leq T$, T={2, 150},

$$f(t) = \cos(t)-4t\cos(2t)+4e^t\cos\left(-2+2t^3\right)+2(1+t)\sin(-2+2t) +8\sin\left(2t^2\right),$$

and the exact solution of this problem is $y(t)=\cos(2t)$. After solving the problem for $N=6,\ 10$ and 14, As N is increased, the absolute errors decay as seen in Figure 7. The oscillation of the Padé-matching polynomial $y_{14}(t)$ on [0,150] is displayed along with the exact solution in Figure 8. The phase plane of the matching polynomial

FIGURE 6. Comparison of the exact and matching polynomial solutions for Application 4.3.



solution on [0,2] is plotted in Figure 9. As seen from Figure 9, this solution is stable on the given time interval. Table 5 illustrates that the method based on K_n obtains better results than the method based on P_n. On the other hand, the convergence consistency is obtained via the method based on K_n as

$$\{Cr_{18}, Cr_{19}, Cr_{20}, Cr_{21}, Cr_{22}\} = \{2.41,\ 1.02,\ 1.67,\ 0.81,\ 1.85\} > 0. \quad (4.3)$$

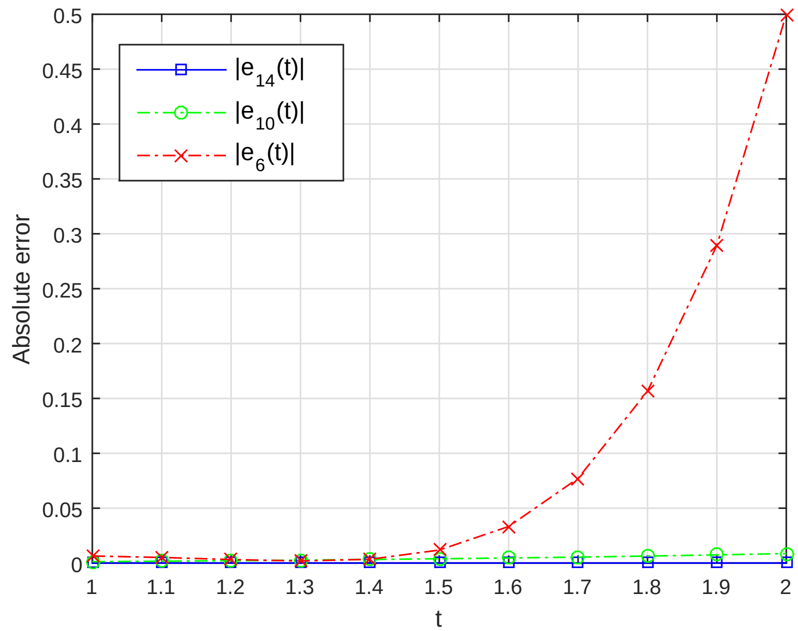
TABLE 5. CPU time and L_∞ error of different graph-based method in terms of N for Application 4.4.

$y_N^{K,P}(t)$	$y_{100}^{K}(t)$	$y_{100}^{P}(t)$
Time (sec.)	0.390	25.578
L_∞ error	$2.70e-11$	$5.75e-09$

Application 4.5. [44] Consider the neutral delay differential equation

$$y'(t) + \sqrt{\cos t}\, y'\left(\sqrt{t}\right) + \left(\sin\sqrt{t} + e^t\right) y(\sin t) = f(t),\ 0 \le t \le 1,$$

FIGURE 7. The behavior of the absolute errors with respect to N and t for Application 4.4.



subject to the initial condition $y(0) = 1$. Here, the exact solution of this problem is $y(t) = e^t$ and

$$f(t) = e^t + e^{\sqrt{t}}\sqrt{\cos t} + e^{\sin t}\left(\sin\sqrt{t} + e^t\right).$$

Solving this problem, we get the numerical results of the solutions in terms of N and M. Table 6 indicates that the absolute and L_∞ errors are much better than those obtained by Haar wavelet series method [44]. In addition, the residual error analysis provides the improved numerical results as seen in Table 6. The convergence consistency of the method based on K_n yields

$$\{Cr_4, Cr_5, Cr_6, Cr_7, Cr_8, Cr_9, Cr_{10}\} = \{4.29,\ 4.52,\ 4.18,\ 6.69,\ 2.27,\ 6.40,\ 2.73\} > 0. \quad (4.4)$$

5. A descriptive algorithm

We describe an algorithm, which is designed to form a basis of the computer program module. Note that this algorithm can be easily conjoined with Laplace-Padé algorithm.

Step 1: Start

Figure 8. Oscillation of the Padé-matching polynomial and exact solutions on time interval [0,150] for Application 4.4.

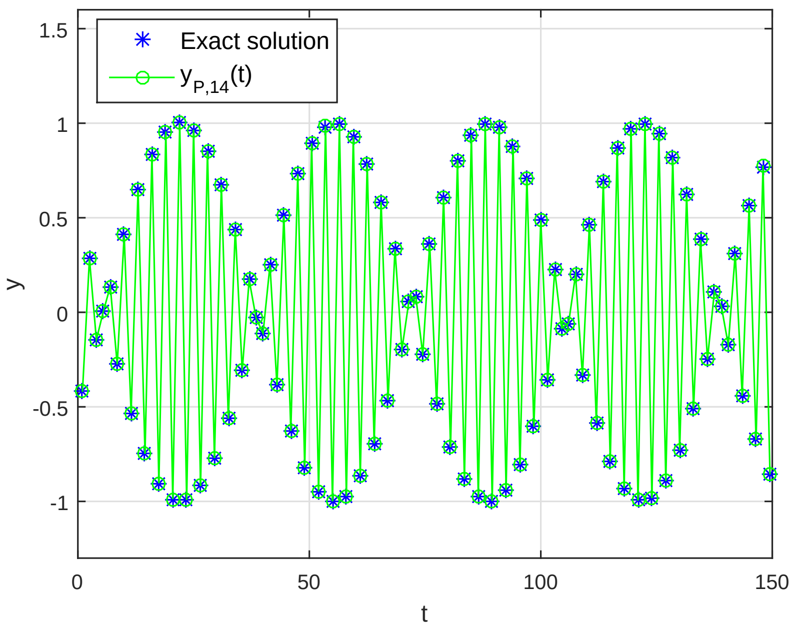


Figure 9. The phase plane of the matching polynomial solution $y_{14}(t)$ on time interval [0,2] for Application 4.4.

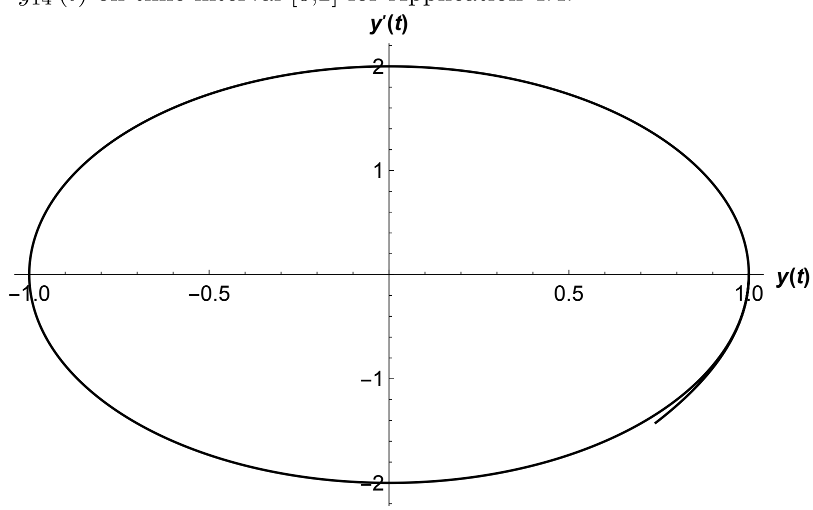


TABLE 6. Comparison of the absolute and L_∞ errors for Application 4.5.

t_i	$\left\|e_4^K(t_i)\right\|$ CLCPs	$\left\|E_{4,5}^K(t_i)\right\|$ CLCPs	$\left\|E_{4,6}^K(t_i)\right\|$ CLCPs	$\left\|E_{4,6}^K(t_i)\right\|$ SCPs	Haar wavelet meth. M=4 [44]
0.0625	$1.18e-05$	$8.72e-07$	$4.08e-08$	$4.24e-09$	$2.00e-04$
0.1875	$5.21e-05$	$2.42e-06$	$4.00e-08$	$6.87e-08$	$9.60e-02$
0.3125	$5.15e-05$	$5.40e-07$	$5.72e-08$	$1.18e-07$	$1.88e-02$
0.4375	$7.39e-06$	$1.47e-06$	$1.46e-08$	$2.99e-08$	$2.70e-02$
0.5625	$3.48e-05$	$5.04e-07$	$8.92e-08$	$3.80e-08$	$1.01e-02$
0.6875	$2.62e-05$	$1.95e-06$	$4.71e-08$	$3.26e-08$	$6.00e-03$
0.8125	$3.52e-05$	$1.38e-06$	$5.25e-08$	$4.58e-08$	$4.00e-03$
0.9375	$4.70e-05$	$1.51e-06$	$4.88e-08$	$3.60e-08$	$1.10e-03$
L_∞		$\|\|e_{16}\|\|_\infty = 4.44e-016$			$\|\|e_{16}\|\|_\infty = 1.89e-004$ [44]

Step 2: Determine a graph family (G_n) and input(N, a, b)
Step 3: $M_n(G_n, t) \leftarrow p(G_n, 0), p(G_n, 1), \ldots, p(G_n, m)$
Step 4: $\boldsymbol{K}^T \leftarrow$ takecoefficients $\left(t^0, t^1, \ldots, t^n\right)$
Step 5: $\boldsymbol{W} \leftarrow$write($\boldsymbol{P}$, $\boldsymbol{X}_k$, $\boldsymbol{B}$, $\boldsymbol{Q}$, $\boldsymbol{R}^{(j)}$, $\boldsymbol{Y}$) and write($\boldsymbol{F}$)
Step 6: $\tilde{\boldsymbol{W}} \leftarrow$ join $(\boldsymbol{W},\ \boldsymbol{U}_i)$
Step 7: $\tilde{\boldsymbol{F}} \leftarrow$ join $(\boldsymbol{F},\ \tau_i)$
Step 8: $\boldsymbol{Y} \leftarrow$ linearsolve $\left(\tilde{\boldsymbol{W}}\ ;\ \tilde{\boldsymbol{F}}\right)$
Step 9: $y_N(t) \leftarrow \boldsymbol{M}_n(G_n, t)\ \boldsymbol{Y}$
Step 10: Stop

6. Conclusions

A novel method based on a simple graph and matrix-collocation have been applied to CFNDEs. The residual error analysis and Laplace-Padé method have also been employed along with the proposed method. It is obviously seen that the vulnerability of a graph G has an effect on both the present method and the numerical results. The method based on K_n, which is a durable and dependable graph in terms of vulnerability, has enabled us to obtain more consistent results. Therefore, CPU time(s) and errors of complete graph-based method are far better than those obtained by path graph-based method as seen in Tables 2-5. Tables 1 and 6 show that the present results are better than those obtained by other existing methods. It can be easily noticed from Applications 4.2 and 4.4 that Laplace-Padé method is in good harmony with the graph-matrix collocation method. Thanks to this hybrid method, the behavior of the oscillatory approximate solution has been determined sensitively as seen form Figures 4 and 8. We have observed the stable behavior of the matching polynomial solutions in phase planes (see Figures 5 and 9). In addition, we have investigated the convergence consistency of the method as can be seen in the numerical results (4.1)-(4.4).

On the other hand, CLCPs and SCPs have been compared in the applications and so we overwhelmingly deduce that CLCPs and complete graph are eligible for the efficiency and accuracy of the method. The main advantages of the present method is efficient, consistent, fast and simple according to the other existing methods. It would be interesting to construct the matrix collocation method for different graph structures. As a future work, these graph-based methods are planned to be employed for solving other problems, such as integro-differential, partial differential, fractional differential and integral equations.

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