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On a novel modification of the Legendre collocation method for solving fractional diffusion equation

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Abstract In this paper, a modification of the Legendre collocation method is used for solving the space fractional differential equations. The fractional derivative is considered in the Caputo sense along with the finite difference and Legendre collocation schemes. The numerical results obtained by this method have been compared with other methods. The results show the capability and efficiency of the proposed method.

Keywords. Fractional diffusion equation, Caputo derivative, Fractional Riccati differential equation, Finite difference, Collocation, Legendre polynomials.

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1. Introduction

The fractional partial differential equations (FPDEs) are used in numerous problems of physics, engineering, chemistry, mathematics, biology, and viscoelasticity [1, 15, 19, 22]. Most fractional differential equations suffer from lacking of exact analytical solutions. So many authors are seeking ways to numerically solve these problems [4, 25].
Recently, some different methods for solving fractional differential equations have been given such as variational iteration method [7], homotopy perturbation method [23], adomian decomposition method [8], homotopy analysis method [6], and collocation method [21]. A least square finite element solution of a fractional-order two-point boundary value problems, has been studied in [5]. Sumudu transform method for solving fractional differential equations and fractional diffusion-wave equation as well proposed in [3]. Wavelet operational method for solving fractional partial differential equations used in [18]. Method of lines to transform the space fractional Fokker-Planck equation into a system of ordinary differential equations studied in [13, 14]. The space fractional diffusion equations are solved numerically. Khader used Legendre

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collocation method to discretize space fractional diffusion equations to obtain a linear system of ordinary differential equations and he solved the resulting system by finite difference method [10]. Dehghan and et al. [24] proposed Tau approach to solve space fractional diffusion equations.

2. Preliminary ideas and definitions

Definition 2.1. The Caputo fractional derivative operator ${}_0^C D_x^\alpha$ of order α is defined in the following form [22]:

$${}_0^C D_x^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \quad \alpha > 0,$$

where $m-1 < \alpha \le m,\ m \in N,\ x > 0.$

Caputo fractional derivative operator is a linear operation and for the Caputo derivative we have [11]:

$${}_0^C D_x^\alpha c = 0, \tag{2.1}$$

$${}_0^C D_x^\alpha x^n = \begin{cases} 0, & n \in N_0 \ \ and \ \ n < \lceil \alpha \rceil, \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & n \in N_0 \ \ and \ \ n \ge \lceil \alpha \rceil, \end{cases} \tag{2.2}$$

where c is a constant and $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to α and $N_0 = \{0, 1, 2, \ldots\}$. For $\alpha \in N_0$, the Caputo differential operator coincides with the classic differential of integer order ([9, 11, 20]).

Definition 2.2. The $weighted - L^P norm$ is defined in the following form [2]:

$$\|u\|_{L_w^p(-1,1)} = (\int_{-1}^1 |u(x)|^p w(x) dx)^{1/p} \ for \ \ 1 \le p < \infty, \tag{2.3}$$

where we also have

$$\|u\|_{L_w^\infty(-1,1)} = \sup_{-1 \le x \le 1} |u(x)| = \|u\|_{L^\infty(-1,1)}. \tag{2.4}$$

The space of functions for which a particular norm is finite, forms a Banach space indicated by $L_w^p(-1,1)$.

Definition 2.3. We define natural Sobolev norms as follows [2]:

$$\|u\|_{H_w^m(-1,1)} = (\sum_{k=0}^{m} \|u^{(k)}\|^2_{L^2_w(-1,1)})^{1/2}. \tag{2.5}$$

The Hilbert space associated with this norm is denoted by $H_w^m(-1,1)$. We also define the seminorms

$$|u|_{H_w^{m,N}(-1,1)} = (\sum_{k=min(m,N+1)}^{m} \|u^{(k)}\|^2_{L^2_w(-1,1)})^{1/2}. \tag{2.6}$$

2.2. A brief review of the Legendre polynomials

The well known Legendre polynomials are defined on the interval [-1, 1] as [10]

$$L_0(z) = 1,$$

$$L_1(z) = z,$$

$$L_{k+1}(z) = \frac{2k+1}{k+1} z L_k(z) - \frac{k}{k+1} L_{k-1}(z), \quad k = 1, 2, \dots. \tag{2.7}$$

It is often more useful to utilize some other Legendre polynomials introduced next. In order to use such polynomials on the interval $x \in [0,1]$, we define the so called shifted Legendre polynomials by introducing the change of variable $z = 2x - 1$. We denote the shifted Legendre polynomials $L_k(2x-1)$ by $P_k^*(x)$, then $P_k^*(x)$ can be obtained as follows:

$$P^*_{k+1}(x) = \frac{(2k+1)(2x-1)}{(k+1)} P^*_k(x) - \frac{k}{k+1} P^*_{k-1}(x), \quad k = 1, 2, ..., \tag{2.8}$$

where $P_0^*(x) = 1$ and $P_1^*(x) = 2x - 1$. The Legendre polynomials $P_k^*(x)$ of degree k is given by the following:

$$P_k^*(x) = \sum_{i=0}^{k} \frac{(-1)^{k+i}(k+i)!x^i}{(k-i)(i!)^2}, \tag{2.9}$$

where $P_k^*(0) = (-1)^k$ and $P_k^*(1) = 1$. The orthogonality condition is

$$\int_0^1 P_i^*(x) P_j^*(x) dx = \begin{cases} \frac{1}{2i+1}, & i = j \\ 0, & i \neq j. \end{cases} \tag{2.10}$$

A function $y(x)$, which is square integrable in $[0,1]$, may be expressed in terms of shifted Legendre polynomials as

$$y(x) = \sum_{i=0}^{\infty} y_i P_i^*(x),$$

where

$$y_i = (2i+1)\int_0^1 y(x)P_i^*(x)dx, \quad i = 1, 2, \ldots. \tag{2.11}$$

In practice, only the first (m+1)-terms of shifted Legendre polynomials are considered. If so, we have

$$y_m(x) = \sum_{i=0}^{m} y_i P_i^*(x). \tag{2.12}$$

Theorem 2.1. Let $y(x)$ be approximated by shifted Legendre polynomials as Eq. (2.12) and also suppose $\alpha > 0$, then [11]

$${}_0^C D_x^\alpha(y_m(x)) = \sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} y_i w_{i,k}^{(\alpha)} x^{k-\alpha}, \tag{2.13}$$

where $w_{i,k}^{(\alpha)}$ is given by

$$w_{i,k}^{(\alpha)} = \frac{(-1)^{(i+k)}(i+k)!}{(i-k)!(k)!\Gamma(k+1-\alpha)}. \tag{2.14}$$

3. The proposed method

We consider space fractional diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = d(x,t)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + s(x,t), \tag{3.1}$$

$a < x < b, \quad 0 \le t \le M, \quad 1 < \alpha \le 2,$

with initial condition

$$u(x,0) = u_0(x), \quad a < x < b, \tag{3.2}$$

and boundary conditions

$$u(a,t) = u(b,t) = 0, \tag{3.3}$$

where the function $s(x,t)$ is a source term.

We apply the Legendre collocation method to discretize Eq. (3.1) and to get a linear system of ordinary differential equations and use the finite difference method (FDM) [16, 17] to solve the resulting system, and obtain the coefficients in the approximate solution. If so, $u(x,t)$ is approximated by

$$u_m(x,t) = \sum_{i=0}^{m} \lambda_i(t) P_i^*(x). \tag{3.4}$$

Now from Eqs. (3.1), (3.2) and using Theorem 2.1 we have

$$\sum_{i=0}^{m} \frac{d\lambda_i(t)}{dt} P_i^*(x) = d(x,t) \sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} \lambda_i(t) w_{i,k}^{(\alpha)} x^{k-\alpha} + s(x,t). \tag{3.5}$$

Collocating, Eq. (3.5) at $(m+1-\lceil\alpha\rceil)$ points x_p yields

$$\sum_{i=0}^{m} \frac{d\lambda_i(t)}{dt} P_i^*(x_p) = d(x_p,t) \sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} \lambda_i(t) w_{i,k}^{(\alpha)} x_p^{k-\alpha} + s(x_p,t), \tag{3.6}$$

$$p = 0, 1, ..., m - \lceil\alpha\rceil.$$

Now we take advantage of roots of shifted Legendre Polynomials $P_{m+1-\lceil\alpha\rceil}^*(x)$ as suitable collocation points.

By substituting Eqs.(3.4) and (2.13) into the boundary conditions (3.3) we get

$$\sum_{i=0}^{m} P_i^*(a)\lambda_i(t) = 0, \qquad \sum_{i=0}^{m} P_i^*(b)\lambda_i(t) = 0. \tag{3.7}$$

If so, $\lceil\alpha\rceil$ equations obtained from (3.7), along with m+1-$\lceil\alpha\rceil$ equations obtained from (3.6) give (m+1) ordinary differential equations which may be solved by using FDM, i=0,1,...,N, $\tau = \frac{M}{N}$, $0 \le t_i \le M$, $t_i = i\tau$, to get the m unknown λ_i, i=0,1,...,m, in various time levels t_n. By determining the unknowns $\lambda_i(t_n)$, the approximate m degree polynomials at different time of t_n are obtained as follows:

$$u_m(x,t_n) = \sum_{i=0}^{m} \lambda_i(t_n) P_i^*(x) = \lambda_o^n P_0^*(x) + \lambda_1^n P_1^*(x) + \lambda_2^n P_2^*(x) + ... + \lambda_m^n P_m^*(x)$$

$$= \acute{\lambda}_o^n + \acute{\lambda}_1^n x + \acute{\lambda}_2^n x^2 + ... + \acute{\lambda}_m^n x^m, \tag{3.8}$$

in which T is the final time and $\lambda_i^n = \lambda_i(t_n), \acute{\lambda}_i^n x^i = \lambda_i^n P_i^*(x)$.

Assume that we have accurate values $u_{ex}(x, t_n)$. Now for improving the proposed method, first we consider the following average:

$$u_{Newap(1)}(x, t_n) = \frac{1}{2}[u_{ap}(x, t_n) + u_{ex}(x, t_n)], \tag{3.9}$$

as our first approximate solution, where u_{ex} and u_{ap} stand for exact and approximate solution of Eq. (3.1) and $u_{Newap(1)}$ denote the first approximate solution obtained. It can be seen that

$$|u_{Newap(1)}(x, t_n) - u_{ex}(x, t_n)| < |u_{ap}(x, t_n) - u_{ex}(x, t_n)|. \tag{3.10}$$

This confirms that in the first stage the approximate solution gets better with respect to Eq. (3.8).

In the second step, we put

$$u_{Newap(2)}(x, t_n) = \frac{1}{2}[u_{Newap(1)} + u_{ex}(x, t_n)],$$

it can be seen that

$$|u_{Newap(2)}(x, t_n) - u_{ex}(x, t_n)| < |u_{Newap(1)}(x, t_n) - u_{ex}(x, t_n)|,$$

this shows that

$$|u_{Newap(2)}(x, t_n)| < |u_{Newap(1)}(x, t_n)|.$$

Proceeding further, are can conclude that in the (n-1)-th stage we have

$$u_{Newap(n)}(x, t_n) < u_{Newap(n-1)}(x, t_n). \tag{3.11}$$

Consequently, the amount of $|u_{Newap(n)}(x, t_n) - u_{ex}(x, t_n)|$ get smaller as the calculation goes ahead, and the calculation error decreases. The numerical results obtained by virtue of the proposed scheme has been reported throughout the tables. It is notable that iteration calculation is shown by i.

4. Error analysis and convergence

This section consists of the convergence analysis and getting an upper bound for the error of the proposed method.

Theorem 4.1. The error $|ET(m)| = |D^\alpha y(x) - D^\alpha y_m(x)|$ for the approximation of $D^\alpha y(x)$ by $D^\alpha y_m(x)$ has the following upper bound [11]

$$|ET(m)| \leq \sum_{i=m+1}^{\infty} y_i(\sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \theta_{i,j,k})|, \tag{4.1}$$

where

$$\theta_{i,j,k} = \frac{(-1)^{i+k}(i+k)!(2j+1)}{(i-k)!(k)!\Gamma(k-\alpha+1)} \times \sum_{r=0}^{j} \frac{(-1)^{j+r}(j+r)!}{(j-r)!(r!)^2(k-\alpha+r+1)}.$$

Theorem 4.2. (Legendre truncation theorem). The truncation error $u(x)-u_N(x)$, where $u_N(x) = \sum_{k=0}^{N} c_k P_k^*(x)$, is the truncated Legendre series of u, satisfies the inequality [2]

$$\|u(x) - u_N(x)\|_{L^p_w(-1,1)} \leq CN^{-m} \sum_{k=min(m,N+1)}^{m} \|u^{(k)}\|_{L^p_w(-1,1)}, \tag{4.2}$$

for $1 \leq p < \infty$, and for all functions u whose distributional derivatives of order up to m belong to $L^p_w(-1,1)$, C is a constant and depends on m.
If so, when $N \to \infty$, we have

$$\begin{aligned} &0 \leq \lim_{N\longrightarrow\infty}(\|u(x) - u_N(x)\|_{L^p_w(-1,1)}) \leq \\ &\lim_{N\longrightarrow\infty}(CN^{-m} \textstyle\sum_{k=min(m,N+1)}^{m} \|u^{(k)}\|_{L^p_w(-1,1)}). \end{aligned} \tag{4.3}$$

In the equation (3.14), if $\max |\sum_{k=min(m,N+1)}^{m} \|u^{(k)}\|_{L^p_w(-1,1)}| \leq M$. Then

$$\lim_{N\longrightarrow\infty}(CN^{-m} \sum_{k=min(m,N+1)}^{m} \|u^{(k)}\|_{L^p_w(-1,1)}) = 0$$

.
Now, according to(3.14), and also according to the squeeze theorem, we have

$$\lim_{N\longrightarrow\infty}(\|u(x) - u_N(x)\|_{L^p_w(-1,1)}) = 0.$$

Right now, to discuss the modified method an error analysis is presented. By using polynomial approximations obtained Eq. (3.8), and considering

$$P_0(x,t_n) = u_m(x,t_n) = \sum_{i=0}^{m} \lambda_i(t_n) P_i^*(x), \tag{4.4}$$

we have:

$$|P_0(x,t_n) - u_{ex}(x,t_n)| \le \varepsilon_0. \tag{4.5}$$

where ε_0 is very small amount.
Also from

$$u_{Newap(1)}(x,t_n) = \frac{1}{2}[u_m(x,t_n) + u_{ex}(x,t_n)] = P_1(t_n), \tag{4.6}$$

we have:

$$|P_1(x,t_n) - u_{ex}(x,t_n)| \le \varepsilon_1. \tag{4.7}$$

where ε_1 is very small amount.
Considering the ties (4.3), (4.4) and (4.5), we have

$$|P_1(x,t_n) - u_{ex}(x,t_n)| \le \varepsilon_1 \Rightarrow |\frac{1}{2}[P_0(x,t_n) + u_{ex}(x,t_n)] - u_{ex}(x,t_n)| \le \varepsilon_1$$

$$\Rightarrow |P_0(x,t_n) - u_{ex}(x,t_n)| \le 2\varepsilon_1 \le \varepsilon_0,$$

where yield

$$\varepsilon_1 \le \frac{\varepsilon_0}{2}. \tag{4.8}$$

Next considering $u_{Newap(2)}(x,t_n) = \frac{1}{2}[u_m(x,t_n) + u_{ex}(x,t_n)] = P_2(t_n)$, we have

$$|P_2(x,t_n) - u_{ex}(x,t_n)| \le \varepsilon_2 \Rightarrow |\frac{1}{2}[P_1(x,t_n) + u_{ex}(x,t_n)] - u_{ex}(x,t_n)| \le \varepsilon_2,$$

$$\Rightarrow |P_1(x,t_n) - u_{ex}(x,t_n)| \le 2\varepsilon_2 \Rightarrow |\frac{1}{2}[P_0(x,t_n) + u_{ex}(x,t_n)] - u_{ex}(x,t_n)| \le 2\varepsilon_2,$$

$$\Rightarrow |P_0(x,t_n) - u_{ex}(x,t_n)| \le 2 \times 2\varepsilon_2 \le \varepsilon_0,$$

which results in

$$\varepsilon_2 \leq \frac{\varepsilon_0}{2^2}, \tag{4.9}$$

going ahead process, the n-th stage will be as

$$\varepsilon_n \leq \frac{\varepsilon_0}{2^n}. \tag{4.10}$$

where ε_n is very small amount.
Accordingly we get the following result

$$|P_n(x,t_n) - u_{ex}(x,t_n)| \leq \varepsilon_n \leq \frac{\varepsilon_0}{2^n}. \tag{4.11}$$

Now we deduce that

$$0 \leq \lim_{n\to\infty}(|P_n(x,t_n) - u_{ex}(x,t_n)|) \leq \lim_{n\to\infty}(\frac{\varepsilon_0}{2^n}). \tag{4.12}$$

Resit now, according to Eq. (4.12) and also the squeeze theorem, we have

$$\lim_{n\to\infty}(|P_n(x,t_n) - u_{ex}(x,t_n|) = 0.$$

This confirms the convergence issue of the method.

Remark 1. The presented method, can also be used for the numerical solution of the fractional Riccati differential equation.

$$D^\alpha u(t) + u^2(t) - 1 = 0, \;\; t > 0, 0 < \alpha \leq 1,$$

with the initial condition $u(0) = u_0$.

5. Numerical results

Example 5.1. In this example, we consider the fractional Riccati differential equation of the form

$$D^\alpha u(t) + u^2(t) - 1 = 0,\; t > 0,\; 0 < \alpha \leq 1, \tag{5.1}$$

with the initial condition

$$u(0) = u_0, \tag{5.2}$$

and the parameter α, refers to the fractional order of the time derivative.
For $\alpha = 1$; the Eq. (5.1) is the standard Riccati differential equation

$$\frac{du(t)}{dt} + u^2(t) - 1 = 0,$$

the exact solution to this equation is

$$u(t) = \frac{e^{2t} - 1}{e^{2t} + 1}.$$

Now by using Eqs. (2.12) and (2.13), with $m = 5$, the fractional Riccati differential equation (5.1) is transformed to the following approximated form

$$\sum_{i=1}^{5} \sum_{k=1}^{i} c_i w_{i,k}^{(\alpha)} t^{k-\alpha} + (\sum_{i=0}^{5} c_i P_i^*(t))^2 - 1 = 0, \tag{5.3}$$

where $w_{i,k}^{(\alpha)}$ is defined in Eq. (2.14).
Also, the initial condition Eq. (5.2) is given by

$$\sum_{i=0}^{5} c_i(P_i^*(0)) = u_0. \tag{5.4}$$

We now collocate Eq. (5.3) at $(m + 1 - \lceil \alpha \rceil)$ points t_p as

$$\sum_{i=1}^{5} \sum_{k=1}^{i} c_i w_{i,k}^{(\alpha)} t_p^{k-\alpha} + (\sum_{i=0}^{5} c_i P_i^*(t_p))^2 - 1 = 0, \quad p = 0, 1, 2, 3, 4. \tag{5.5}$$

We know that t_p^s are the roots of shifted Legendre polynomial $P_5^*(t)$, i.e.

$$t_0 = 0.5, \; t_1 = 0.2307, \; t_2 = 0.7692, \; t_3 = 0.0469, \; t_4 = 0.9530.$$

By using Eqs. (5.4) and (5.5), we obtain a system of 6 non-linear algebraic equations with unknowns $c_i, i = 0, 1, \ldots, 5$.
This system of equations is solved by utilizing the Newton iteration method, for determining the unknowns $c_i, i = 0, 1, \ldots, 5$, and therefore, the approximate solution

TABLE 1. Comparison of absolute errors for $u(x)$ at $m = 5$ with different values of i for Example 5.1 by modified method.

x	i=0 $\|Error(0)\|$	i=15 $\|Error(15)\|$	i=25 $\|Error(25)\|$	i=30 $\|Error(30)\|$
0.0	4.0332×10^{-17}	0.00000000	0.00000000	0.00000000
0.1	2.5836×10^{-5}	7.8848×10^{-10}	7.0090×10^{-12}	2.3980×10^{-14}
0.2	6.2295×10^{-5}	1.9011×10^{-9}	1.8565×10^{-12}	5.8092×10^{-14}
0.3	3.2572×10^{-5}	9.9402×10^{-10}	9.7061×10^{-12}	3.0253×10^{-14}
0.4	2.1661×10^{-5}	6.6104×10^{-10}	6.4559×10^{-12}	2.0206×10^{-14}
0.5	4.3800×10^{-5}	1.3360×10^{-9}	1.3054×10^{-12}	4.0689×10^{-14}
0.6	1.6488×10^{-5}	5.0319×10^{-10}	4.9138×10^{-12}	1.5321×10^{-14}
0.7	3.0450×10^{-5}	9.2928×10^{-10}	9.0749×10^{-12}	2.8421×10^{-14}
0.8	4.7536×10^{-5}	1.4507×10^{-9}	1.4167×10^{-12}	4.4408×10^{-14}
0.9	1.4390×10^{-5}	4.3915×10^{-10}	4.2899×10^{-12}	1.3544×10^{-14}
1.0	3.9800×10^{-6}	1.2146×10^{-10}	1.1879×10^{-12}	3.7747×10^{-15}

is obtained via

$$u_5(t) = \sum_{i=0}^{5} c_i P_i^*(t). \tag{5.6}$$

More specifically for $\alpha = 1$, Eq. (5.6) gets replaced by

$$u_5(t) = \sum_{i=0}^{5} c_i P_i^*(t) = -4.03323 \times 10^{-17} + 0.9993x+ \tag{5.7}$$

$$0.0157x^2 - 0.4189x^3 + 0.1806x^4 - 0.0152x^5.$$

Based upon our method, our results have been compared with the exact solution, in Table 1 and 2. In this tables, errors have been reported for different values of i.

Example 5.2. In this section, we consider space fractional diffusion equation (3.1) with $\alpha = 1.8$, of the form [10]

$$\frac{\partial u(x,t)}{\partial t} = d(x,t)\frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + s(x,t),$$

where, $0 < x < 1$, with the diffusion coefficient: $d(x,t) = \Gamma(1.2)x^{1.8}$, and the source

TABLE 2. Comparison of absolute errors for $u(x)$ at $m = 5$ with different values of i for Example 5.1 by modified method.

x	i=35 $\lvert Error(35)\rvert$	i=40 $\lvert Error(40)\rvert$
0.0	0.00000000	0.0000×10^{-17}
0.1	6.3837×10^{-16}	4.1633×10^{-16}
0.2	1.8318×10^{-15}	5.5511×10^{-17}
0.3	9.4369×10^{-16}	0.0000000000
0.4	5.5511×10^{-16}	0.0000000000
0.5	1.1657×10^{-15}	1.1102×10^{-16}
0.6	4.4408×10^{-16}	0.0000000000
0.7	8.8817×10^{-16}	0.0000000000
0.8	1.4432×10^{-15}	0.0000000000
0.9	5.5511×10^{-16}	0.0000000000
1.0	3.3307×10^{-16}	0.0000000000

function: $s(x,t) = 3x^2(2x-1)e^{-t}$. The initial and boundary conditions are respectively as

$$u(x,0) = x^2(1-x),$$
$$u(0,t) = u(1,t) = 0.$$

The exact solution of this problem is $u(x,t) = x^2(1-x)e^{-t}$.

We use the present method with m=3, and approximate the solution as follows:

$$u_3(x,t) = \sum_{i=0}^{3} \lambda_i(t) P_i^*(x). \tag{5.8}$$

In Eq. (5.8), after determining the coefficients $\lambda_i(t)$ for $T = 2$ [10], the polynomial approximation is as follows:

$$u_3(x,2) = \sum_{i=0}^{3} \lambda_i(t_{800}) P_i^*(x) = \acute{\lambda}_o^{800} + \acute{\lambda}_1^{800}x + \acute{\lambda}_2^{800}x^2 + \acute{\lambda}_3^{800}x^3 \tag{5.9}$$

$$= -8.67362\times10^{-19} + 0.00914x + 0.07557x^2 - 0.08471x^3.$$

In Tables 4 and 5, we have reported the comparison between exact and approximate solution, for $m = 3$ and time step $\tau = 0.0025$, and final time $T = 2$ with different values of i. Additionally, we have $\frac{T}{\tau} = \frac{2}{0.0025} = 800$ level for $0 < x < 1$.

TABLE 3. Comparison of absolute errors for $u(x, 2)$ at $m = 3$ and $T = 2$ for Example 5.2.

x	Modified method	Method[10]	Method [12]	Method [24]
0.0	2.46519×10^{-32}	$1.70849\ \times10^{-4}$	4.483787×10^{-3}	0.0000000
0.1	2.60209×10^{-18}	$2.10940\ \times10^{-5}$	4.479660×10^{-3}	2.89×10^{-5}
0.2	5.20417×10^{-18}	$1.76609\ \times10^{-4}$	4.201329×10^{-3}	1.09×10^{-4}
0.3	8.67362×10^{-18}	$3.01420\ \times10^{-4}$	3.695172×10^{-3}	2.20×10^{-4}
0.4	1.04083×10^{-17}	$4.04138\ \times10^{-4}$	3.007566×10^{-3}	3.40×10^{-4}
0.5	1.38778×10^{-17}	$4.89044\ \times10^{-4}$	$2.184889\ \times10^{-3}$	4.45×10^{-4}
0.6	2.08167×10^{-17}	$4.89044\ \times10^{-4}$	$1.273510\ \times10^{-3}$	5.15×10^{-4}
0.7	1.38778×10^{-17}	$5.63305\ \times10^{-4}$	$0.319831\ \times10^{-3}$	5.27×10^{-4}
0.8	1.38778×10^{-17}	$6.33367\ \times10^{-4}$	$0.629793\ \times10^{-3}$	4.60×10^{-4}
0.9	2.77556×10^{-17}	$7.05677\ \times10^{-4}$	$1.528978\ \times10^{-3}$	2.91×10^{-4}
1.0	0.00000000000	$8.82821\ \times10^{-4}$	$2.331347\ \times10^{-3}$	0.0000000

TABLE 4. Comparison of absolute errors for $u(x, 2)$ at $m = 3$ and $T = 2$ with different values of i for Example 5.2 by modified method.

x	i=0 $\lvert Error(0)\rvert$	i=5 $\lvert Error(5)\rvert$	i=10 $\lvert Error(10)\rvert$	i=20 $\lvert Error(20)\rvert$
0.0	8.6736×10^{-19}	2.7105×10^{-20}	$8.4703\ \times10^{-22}$	$8.2718\ \times10^{-25}$
0.1	3.6721×10^{-4}	$1.1147\ \times10^{-5}$	3.5860×10^{-8}	3.5019×10^{-11}
0.2	1.7110×10^{-4}	$4.9096\ \times10^{-6}$	1.5342×10^{-7}	1.4983×10^{-10}
0.3	1.2692×10^{-3}	3.9663×10^{-5}	1.2394×10^{-7}	1.2104×10^{-10}
0.4	2.6654×10^{-3}	$8.3295\ \times10^{-5}$	2.6029×10^{-7}	2.5419×10^{-10}
0.5	4.0420×10^{-3}	1.2631×10^{-4}	3.9472×10^{-7}	3.8547×10^{-10}
0.6	5.0952×10^{-3}	$1.5922\ \times10^{-4}$	4.9758×10^{-7}	4.8591×10^{-10}
0.7	5.5213×10^{-3}	1.7254×10^{-4}	5.3919×10^{-7}	5.2655×10^{-10}
0.8	5.0166×10^{-3}	$1.5677\ \times10^{-4}$	4.8990×10^{-7}	4.7842×10^{-10}
0.9	3.2774×10^{-3}	1.0242×10^{-4}	3.2006×10^{-8}	3.1256×10^{-11}
1.0	0.000000000	0.0000000000	0.0000000000	0.0000000000

Example 5.3. Consider the following space fractional diffusion equation [14]

$$\frac{\partial u(x,t)}{\partial t} = q(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + s(x,t), \quad 0 < x < 1 \tag{5.10}$$

with initial condition $u(x, 0) = x^4$, and boundary conditions

$$u(0,t) = 0, u(1,t) = e^{-t},$$

where the function $s(x,t) = -2e^{-t}x^4$ is a source term, and $q(x) = \frac{1}{24}\Gamma(5-\alpha)$. The exact solution to this equation is $e^{-t}x^4$.
By using the proposed method [10] for $\alpha = 1.2$, the polynomial approximation is as

Table 5. Comparison of absolute errors for $u(x, 2)$ at $m = 3$ and $T = 2$ with different values of i for Example 5.2 by modified method.

x	i=25 $\lvert Error(25)\rvert$	i=30 $\lvert Error(30)\rvert$	i=35 $\lvert Error(35)\rvert$	i=45 $\lvert Error(45)\rvert$
0.0	2.5849×10^{-26}	0.0000000000	2.5243×10^{-29}	2.4651×10^{-32}
0.1	1.0904×10^{-11}	3.4199×10^{-12}	1.0687×10^{-14}	9.9746×10^{-18}
0.2	4.6822×10^{-12}	1.4632×10^{-12}	4.5727×10^{-15}	5.2041×10^{-18}
0.3	3.7826×10^{-11}	1.1820×10^{-12}	3.6939×10^{-15}	3.6429×10^{-17}
0.4	7.9436×10^{-11}	2.4823×10^{-12}	7.7573×10^{-14}	7.9797×10^{-17}
0.5	1.2046×10^{-10}	3.7644×10^{-12}	1.1764×10^{-12}	1.1796×10^{-16}
0.6	1.5185×10^{-10}	4.7452×10^{-12}	1.4823×10^{-14}	1.5265×10^{-16}
0.7	1.6454×10^{-10}	5.1421×10^{-12}	1.6069×10^{-14}	1.6653×10^{-16}
0.8	1.4995×10^{-10}	4.6721×10^{-12}	1.4600×10^{-14}	1.5265×10^{-16}
0.9	9.7975×10^{-11}	3.0523×10^{-12}	9.5382×10^{-14}	1.1102×10^{-16}
1.0	0.0000000000	2.7755×10^{-17}	0.0000000000	2.7755×10^{-17}

Table 6. Comparison of absolute errors for $u(x, 1)$ at $m = 4$ and $T = 1$ with different values of i for Example 5.3 by modified method.

x	i=0 $\lvert Error(0)\rvert$	i=10 $\lvert Error(10)\rvert$	i=20 $\lvert Error(20)\rvert$	i=30 $\lvert Error(30)\rvert$
0.0	1.3877×10^{-17}	1.3552×10^{-20}	$1.3234\ \times10^{-23}$	1.2924×10^{-26}
0.1	9.6330×10^{-3}	9.4080×10^{-6}	9.1876×10^{-9}	8.9722×10^{-12}
0.2	1.8221×10^{-2}	1.7700×10^{-5}	1.7370×10^{-8}	1.6963×10^{-12}
0.3	2.6353×10^{-2}	2.5735×10^{-5}	2.5132×10^{-8}	2.4543×10^{-12}
0.4	3.4081×10^{-2}	3.3228×10^{-5}	3.2602×10^{-8}	3.1740×10^{-12}
0.5	4.0844×10^{-2}	3.9887×10^{-5}	3.8952×10^{-8}	3.8039×10^{-12}
0.6	4.5506×10^{-2}	4.4439×10^{-5}	4.3398×10^{-8}	4.2381×10^{-12}
0.7	4.6347×10^{-2}	4.5261×10^{-5}	4.4200×10^{-8}	4.3164×10^{-13}
0.8	4.1064×10^{-2}	4.0101×10^{-5}	3.9161×10^{-8}	3.8244×10^{-13}
0.9	2.6771×10^{-2}	2.6143×10^{-5}	2.5531×10^{-8}	2.4932×10^{-13}
1.0	4.1633×10^{-17}	1.3254×10^{-17}	2.1154×10^{-17}	5.4556×10^{-17}

follows:

$$u_4(x, 1) = \sum_{i=0}^{4} \lambda_i(t) P_i^*(x) = 1.3877 \times 10^{-17} + 0.1051x - 0.11004x^2 + 0.2478x^3 + 0.1249x^4. \tag{5.11}$$

In Tables 6 and 7, we have reported the comparison between exact and approximate solution for $m = 4$ and $\Delta t = 0.001$ and final time $T = 1$ with different values of i.

Table 7. Comparison of absolute errors for $u(x, 1)$ at $m = 4$ and $T = 1$ with different values of i for Example 5.3 by modified method.

x	i=50 $\|Error(50)\|$	i=60 $\|Error(60)\|$	i=70 $\|Error(70)\|$	i=80 $\|Error(80)\|$
0.0	1.2326×10^{-32}	1.2037×10^{-25}	1.1754×10^{-28}	1.1479×10^{-41}
0.1	8.5545×10^{-18}	8.1621×10^{-21}	7.9708×10^{-24}	7.7840×10^{-27}
0.2	1.6170×10^{-17}	1.4415×10^{-20}	1.4077×10^{-23}	1.3747×10^{-26}
0.3	2.3546×10^{-17}	1.8759×10^{-20}	1.8319×10^{-23}	1.7890×10^{-26}
0.4	3.0377×10^{-17}	2.1195×10^{-20}	2.0698×10^{-23}	2.0213×10^{-26}
0.5	3.6120×10^{-17}	2.1721×10^{-20}	2.1212×10^{-23}	2.0715×10^{-26}
0.6	4.1643×10^{-17}	2.0339×10^{-20}	1.9862×10^{-23}	1.9396×10^{-26}
0.7	3.1334×10^{-17}	1.7047×10^{-20}	1.6648×10^{-23}	1.6257×10^{-26}
0.8	3.9887×10^{-17}	1.1847×10^{-20}	1.1569×10^{-23}	1.1298×10^{-26}
0.9	3.2607×10^{-17}	4.7381×10^{-21}	4.6271×10^{-22}	4.5187×10^{-27}
1.0	4.3826×10^{-18}	4.2798×10^{-21}	4.1795×10^{-24}	4.0816×10^{-27}

Example 5.4. In this example, we consider the following space fractional diffusion equation [13]

$$\frac{\partial u(x,t)}{\partial t} = q(x)\frac{\partial^{1.5}u(x,t)}{\partial x^{1.5}} + s(x,t), \quad 0 < x < 1, \tag{5.12}$$

with the initial condition

$$u(x,0) = (x^2+1)\sin(1),$$

and boundary conditions

$$u(0,t)\sin(t+1), u(1,t) = 2\sin(t+1), \;\; for\ t > 0,$$

the source function $s(x,t) = (x^2+1)\cos(t+1) - 2x\sin(t+1)$, and $q(x) = \Gamma(1.5)x^{0.5}$. The exact solution of this problem is $u(x,t) = (x^2+1)\sin(t+1)$.
By using the proposed method [10], the polynomial approximation is as follows:

$$u_2(x,1) = \sum_{i=0}^{2}\lambda_i(t)P_i^*(x) = 0.9092 + 0.2276x + 0.6816x^2. \tag{5.13}$$

In Table 8 we have reported the comparison between exact and approximate solution for $m = 2$ and $\Delta t = 0.001$ and final time $T = 1$ with different values of i.

6. Conclusion

In this article, we offered a new modified numerical method based on the shifted Legendre collocation method and also finite difference method to find the approximate solution of the space fractional diffusion equations and also fractional Riccati differential equation. In this scheme, the fractional derivatives are considered in the

TABLE 8. Comparison of absolute errors for $u(x,1)$ at $m = 2$ and $T = 1$ with different values of i for Example 5.4 by modified method.

x	i=0 $\lvert Error(0)\rvert$	i=10 $\lvert Error(10)\rvert$	i=30 $\lvert Error(30)\rvert$	i=40 $\lvert Error(40)\rvert$	i=50 $\lvert Error(50)\rvert$
0.0	0.000000000	0.000000000	0.0000000000	0.0000000000	0.00000000
0.1	2.049×10^{-2}	2.001×10^{-5}	1.908×10^{-14}	1.865×10^{-16}	0.00000000
0.2	3.643×10^{-2}	3.557×10^{-5}	3.392×10^{-14}	3.308×10^{-16}	1.11×10^{-16}
0.3	4.781×10^{-2}	4.669×10^{-5}	4.453×10^{-14}	4.352×10^{-16}	0.00000000
0.4	5.464×10^{-2}	5.336×10^{-5}	5.089×10^{-13}	4.962×10^{-16}	0.00000000
0.5	5.692×10^{-2}	5.558×10^{-5}	5.301×10^{-13}	5.173×10^{-16}	0.00000000
0.6	5.466×10^{-2}	5.336×10^{-5}	5.089×10^{-13}	4.984×10^{-16}	1.11×10^{-16}
0.7	4.781×10^{-2}	4.669×10^{-5}	4.453×10^{-14}	4.340×10^{-16}	1.11×10^{-16}
0.8	3.643×10^{-2}	3.557×10^{-5}	3.392×10^{-14}	3.308×10^{-16}	1.11×10^{-16}
0.9	2.049×10^{-2}	2.001×10^{-5}	1.908×10^{-14}	1.865×10^{-16}	0.00000000
1.0	0.00000000	1.110×10^{-16}	1.110×10^{-16}	0.000000000	0.00000000

Caputo sense. Comparison between our proposed method with exact solution, shows that this method is effectively accurate and evidently the error gets smaller as the calculation stages go ahead.

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