Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 7, No. 2, 2019, pp. 252-265

Linear fractional fuzzy differential equations with Caputo derivative

Rahman Abdollahi
Department of Mathematics, University of
Mohaghegh Ardabili, Ardabil, Iran.
E-mail: r.abdollahimath@gmail.com

Mohamadbagher Farshbaf Moghimi*
Department of Mathematics, University of
Mohaghegh Ardabili, Ardabil, Iran.
E-mail: mbfmoghimi@yahoo.com

Alireza Khastan
Department of Mathematics, Institute for Advanced
Studies in Basic Sciences (IASBS), Zanjan, Iran.
E-mail: khastan@iasbs.ac.ir

Mohammad Reza Hooshmandasl
Department of Computer Science,
Yazd University, Yazd, Iran.
E-mail: hooshmandasl@yazd.ac.ir

Abstract In this paper, we study linear fractional fuzzy differential equations involving the Caputo generalized Hukuhara derivative. Using the fuzzy Laplace transform, we present the general form of solutions in terms of Mittag-Leffler functions. Finally, some examples are provided to illustrate our results.

Keywords. Caputo fractional derivative, Fuzzy fractional differential equation, Fuzzy initial value problem, Fuzzy Laplace transform, Generalized Hukuhara differentiability.

2010 Mathematics Subject Classification. 39A14, 34L40.

1. Introduction

The differential equations involving fractional derivative operators, appear to be important in modelling of several complex phenomena in different fields of science and engineering, for instance in electrical circuits, biology, biomechanics, electrochemistry, control and electromagnetic processes [1, 18]. There are several approaches to define the derivative and integral of fractional order. For example, the Grünwald-Letnikov definition of derivative and integral starts from classical definitions of derivatives and integrals based on infinitesimal division and limit. The disadvantages of this approach are its technical difficulty of the computations and the proofs and large restrictions on functions. Fortunately, there are others, more elegant approaches like the Riemann–Liouville definition which includes the results of the previous one as a special case. It turns out that the Riemann–Liouville derivatives have certain disadvantages when

Received: 30 April 2018 ; Accepted: 12 March 2019.
$*$ Corresponding author.

trying to model real-world phenomena with fractional differential equations [18]. In this paper, we focus on the Caputo definition for the fractional derivative.
Recently, Agarwal et al. [2] have proposed a concept of solution for fractional differential equations with uncertainty. They combined the notion of fractional with uncertainty. They have considered the Riemann-Liouville differentiability concept based on the Hukuhara differentiability. In [4], the authors presented the global solutions for nonlinear fuzzy fractional integral and integrodifferential equations. In [3], Agarwal et al. presented a Schauder fixed point theorem in semilinear spaces and its application to fuzzy fractional differential equations (FFDEs). Using this theorem, the authors in [16] have presented an existence result for a class of FFDEs. In [5], the authors have extended the definition of generalized Hukuhara differentiability to the fractional case. In [24], the concept of Caputo Hukuhara differentiability was applied to solve the fractional differential equations with uncertainty.
The Laplace transform is a very useful tool for solving linear ordinary differential equations with constant coefficients, since it converts linear differential equations to linear algebraic equations which can be solved easily [12]. The final step, the inverse transform of the result, is usually the most complicated part of this approach. The situation with linear fractional differential equations with constant coefficients is completely analogous [18]. In [7], the authors introduced the fuzzy Laplace transform and applied it to solve some fuzzy differential equations. Later, several authors used the fuzzy Laplace transform to solve fuzzy differential equations and fuzzy fractional differential equations (see [24] and the references therein).

In this paper, we study the linear fuzzy fractional differential equations under Caputo sense and present the explicit solutions of this problem in the general case. The paper is organized as follows. In section 2, we recall some basic knowledge of fuzzy calculus, fractional calculus and fuzzy Laplace transform. In section 3, we present main results of this paper and in section 4 some examples are given.

2. Preliminaries

In this section, we give some definitions and introduce the necessary notations which will be used throughout the paper, see, for example, [10].

Definition 2.1. A fuzzy number is a fuzzy set such as $u : \mathbb{R} \longrightarrow [0,1]$, satisfying the following properties:
(i) u is normal, that is, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,
(ii) u is a fuzzy convex set, i.e. $u((1-\lambda)x + \lambda y) \geq \min\{u(x), u(y)\}, \quad \forall x, y \in \mathbb{R}, \lambda \in [0,1]$,
(iii) u is upper semi-continuous,
(iv) $[u]^0 = cl\{x \in \mathbb{R}; u(x) > 0\}$ is compact.

The set of all fuzzy numbers is denoted by $\mathbb{R}_F$. Given a fuzzy number $u \in$ and $0 < r \leq 1$, we obtain the r-level set of u by $[u]^r = \{s \in \mathbb{R} \,|\, u(s) \geq r\}$ and the support of u as $[u]^0 = cl\{s \in \mathbb{R} | u(s) > 0\}$. For any $r \in [0,1]$, due to the properties imposed on the set of fuzzy numbers, we have that $[u]^r$ is a bounded closed interval. The notation $[u]^r = [\underline{u}^r, \overline{u}^r]$, denotes explicitly the r-level set of u.

For given $u, v \in$ and $\lambda \in \mathbb{R}$, we define the sum $u+v$ and the product λu by the standard level-set operations $[u+v]^r = [u]^r + [v]^r$, $[\lambda u]^r = \lambda [u]^r, \forall r \in [0,1]$, where $[u]^r + [v]^r$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda[u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$. The metric structure is given by the Hausdorff distance $D : \times \to \mathbb{R}_+ \cup \{0\}$,

$$D[u, v] = \sup_{r\in[0,1]} \max\{|\underline{u}^r - \underline{v}^r|, |\overline{u}^r - \overline{v}^r|\}, \quad \forall u, v \in .$$

Remark 2.2. The following properties are well-known [11]

(1) $(, D)$ is a complete metric space.
(2) $D[u + w, v + w] = D[u, v], \quad \forall u, v, w \in \mathbb{R}_F$,
(3) $D[ku, kv] = |k|D[u, v], \quad \forall k \in \mathbb{R}$,
(4) $D[u + v, w + e] \le D[u, w] + D[v, e], \quad \forall u, v, w, e \in \mathbb{R}_F$,

Definition 2.3. ([11]) Let $u, v \in \mathbb{R}_F$. If there exist $w \in \mathbb{R}_F$ such that $u = v + w$, then w is called the Hukuhara difference of u and v, and it is denoted by $u \ominus v$.

Definition 2.4. ([17]) Let $F : (a, b) \longrightarrow \mathbb{R}_F$ be a fuzzy-valued function. For fix $t_0 \in (a, b)$, we say that F is generalized Hukuhara differentiable at t_0, if there exists an element $F'(t_0) \in \mathbb{R}_F$ such that either

(1) for all $h > 0$ sufficiently close to 0, the H-differences $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ exist and the limits (in the metric D)

$$\lim_{h\to 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t_0)$$

or

(2) for all $h > 0$ sufficiently close to 0, the H-differences $F(t_0) \ominus F(t_0 + h)$, $F(t_0 - h) \ominus F(t_0)$ exist and the limits (in the metric D)

$$\lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0 + h)}{-h} = \lim_{h\to 0^+} \frac{F(t_0 - h) \ominus F(t_0)}{-h} = F'(t_0).$$

Remark 2.5. In the previous definition, case (1) corresponds to the H-derivative introduced in [23], so this differentiability concept is a generalization of the Hukuhara derivative.

Definition 2.6. Let $F : (a, b) \longrightarrow \mathbb{R}_F$. We say that F is (1)-differentiable on (a, b) if F is differentiable in the sense (1) of Definition 2.4, and similarly for (2)-differentiable on (a, b).

Theorem 2.7. *([17]) Let $F : (a, b) \longrightarrow \mathbb{R}_F$ and put $[F(t)]^r = [\underline{F}(t; r), \overline{F}(t; r)]$ for each $r \in [0, 1]$.*

(i) If F is (1)-differentiable, then $\underline{F}$ and $\overline{F}$ are differentiable functions and $[F'(t)]^r = [\underline{F}'(t; r), \overline{F}'(t; r)]$.

(ii) If F is (2)-differentiable, then $\underline{F}$ and $\overline{F}$ are differentiable functions and $[F'(t)]^r = [\overline{F}'(t; r), \underline{F}'(t; r)]$.

Let $T \subset \mathbb{R}$ be an interval. We denote by $C(T, \mathbb{R}_F)$ the space of all continuous fuzzy functions on T. Also, we denote by $L^1(T, \mathbb{R}_F)$ the space of all fuzzy functions $f : T \longrightarrow \mathbb{R}_F$ which are Lebesgue integrable on the bounded interval T of $\mathbb{R}$.

Theorem 2.8. *([26]) Let $f(t)$ be a fuzzy-valued function on $[a, \infty)$, and it is represented by*

$$\big(\underline{f}(t;r), \overline{f}(t;r)\big).$$

For any fixed $r \in [0,1]$, assume $\underline{f}(t;r)$ and $\overline{f}(t;r)$ are Riemann-integrable on $[a,b]$ for every $b \geq a$, and assume that there are two positive functions $\underline{M}(r)$ and $\overline{M}(r)$, such that $\int_a^b |\underline{f}(t;r)|dt \leq \underline{M}(r)$ and $\int_a^b |\overline{f}(t;r)|dt \leq \overline{M}(r)$ for every $b \geq a$; then, $f(t)$ is improper fuzzy Riemann-integrable on $[a, \infty)$. The improper fuzzy Riemann-integral is a fuzzy number, and we have:

$$\int_a^\infty f(t;r)dt = \left[\int_a^\infty \underline{f}(t;r)dt, \int_a^\infty \overline{f}(t;r)dt\right]. \tag{2.1}$$

Definition 2.9. ([16]) Let $u \in C((0,a], \mathbb{R}_F) \cap L^1((0,a), \mathbb{R}_F)$. The fuzzy fractional integral of order $q > 0$ of u, is defined as

$$I^q u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s)\, ds, \quad t \in (0,a), \tag{2.2}$$

provided the integral in the right-hand side is defined for a.e. $t \in (0,a)$.

Definition 2.10. Let $f : [0,a] \longrightarrow \mathbb{R}_F$ be a generalized Hukuhara differentiable function and

$$f' \in C\big((0,a], \mathbb{R}_F\big) \bigcap L^1\big((0,a), \mathbb{R}_F\big).$$

The Caputo fractional H-derivative of fuzzy-valued function f is defined as

$$({}^C D^q f)(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{f'(s)}{(t-s)^q} ds, \tag{2.3}$$

where $0 < q < 1$.

Definition 2.11. ([24]) Let $f : [0,a] \longrightarrow \mathbb{R}_F$ be a generalized Hukuhara differentiable function and $f' \in C\big((0,a], \mathbb{R}_F\big) \bigcap L^1\big((0,a), \mathbb{R}_F\big)$. We say that f is ${}^C[1-q]-$differentiable if $\big({}^C D^q f\big)(t;r) = \big[\big({}^C D^q \underline{f}\big)(t;r), \big({}^C D^q \overline{f}\big)(t;r)\big]$ and is ${}^C[2-q]-$differentiable if

$$\big({}^C D^q f\big)(t;r) = \big[\big({}^C D^q \overline{f}\big)(t;r), \big({}^C D^q \underline{f}\big)(t;r)\big].$$

Theorem 2.12. *Let $f : [0,a] \longrightarrow \mathbb{R}_F$ be generalized differentiable and*

$$f' \in C\big((0,a], \mathbb{R}_F\big) \bigcap L^1\big((0,a), \mathbb{R}_F\big).$$

Then,
(i) f is [1-q]-differentiable at t_0 iff f is ${}^C[1-q]-$differentiable at t_0.
(ii) f is [2-q]-differentiable at t_0 iff f is ${}^C[2-q]-$differentiable at t_0.

Proof. (i) Let f be [1-q]-differentiable at t_0. Then we have

$$\left[({}^C D^q f)(t_0)\right]^\alpha = \left[I^{1-q} f'(t_0)\right]^\alpha = \left[I^{1-q}(\underline{f}_\alpha)'(t_0), I^{1-q}(\overline{f}_\alpha)'(t_0)\right] = \left[({}^C D^q \underline{f}_\alpha)(t_0), ({}^C D^q \overline{f}_\alpha)(t_0)\right]$$

that is, f is ${}^C[1-q]-$differentiable at t_0. Conversely, if f be ${}^C[1-q]-$differentiable at t_0, then

$$\begin{aligned} \left[I^{q-1}({}^C D^q f(t_0))\right]^\alpha &= \left[I^{q-1}({}^C D^q \underline{f}_\alpha)(t_0), I^{q-1}({}^C D^q \overline{f}_\alpha)(t_0)\right] \\ &= \left[I^{q-1}(I^{1-q}(\underline{f}_\alpha)'(t_0)), I^{q-1}(I^{1-q}(\overline{f}_\alpha)'(t_0))\right] \\ &= \left[(\underline{f}_\alpha)'(t_0), (\overline{f}_\alpha)'(t_0)\right], \end{aligned}$$

by Definition 2.10. On the other hand,

$$\left[I^{q-1}({}^C D^q f(t_0))\right]^\alpha = \left[f'(t_0)\right]^\alpha,$$

so, f is [1-q]-differentiable at t_0.
(ii) Let f be [2-q]-differentiable at t_0. Then we have

$$\begin{aligned} \left[({}^C D^q f)(t_0)\right]^\alpha &= \left[I^{1-q} f'(t_0)\right]^\alpha \\ &= \left[I^{1-q}(\overline{f}_\alpha)'(t_0), I^{1-q}(\underline{f}_\alpha)'(t_0)\right] \\ &= \left[({}^C D^q \overline{f}_\alpha)(t_0), ({}^C D^q \underline{f}_\alpha)(t_0)\right], \end{aligned}$$

that is, f is ${}^C[2-q]-$differentiable at t_0. Conversely, if f be ${}^C[2-q]-$differentiable at t_0, then

$$\begin{aligned} \left[I^{q-1}({}^C D^q f(t_0))\right]^\alpha &= \left[I^{q-1}({}^C D^q \overline{f}_\alpha)(t_0), I^{q-1}({}^C D^q \underline{f}_\alpha)(t_0)\right] \\ &= \left[I^{q-1}(I^{1-q}(\overline{f}_\alpha)'(t_0)), I^{q-1}(I^{1-q}(\underline{f}_\alpha)'(t_0))\right] \\ &= \left[(\overline{f}_\alpha)'(t_0), (\underline{f}_\alpha)'(t_0)\right], \end{aligned}$$

by Definition 2.10, that is, f is $[2-q]-$differentiable at t_0. □

Definition 2.13. ([24]) Let $f(t)$ be a continuous fuzzy-valued function. Suppose that $f(t)e^{-pt}$ is improper fuzzy Riemann-integrable on $[0,\infty)$. Then $\int_0^\infty f(t)e^{-pt}dt$ is the fuzzy Laplace transform and can be denoted as

$$\mathbf{L}\{f(t)\} = \int_0^\infty f(t)e^{-pt}dt, \quad (p > 0 \quad and \quad integer). \tag{2.4}$$

From Theorem 2.8, it is easy to see that for all $r \in [0,1]$, we have

$$\int_0^\infty f(t;r)e^{-pt}dt = \left[\int_0^\infty \underline{f}(t;r)e^{-pt}dt, \int_0^\infty \overline{f}(t;r)e^{-pt}dt\right].$$

By virtue of the definition of the classical Laplace transform, one can obtain easily

$$\ell\{\underline{f}(t;r)\} = \int_0^\infty \underline{f}(t;r)e^{-pt}dt, \quad \ell\{\overline{f}(t;r)\} = \int_0^\infty \overline{f}(t;r)e^{-pt}dt.$$

Therefore, we conclude that

$$\mathbf{L}\{f(t;r)\} = \left[\ell\{\underline{f}(t;r)\}, \ell\{\overline{f}(t;r)\}\right].$$

Theorem 2.14. *([24]) Let $f(t), g(t)$ be continuous fuzzy-valued functions. Suppose that $c_1, c_2 \in \mathbb{R}$ are constants. Then*

$$\boldsymbol{L}\big\{c_1 f(t) + c_2 g(t)\big\} = c_1 \boldsymbol{L}\{f(t)\} + c_2 \boldsymbol{L}\{g(t)\}.$$

Lemma 2.15. *([24]) Let $f(t)$ be a continuous fuzzy-valued function on $[0, \infty)$ and $\lambda \in R$. Then*

$$\boldsymbol{L}\{\lambda f(t)\} = \lambda \boldsymbol{L}\{f(t)\}.$$

Theorem 2.16. *([24]) Let $f(t)$ be a continuous fuzzy-valued function and $\boldsymbol{L}\{f(t)\} = F(p)$; then*

$$\boldsymbol{L}\{e^{at} f(t)\} = F(p - a),$$

where e^{at} is a real-valued function.

Theorem 2.17. *([24]) Let $f(t)$ be a continuous fuzzy-valued function on $[0, \infty)$. Then we have*

$$\boldsymbol{L}\{{}^{C}D^{q} f(t)\} = s^{q} \boldsymbol{L}\{f(t)\} \ominus f(0), \tag{2.5}$$

if f is ${}^{C}[1-q]-$differentiable and

$$\boldsymbol{L}\{{}^{C}D^{q} f(t)\} = -f(0) \ominus (-s^{q} \boldsymbol{L}\{f(t)\}), \tag{2.6}$$

if f is ${}^{C}[2-q]-$differentiable.

3. Caputo Fractional Fuzzy Differential Equations

In this section, we consider the fuzzy fractional initial value problem (FFIVP)

$$\begin{cases} {}^{C}D^{q} x(t) = \lambda x(t) + b(t), \\ x(0) = x_0 \in \mathbb{R}_F, \end{cases} \tag{3.1}$$

where $\lambda \in \mathbb{R}$ and $b(t) \in \mathbb{R}_F$.
We solve FFIVP (3.1) by appling the fuzzy Laplace transform. We discuss this problem in three cases $\lambda > 0$, $\lambda < 0$ and $\lambda = 0$, and present the general form of its ${}^{C}[1-q]$ and ${}^{C}[2-q]$ solutions.

Case I. Let $\lambda > 0$. Suppose that $x(t)$ be ${}^{C}[1-q]-$differentiable. If we apply the fuzzy Laplace transform to both sides of (3.1), using Theorem 2.17, we obtain

$$s^{q} \mathbf{L}\{x(t)\} \ominus x(0) = \lambda \mathbf{L}\{x(t)\} + \mathbf{L}\{b(t)\}. \tag{3.2}$$

So, we have

$$s^{q} \ell\{\underline{x}(t)\} - \underline{x}(0) = \lambda \ell\{\underline{x}(t)\} + \ell\{\underline{b}(t)\}$$

and

$$s^{q} \ell\{\overline{x}(t)\} - \overline{x}(0) = \lambda \ell\{\overline{x}(t)\} + \ell\{\bar{b}(t)\}.$$

Then, we deduce

$$\ell\{\underline{x}(t)\} = \frac{\underline{x}(0) + \ell\{\underline{b}(t)\}}{s^{q} - \lambda}.$$

Now, by applying the inverse Laplace transform, we have

$$\underline{x}(t) = \underline{x}(0)\ell^{-1}\{\frac{1}{s^q - \lambda}\} + \ell^{-1}\left\{\ell\{\underline{b}(t)\} \cdot \frac{1}{s^q - \lambda}\right\}.$$

By convolution theorem [12], we have

$$\underline{x}(t) = \underline{x}(0)t^{q-1}E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)\underline{b}(s)ds,$$

where $E_{q,q}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(qk+q)}$ is the classical Mittag-Leffler function. Similarly, we obtain

$$\overline{x}(t) = \overline{x}(0)t^{q-1}E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)\overline{b}(s)ds.$$

Therefore, the solution is given by

$$x(t) = x(0)t^{q-1}E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)b(s)ds. \tag{3.3}$$

Let $x(t)$ be ${}^C[2-q]-$differentiable, then by Theorem 2.17 and applying the fuzzy Laplace transform to both sides of (3.1), we have

$$-x(0) \ominus (-s^q\mathbf{L}\{x(t)\}) = \lambda\mathbf{L}\{x(t)\} + \mathbf{L}\{b(t)\}. \tag{3.4}$$

Then

$$(s^q\ell\{\overline{x}(t)\} - \overline{x}(0), s^q\ell\{\underline{x}(t)\} - \underline{x}(0)) = \left(\lambda\ell\{\underline{x}(t)\} + \ell\{\underline{b}(t)\}, \lambda\ell\{\overline{x}(t)\} + \ell\{\overline{b}(t)\}\right).$$

So, we get

$$\begin{cases} s^q\ell\{\underline{x}(t)\} - \lambda\ell\{\overline{x}(t)\} = \underline{x}(0) + \ell\{\overline{b}(t)\}, \\ s^q\ell\{\overline{x}(t)\} - \lambda\ell\{\underline{x}(t)\} = \overline{x}(0) + \ell\{\underline{b}(t)\}. \end{cases}$$

Therefore, using the theory of ordinary differential equations and classic Laplace transform [12], we obtain

$$\ell\{\underline{x}(t)\} = \frac{s^q\left[\underline{x}(0) + \ell\{\overline{b}(t)\}\right]}{s^{2q} - \lambda^2} + \frac{\lambda\left[\overline{x}(0) + \ell\{\underline{b}(t)\}\right]}{s^{2q} - \lambda^2}, \tag{3.5}$$

and

$$\ell\{\overline{x}(t)\} = \frac{s^q\left[\overline{x}(0) + \ell\{\underline{b}(t)\}\right]}{s^{2q} - \lambda^2} + \frac{\lambda\left[\underline{x}(0) + \ell\{\underline{b}(t)\}\right]}{s^{2q} - \lambda^2}. \tag{3.6}$$

Now, by applying the inverse Laplace transform to both sides of (3.5), we get

$$\underline{x}(t) = \ell^{-1}\left\{\frac{s^q\left[\underline{x}(0) + \ell\{\overline{b}(t)\}\right]}{s^{2q} - \lambda^2}\right\} + \ell^{-1}\left\{\frac{\lambda\left[\overline{x}(0) + \ell\{\underline{b}(t)\}\right]}{s^{2q} - \lambda^2}\right\}. \tag{3.7}$$

It is easy to see that

$$\ell^{-1}\left\{\frac{s^q\left[\underline{x}(0)+\ell\{\overline{b}(t)\}\right]}{s^{2q}-\lambda^2}\right\} = \ell^{-1}\left\{s^{-q}\left[\underline{x}(0)+\ell\{\overline{b}(t)\}\right]\left[1-s^{-2q}\lambda^2\right]^{-1}\right\}$$

$$= \ell^{-1}\left\{s^{-q}\left[\underline{x}(0)+\ell\{\overline{b}(t)\}\right]\sum_{k=0}^{\infty}(s^{-2q}\lambda^2)^k\right\}$$

$$= \ell^{-1}\left\{\underline{x}(0)\sum_{k=0}^{\infty}\lambda^{2k}s^{-2qk-q}\right\} + \ell^{-1}\left\{\ell\{\overline{b}(t)\}\sum_{k=0}^{\infty}\lambda^{2k}s^{-2qk-q}\right\}$$

$$= \underline{x}(0)\sum_{k=0}^{\infty}\lambda^{2k}\ell^{-1}\left\{s^{-2qk-q}\right\} + \overline{b}(t) * \sum_{k=0}^{\infty}\lambda^{2k}\ell^{-1}\left\{s^{-2qk-q}\right\}$$

$$= \underline{x}(0)\sum_{k=0}^{\infty}\lambda^{2k}\frac{t^{2qk+q-1}}{\Gamma(2qk+q)} + \overline{b}(t) * \sum_{k=0}^{\infty}\lambda^{2k}\frac{t^{2qk+q-1}}{\Gamma(2qk+q)}$$

$$= \underline{x}(0)t^{q-1}E_{2q,q}(\lambda^2 t^{2q}) + \int_0^t (t-s)^{q-1}E_{2q,q}(\lambda^2(t-s)^{2q})\overline{b}(s)ds,$$

and

$$\lambda\ell^{-1}\left\{\frac{\overline{x}(0)+\ell\{\underline{b}(t)\}}{s^{2q}-\lambda^2}\right\} = \lambda\ell^{-1}\left\{\frac{\overline{x}(0)}{s^{2q}-\lambda^2}\right\} + \lambda\ell^{-1}\left\{\ell\{\underline{b}(t)\}\cdot\frac{1}{s^{2q}-\lambda^2}\right\}$$

$$= \lambda\overline{x}(0)t^{2q-1}E_{2q,2q}(\lambda^2 t^{2q}) + \lambda\int_0^t (t-s)^{2q-1}E_{2q,2q}(\lambda^2(t-s)^{2q})\underline{b}(s)ds,$$

where $*$ stands for convolution operator. So, by (3.7), we have

$$\begin{aligned}\underline{x}(t) &= \underline{x}(0)t^{q-1}E_{2q,q}(\lambda^2 t^{2q}) + \int_0^t (t-s)^{q-1}E_{2q,q}(\lambda^2(t-s)^{2q})\overline{b}(s)ds \\ &+ \lambda\overline{x}(0)t^{2q-1}E_{2q,2q}(\lambda^2 t^{2q}) + \lambda\int_0^t (t-s)^{2q-1}E_{2q,2q}(\lambda^2(t-s)^{2q})\underline{b}(s)ds,\end{aligned}$$

and similarly,

$$\begin{aligned}\overline{x}(t) &= \overline{x}(0)t^{q-1}E_{2q,q}(\lambda^2 t^{2q}) + \int_0^t (t-s)^{q-1}E_{2q,q}(\lambda^2(t-s)^{2q})\underline{b}(s)ds \\ &+ \lambda\underline{x}(0)t^{2q-1}E_{2q,2q}(\lambda^2 t^{2q}) + \lambda\int_0^t (t-s)^{2q-1}E_{2q,2q}(\lambda^2(t-s)^{2q})\overline{b}(s)ds.\end{aligned}$$

So, the ${}^{C}[2-q]-$solution is given by

$$x(t) = x(0)t^{q-1}E_{2q,q}(\lambda^2 t^{2q}) \ominus -\lambda x(0)t^{2q-1}E_{2q,2q}(\lambda^2 t^{2q})$$

$$+\lambda\int_0^t (t-s)^{2q-1}E_{2q,2q}(\lambda^2(t-s)^{2q})b(s)ds$$

$$\ominus -\int_0^t (t-s)^{q-1}E_{2q,q}(\lambda^2(t-s)^{2q})b(s)ds,$$

provided the H-differences exist. So, we have the following result.

Theorem 3.1. *Let $\lambda > 0$. Then*
(i) the ${}^{C}[1-q]-$*solution of FFIVP (3.1) is given by*

$$x(t) = x(0)t^{q-1}E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)b(s)ds,$$

(ii) the ${}^{C}[2-q]-$*solution of FFIVP (3.1) is given by*

$$\begin{aligned} x(t) &= x(0)t^{q-1}E_{2q,q}(\lambda^2 t^{2q}) \ominus -\lambda x(0)t^{2q-1}E_{2q,2q}(\lambda^2 t^{2q}) \\ &\quad + \lambda\int_0^t (t-s)^{2q-1}E_{2q,2q}(\lambda^2(t-s)^{2q})b(s)ds \\ &\quad \ominus - \int_0^t (t-s)^{q-1}E_{2q,q}(\lambda^2(t-s)^{2q})b(s)ds, \end{aligned}$$

provided the H-differences exist.

Case II. Let $\lambda < 0$. First, we suppose that $x(t)$ be ${}^{C}[1-q]-$differentiable. By Theorem 2.17 and applying the fuzzy Laplace transform to both sides of (3.1), we have

$$\Big(s^q\ell\{\underline{x}(t)\} - \underline{x}(0), s^q\ell\{\overline{x}(t)\} - \overline{x}(0)\Big) = \Big(\lambda\ell\{\overline{x}(t)\} + \ell\{\underline{b}(t)\}, \lambda\ell\{\underline{x}(t)\} + \ell\{\overline{b}(t)\}\Big).$$

Therefore, we have

$$s^q\ell\{\underline{x}(t)\} - \lambda\ell\{\overline{x}(t)\} = \underline{x}(0) + \ell\{\underline{b}(t)\},$$

and

$$s^q\ell\{\overline{x}(t)\} - \lambda\ell\{\underline{x}(t)\} = \overline{x}(0) + \ell\{\overline{b}(t)\}.$$

Then, we obtain

$$\ell\{\underline{x}(t)\} = \frac{s^q\big[\underline{x}(0) + \ell\{\underline{b}(t)\}\big]}{s^{2q} - \lambda^2} + \frac{\lambda\big[\overline{x}(0) + \ell\{\overline{b}(t)\}\big]}{s^{2q} - \lambda^2},$$

and

$$\ell\{\overline{x}(t)\} = \frac{s^q\big[\overline{x}(0) + \ell\{\overline{b}(t)\}\big]}{s^{2q} - \lambda^2} + \frac{\lambda\big[\underline{x}(0) + \ell\{\underline{b}(t)\}\big]}{s^{2q} - \lambda^2}.$$

So, similarly to the **Case I**, we have

$$\begin{aligned} \underline{x}(t) &= \underline{x}(0)t^{q-1}E_{2q,q}(\lambda^2 t^{2q}) + \int_0^t (t-s)^{q-1}E_{2q,q}(\lambda^2(t-s)^{2q})\underline{b}(s)ds \\ &+ \lambda\overline{x}(0)t^{2q-1}E_{2q,2q}(\lambda^2 t^{2q}) + \lambda\int_0^t (t-s)^{2q-1}E_{2q,2q}(\lambda^2(t-s)^{2q})\overline{b}(s)ds, \end{aligned}$$

and

$$\begin{aligned} \overline{x}(t) &= \overline{x}(0)t^{q-1}E_{2q,q}(\lambda^2 t^{2q}) + \int_0^t (t-s)^{q-1}E_{2q,q}(\lambda^2(t-s)^{2q})\overline{b}(s)ds \\ &+ \lambda\underline{x}(0)t^{2q-1}E_{2q,2q}(\lambda^2 t^{2q}) + \lambda\int_0^t (t-s)^{2q-1}E_{2q,2q}(\lambda^2(t-s)^{2q})\underline{b}(s)ds, \end{aligned}$$

So, the ${}^{C}[1-q]-$solution of (3.1) for $\lambda < 0$ is given by

$$\begin{aligned} x(t) &= x(0)t^{q-1}E_{2q,q}(\lambda^2 t^{2q}) + \lambda x(0)t^{2q-1}E_{2q,2q}(\lambda^2 t^{2q}) \\ &+ \int_0^t (t-s)^{q-1}E_{2q,q}(\lambda^2(t-s)^{2q})b(s)ds \\ &+ \lambda \int_0^t (t-s)^{2q-1}E_{2q,2q}(\lambda^2(t-s)^{2q})b(s)ds. \end{aligned}$$

Now, we suppose that $x(t)$ is ${}^{C}[2-q]-$differentiable. By Theorem 2.17, we have

$$s^q \ell\{\underline{x}(t)\} - \lambda \ell\{\underline{x}(t)\} = \underline{x}(0) + \ell\{\overline{b}(t)\},$$

and

$$s^q \ell\{\overline{x}(t)\} - \lambda \ell\{\overline{x}(t)\} = \overline{x}(0) + \ell\{\underline{b}(t)\},$$

so, similarly to the **Case I**, we obtain

$$\underline{x}(t) = \underline{x}(0)t^{q-1}E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)\overline{b}(s)ds,$$

and

$$\overline{x}(t) = \overline{x}(0)t^{q-1}E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)\underline{b}(s)ds.$$

Therefore, the ${}^{C}[2-q]-$solution is

$$x(t) = x(0)t^{q-1}E_{q,q}(\lambda t^q) \ominus - \int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)b(s)ds,$$

provided the H-difference exists. Then, we have the following result.

Theorem 3.2. *Let $\lambda < 0$. Then*
(i) the ${}^{C}[1-q]-$solution of FFIVP (3.1) is given by

$$\begin{aligned} x(t) &= x(0)t^{q-1}E_{2q,q}(\lambda^2 t^{2q}) + \lambda x(0)t^{2q-1}E_{2q,2q}(\lambda^2 t^{2q}) \\ &+ \int_0^t (t-s)^{q-1}E_{2q,q}(\lambda^2(t-s)^{2q})b(s)ds \\ &+ \lambda \int_0^t (t-s)^{2q-1}E_{2q,2q}(\lambda^2(t-s)^{2q})b(s)ds, \end{aligned}$$

(ii) the ${}^{C}[2-q]-$solution of FFIVP (3.1) is given by

$$x(t) = x(0)t^{q-1}E_{q,q}(\lambda t^q) \ominus - \int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)b(s)ds,$$

provided the H-difference exists.

Case III. Let $\lambda \equiv 0$. Then, the fractional fuzzy initial value problem (3.1) is as follows

$$\begin{cases} {}^{C}D^q x(t) = b(t), \\ x(0) = x_0 \in \mathbb{R}_F. \end{cases} \tag{3.8}$$

Suppose that $x(t)$ be ${}^{C}[1-q]-$differentiable. Then, by Theorem 2.17, we have

$$s^q \mathbf{L}\{x(t)\} \ominus x(0) = \mathbf{L}\{b(t)\}.$$

So, we obtain

$$s^q \ell\{\underline{x}(t)\} - \underline{x}(0) = \ell\{\underline{b}(t)\},$$

and

$$s^q \ell\{\overline{x}(t)\} - \overline{x}(0) = \ell\{\bar{b}(t)\}.$$

Then, we have

$$\ell\{\underline{x}(t)\} = \frac{\underline{x}(0) + \ell\{\underline{b}(t)\}}{s^q}.$$

Therefore,

$$\underline{x}(t) = \underline{x}(0)\ell^{-1}\{\frac{1}{s^q}\} + \ell^{-1}\left\{\ell\{\underline{b}(t)\} \cdot \frac{1}{s^q}\right\}.$$

Now, by convolution theorem for classic Laplace transform, we have

$$\underline{x}(t) = \underline{x}(0)\frac{t^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\underline{b}(s)ds,$$

and similarly

$$\overline{x}(t) = \overline{x}(0)\frac{t^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\bar{b}(s)ds.$$

So, the ${}^{C}[1-q]-$solution is

$$x(t) = x(0)\frac{t^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}b(s)ds.$$

Let $x(t)$ be ${}^{C}[2-q]-$differentiable. Then, by Theorem 2.17, we have

$$-x(0) \ominus (-s^q \mathbf{L}\{x(t)\}) = \mathbf{L}\{b(t)\}.$$

So, we obtain

$$s^q \ell\{\underline{x}(t)\} - \underline{x}(0) = \ell\{\bar{b}(t)\},$$

and

$$s^q \ell\{\overline{x}(t)\} - \overline{x}(0) = \ell\{\underline{b}(t)\}.$$

Then, we have

$$\ell\{\underline{x}(t)\} = \frac{\underline{x}(0) + \ell\{\bar{b}(t)\}}{s^q},$$

so,

$$\underline{x}(t) = \underline{x}(0)\ell^{-1}\{\frac{1}{s^q}\} + \ell^{-1}\left\{\ell\{\bar{b}(t)\} \cdot \frac{1}{s^q}\right\}.$$

Then, it is easy to see that

$$\underline{x}(t) = \underline{x}(0)\frac{t^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\bar{b}(s)ds,$$

and similarly

$$\overline{x}(t) = \overline{x}(0)\frac{t^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\underline{b}(s)ds.$$

So, the ${}^C[2-q]-$solution is

$$x(t) = x(0)\frac{t^{q-1}}{\Gamma(q)} \ominus -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}b(s)ds,$$

provided the H-difference exists. So, we have proved the following result.

Theorem 3.3. *Let $\lambda \equiv 0$. Then*
(i) the ${}^C[1-q]-$solution of FFIVP (3.8) is given by

$$x(t) = x(0)\frac{t^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}b(s)ds,$$

(ii) the ${}^C[2-q]-$solution of FFIVP (3.8) is given by

$$x(t) = x(0)\frac{t^{q-1}}{\Gamma(q)} \ominus -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}b(s)ds,$$

provided the H-difference exists.

4. Examples

In this section, we present some examples to illustrate our results.
Example 4.1. Let $\lambda = 1$, $b(t) = \hat{0}$. Then, the equation (3.1) is

$$\begin{cases} {}^CD^q x(t) = x(t), \\ x(0) = x_0 \in \mathbb{R}_F. \end{cases} \tag{4.1}$$

So, using Theorem 3.1, the ${}^C[1-q]$solution is

$$x(t) = x(0)t^{q-1}E_{q,q}(t^q),$$

and the ${}^C[2-q]$ solution is

$$x(t) = x(0)t^{q-1}E_{2q,q}(t^{2q}) \ominus -x(0)t^{2q-1}E_{2q,2q}(t^{2q}),$$

provided the H-difference exists.
Example 4.2. Let $\lambda = -1, b(t) = \hat{0}$. Then, we can write FFIVP (3.1) as

$$\begin{cases} {}^CD^q x(t) = -x(t), \\ x(0) = x_0 \in \mathbb{R}_F. \end{cases} \tag{4.2}$$

Using Theorem 3.2, the ${}^C[1-q]-$solution is given by

$$x(t) = x(0)t^{q-1}E_{2q,q}(t^{2q}) - x(0)t^{2q-1}E_{2q,2q}(t^{2q}),$$

and the ${}^C[2-q]$-solution is

$$x(t) = x(0)t^{q-1}E_{q,q}(-t^{2q}). \tag{4.3}$$

5. Conclusion

In this paper, we have studied the solvability of linear fractional fuzzy differential equations, from the point of view of Caputo generalized differentiability. We have obtained the explicit expressions of the solutions. The consideration of more general derivatives in linear fractional fuzzy differential equations, such as the gH-differentiability and g-differentiability [10] is indeed an interesting area for future research.

References

[1] R. P. Agarwal, D. Baleanu, J. J. Nieto, D. F. Torres, and Y. Zhou, *A survey on fuzzy fractional differential and optimal control nonlocal evolution equations*, Journal of Computational and Applied Mathematics, 2017. DOI 10.1016/j.cam.2017.09.039.

[2] R. P. Agarwal, V. Lakshmikantham, and J. J. Nieto, *On the concept of solution for fractional differential equations with uncertainty*, Nonlinear Anal., *72* (2010), 2859–2862.

[3] R. P. Agarwal, S. Arshad, D. O'Regan, and V. Lupulescu, *A Schauder fixed point theorem in semilinear spaces and applications*, Fixed Point Theory and Applications, (2013), 2013:306.

[4] R. Alikhani and F. Bahrami, *Global solutions for nonlinear fuzzy fractional integral and integrodifferential equations*, Communications in Nonlinear Science and Numerical Simulation, *18* (2013), 2007–2017.

[5] T. Allahviranloo, S. Salahshour, and S. Abbasbandy, *Explicit solutions of fractional differential equations with uncertainty*, Soft Computing, *16* (2012), 297–302.

[6] T. Allahviranloo, A. Armand, Z. Gouyandeh, and H. Ghadiri, *Existence and uniqueness of solutions for fuzzy fractional Voltra-Fredholm integro-differential equations*, Journal of Fuzzy Set Valued Analysis, (2013), 1–9.

[7] T. Allahviranloo and M. Barkhordari, *Fuzzy Laplace transforms*, Soft Computing, *14* (2010), 235–243.

[8] S. Arshad and V. Lupulescu, *On the fractional differential equation with uncertainty*, Nonlinear Analysis, *74* (2011), 3685–3693.

[9] S. Arshad, *On existence and uniqueness of solution of fuzzy fractional differential equation*, Iranian Journal of Fuzzy Systems, *10* (2013), 137–151.

[10] B. Bede, *Mathematics of Fuzzy Sets and Fuzzy Logic*, Springer, London, 2013.

[11] B. Bede and S. G. Gal, *Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations*, Fuzzy Sets and Systems, *151* (2005), 581–599.

[12] W. E. Boyce and R. C. Diprima, *Elementary differential equations and boundary value problems*, 4th ed, Wiley, New York, 1986.

[13] D. Delbosco and L. Rodino, *Existence and uniqueness for a nonlinear fractional differential equation*, J. Math. Anal. Appl., *204* (1996), 609–625.

[14] K. Diethelm and N. J. Ford, *Analysis of fractional differential equations*, J. Math. Anal. Appl., *265* (2002), 229–248.

[15] A. Jafarian, A. Khalili Golmankhaneh, and D. Baleanu, *On fuzzy fractional Laplace transformation*, Advances in Mathematical Physics, 2014. DOI 10.1155/2014/295432.

[16] A. Khastan, J. J. Nieto, and R. Rodr íguez-López, *Schauder fixed-point theorem in semilinear spaces and its application to fractional differential equations with uncertainty*, Fixed Point Theory and Applications, 2014.

[17] A. Khastan, J. J. Nieto, and R. Rodríguez-López, *Variation of constant formula for first order fuzzy differential equations*, Fuzzy Sets and Systems, *177* (2011), 20–33.

[18] A. A. Kilbas, M. Rivero, L. Rodrguez-Germ, and J. J. Trujillo, *Caputo linear fractional differential equations*, IFAC Proceedings Volumes, *39* (2006), 52–57.

[19] V. Lakshmikantham, and A. S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal., *69* (2008), 2677–2682.

[20] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, A Wiley-Interscience Publication, John Wiley and Sons, Inc., New York, 1993.
[21] K. B. Oldham and J. Spanier, *The fractional calculus*, Academic Press, New York, 1974.
[22] I. Podlubny, *Fractional differential Equations*, Academic Press, San Diego, 1999.
[23] M. L. Puri and D. A. Ralescu, *Differentials of fuzzy functions*, Journal of Mathematical Analysis and Applications, *91* (1983), 552–558.
[24] S. Salahshour, A. Ahmadian, N. Senu, D. Baleanu, and P. Agarwal, *On analytical solutions of the fractional differential equation with uncertainty*: Application to the Basset problem, Entropy, *17* (2015), 885–902. DOI 10.3390/e17020885.
[25] S. G. Samko, A. A. Kilbas, and O. L. Marichev, *Fractional integrals and derivatives*, Theory and Applications, Gordon and Breach Sci. Publishers, London, New York, 1993.
[26] H. C. Wu, *The improper fuzzy Riemann integral and its numerical integration*, Information Sciences, *111* (1998), 109–137.