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Multiplicity of solutions for a p-Laplacian equation with nonlinear boundary conditions

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Abstract In this paper, we use the three critical points theorem attributed to B. Ricceri in order to establish existence of three distinct solutions for the following boundary value problem:

$$\begin{cases} \Delta_p u = a(x)|u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\nabla u.\nu = \lambda f(x,u) & \text{on } \partial\Omega. \end{cases}$$

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1. Introduction

For $p > 1$ and $\lambda \geq 0$, consider the following boundary value problem:

$$\begin{cases} \Delta_p u = a(x)|u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\nabla u.\nu = \lambda f(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the usual p-Laplacian operator. Our general assumptions are that $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, and ν is the unit outer normal on $\partial\Omega$. In addition, we assume, $a(x) \in C(\overline{\Omega})$ is a positive function and $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is continuous. Problem (1.1) is used to model physical phenomena related to non-Newtonian fluids, flow through porous media, nonlinear elasticity, glaciology, see for example [2], [3], [4] and [6].

In this note we intend to use a three critical points theorem attributed to Ricceri to show that under certain conditions problem (1.1) has three solutions in $W^{1,p}(\Omega)$. This theorem has already been used by other authors to address similar goal. For example see [8], [9] and [12].

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2. Preliminaries

We begin by recalling that $u \in W^{1,p}(\Omega)$ is a weak solution for (1.1) whenever the following integral equation holds:

$$\int_{\Omega}(|\nabla u|^{p-2}\nabla u \nabla \varphi + a(x)|u|^{p-2}u\varphi)\ dx = \lambda \int_{\partial\Omega} f(x,u)\varphi\ d\sigma, \tag{2.1}$$

for all $\varphi \in W^{1,p}(\Omega)$.

We prefer to use another norm in $W^{1,p}(\Omega)$ which is defined as follows:

$$\|u\| = \left(\int_{\Omega}(|\nabla u|^p + a(x)|u|^p)\ dx\right)^{\frac{1}{p}}. \tag{2.2}$$

This norm is equivalent to the usual norm in $W^{1,p}(\Omega)$, see [7, Corollary 2.3]. Henceforth, $\|u\|_p$ and $|u|_p$ denote the norm of u in $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively.

As mentioned earlier our main tool is a three critical points theorem attributed to Ricceri, see [10, 11]. A simplified version of Ricceri's work appears in [5, Theorem 2.1], and that is what we need for problem (1). Here is [5, Theorem 2.1] reformulated incorporating the new norm (2.2):

Theorem 2.1. *Let $J : W^{1,p}(\Omega) \to \mathbb{R}$ be a continuously Gâteaux differentiable functional such that $J(0) = 0$. We also assume:*

(i) *There exist $u_1 \in W^{1,p}(\Omega)$ and $r > 0$ such that*

$$\|u_1\|^p > rp \quad \text{and} \quad \sup_{\|u\| < \sqrt[p]{rp}} < rp\frac{J(u_1)}{\|u_1\|^p}.$$

(ii) *J is sequentially weakly upper semicontinuous.*

(iii) *For some $b > 0$ and for each $\lambda \in [0, b]$:*

$$\lim_{\|u\| \to +\infty}(\|u\|^p - \lambda J(u)) = +\infty.$$

(iv) *The functional $\frac{1}{p}\|.\|^p - \lambda J(.)$ satisfies the Palais-Smale condition on $W^{1,p}(\Omega)$.*

Then, there exists an open interval $\Lambda \subseteq [0, b]$ and a positive real number ρ, such that for each $\lambda \in \Lambda$, the equation

$$\frac{\partial}{\partial u}\left(\frac{1}{p}\|u\|^p - \lambda J(u)\right) = 0$$

admits at least three solutions in $W^{1,p}(\Omega)$ whose norms are less than ρ.

3. Main result

Weak solutions of (1.1) are exactly the critical points of the functional

$$\mathcal{E}_\lambda(u) := \frac{1}{p}\|u\|^p - \lambda \int_{\partial\Omega} F(x,u)\ d\sigma,$$

where $F(x,u) = \int_0^u f(x,t)\ dt$. Let's define $J(u) := \int_{\partial\Omega} F(x,u)\ d\sigma$, for $u \in W^{1,p}(\Omega)$. Now we are ready to state our main result.

Theorem 3.1. *Let $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the following conditions:*

(A1) *For some* $A > 0$ *and* $1 < q < p$, $|f(x,t)| \leq A(1+|t|^{q-1})$, $\forall t \in \mathbb{R}$ *and a.e* $x \in \partial\Omega$.

(A2) *there is* $\gamma > p$ *such that:*

$$\limsup_{\xi \to 0} \left[\frac{1}{|\xi|^\gamma} \sup_{x \in \partial\Omega} \int_0^\xi f(x,t)dt \right] < +\infty;$$

(A3) *there exists* $w \in W^{1,p}(\Omega)$ *such that* $\int_{\partial\Omega} \int_0^{w(x)} f(x,t)\ dt\ d\sigma > 0$.

Then, for every $\alpha > 0$*, there exist an open interval* $\Lambda \subset [0, \alpha]$ *and* $\rho > 0$ *such that for each* $\lambda \in \Lambda$*, problem (1.1) has at least three weak solutions in* $W^{1,p}(\Omega)$ *whose norms are less than* ρ*.*

Proof. From (A1) we infer

$$\begin{aligned} \left| \int_{\partial\Omega} F(x,u)\ d\sigma \right| &\leq A \int_{\partial\Omega} |u|\ d\sigma + \frac{A}{q} \int_{\partial\Omega} |u|^q\ d\sigma \\ &\leq c_1(|u|_q + |u|_q^q), \end{aligned}$$

for some $c_1 > 0$. By the trace imbedding $W^{1,q}(\Omega) \to L^q(\partial\Omega)$, see [1], we have

$$\left| \int_{\partial\Omega} F(x,u)\ d\sigma \right| \leq c_2(\|u\| + \|u\|^q). \tag{3.1}$$

From (3.1) we obtain

$$\|u\|^p - \lambda J(u) \geq \|u\|^p - \lambda c_2(\|u\| + \|u\|^q),$$

thus

$$\lim_{\|u\| \to +\infty} (\|u\|^p - \lambda J(u)) = +\infty, \tag{3.2}$$

since $p > q > 1$. Now we show that $\mathcal{E}_\lambda$ satisfies the Palais-Smale condition. Let $\{u_n\}$ be a sequence in $W^{1,p}(\Omega)$ such that

$$\mathcal{E}_\lambda(u_n) \to \beta, \quad \text{and} \quad \mathcal{E}'_\lambda(u_n) \to 0, \tag{3.3}$$

as $n \to \infty$. From (3.2) and (3.3) we infer that $\{u_n\}$ is a bounded sequence in $W^{1,p}(\Omega)$. Whence, there exists a subsequence of $\{u_n\}$, still denoted $\{u_n\}$, and $v \in W^{1,p}(\Omega)$ such that $u_n \rightharpoonup v$ in $W^{1,p}(\Omega)$. Now by the compact imbeddings $W^{1,p}(\Omega) \to L^p(\Omega)$ and $W^{1,p}(\Omega) \to L^s(\partial\Omega)$, for all $1 < s \leq p$, we deduce

$$u_n \to v \quad \text{in} \quad L^p(\Omega), \tag{3.4}$$

$$u_n \to v \quad \text{in} \quad L^s(\partial\Omega), \tag{3.5}$$

$$u_n(x) \to v(x) \quad \text{a.e. on} \quad \partial\Omega. \tag{3.6}$$

From (3.3) and (3.4) we derive

$$\langle \mathcal{E}'_\lambda(u_n) - \mathcal{E}'_\lambda(v), u_n - v \rangle \to 0, \tag{3.7}$$

as $n \to \infty$. Here $\langle \cdot, \cdot \rangle$ denotes the usual pairing between $W^{-1,p'}(\Omega)$ and $W^{1,p}(\Omega)$. It is readily seen that (3.4), (3.5) and (3.7) yield

$$\int_\Omega \left[|\nabla u_n|^{p-2} \nabla u_n - |\nabla v|^{p-2} \nabla v \right] . \nabla(u_n - v) dx \to 0, \tag{3.8}$$

as $n \to \infty$. At this stage we recall the following formula, see [13],

$$(|A|^{p-2}A - |B|^{p-2}B, A - B) \geq \begin{cases} C\ |A-B|^p & \text{if } p \geq 2, \\ C\ \frac{|A-B|^2}{(|A|+|B|)^{2-p}} & \text{if } p \leq 2, \end{cases} \tag{3.9}$$

where A and B denote vectors in $\mathbb{R}^n$, and $(.,.)$ the usual dot product. First, we assume $p \geq 2$. By (3.9) we have

$$\int_\Omega |\nabla u_n - \nabla v|^p\ dx \leq\ C \int_\Omega \left[|\nabla u_n|^{p-2}\nabla u_n - |\nabla v|^{p-2}\nabla v\right].\nabla(u_n - v)dx. \tag{3.10}$$

Therefore, (3.8) implies

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n - \nabla v|^p\ dx = 0.$$

Next, we assume $p \leq 2$. Let us observe that

$$\int_\Omega |\nabla u_n - \nabla v|^p dx \tag{3.11}$$
$$\leq \left(\int_\Omega \frac{|\nabla(u_n - v)|^2}{(|\nabla u_n| + |\nabla v|)^{2-p}} dx\right)^{\frac{p}{2}} \left(\int_\Omega (|\nabla u_n| + |\nabla v|)^p dx\right)^{\frac{2-p}{2}}.$$

Applying (3.9) to the first integral on the right hand side of (3.11), yields

$$\begin{aligned} \int_\Omega |\nabla u_n - \nabla v|^p dx \quad &\leq \quad C\ \left\{\int_\Omega \left[|\nabla u_n|^{p-2}\nabla u_n - |\nabla v|^{p-2}\nabla v\right].\nabla(u_n - v)dx\right\}^{\frac{p}{2}} \\ &\times \quad \left\{\int_\Omega (|\nabla u_n| + |\nabla v|)^p dx\right\}^{\frac{2-p}{2}}. \end{aligned}$$

Thus, by (3.8) we derive

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n - \nabla v|^p\ dx = 0.$$

Therefore $u_n \to v$ in $W^{1,p}(\Omega)$. So, $\mathcal{E}_\lambda$ satisfies the Palais-Smale condition.

From (A2), there exist $\eta > 0$ and $M > 0$ such that

$$\frac{F(x,\xi)}{|\xi|^\gamma} \leq M, \tag{3.12}$$

for $0 < |\xi| < \eta,\ x \in \partial\Omega$. Assume $\|u\| \leq \sqrt[p]{pr} < \eta$, whence from (3.1) and (3.12) we deduce

$$-C(\|u\| + \|u\|^q) \leq J(u) \leq C\|u\|^\gamma, \quad \text{for some} \quad C > 0. \tag{3.13}$$

Thus

$$0 \leq \sup_{\|u\| \leq \sqrt[p]{pr}} J(u) \leq C(pr)^{\frac{\gamma}{p}},$$

hence

$$\lim_{r\to 0^+} \frac{1}{r} \sup_{\|u\| \leq \sqrt[p]{pr}} J(u) = 0.$$

By (A3) we deduce that w is not identically zero. If $0 < \varepsilon < p\frac{J(w)}{\|w\|^p}$, then there exists $r \in \left(0, \frac{\|w\|^p}{p}\right)$ such that

$$\sup_{\|u\| \leq \sqrt[p]{pr}} J(u) < r\varepsilon < rp\frac{J(w)}{\|w\|^p}.$$

Now from Theorem 2.1 , with $u_1 = w$, we deduce our conclusion. □

Remark 3.2. Here is an example of a function f that satisfies the conditions $(A1)$-$A(3)$ of theorem 3.1;

$$f(x,t) = \begin{cases} |t|^{\gamma-1} & |t| < 1, \\ |t|^{q-1} & |t| \geq 1, \end{cases}$$

where $1 < q < p < \gamma$.

4. Conclusion

In this paper, we used the three critical points theorem to prove the existence of three distinct solutions for an important clas of boundary value problems associate with p-Laplacian equation. This theorem has already been used by other authors to address similar goal. Recently, many authors apply the Ricceri's theorem to prove the existence of solutions for singular elliptic boundary value problems.

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