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Hermite wavelets method for the numerical solution of linear and nonlinear singular initial and boundary value problems

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Abstract In this research article, Modified Hermite wavelets based numerical method is developed for the solution of singular initial and boundary value problems. In the present work we transform the differential equations associated with initial and boundary conditions into system of algebraic equations by expanding the unknown function as a series of Hermite wavelets with unknown coefficients. We solve obtained system of equations using Newton's iterative method through Matlab. Illustrative examples are considered to demonstrate the applicability and accuracy of the proposed technique. Obtained results are compared favorably with the exact solutions. Also, we proved the theorem reveals that, when exact solution can be obtained by the proposed method and theorems regarding convergence and error analysis.

Keywords. Hermite wavelets, Singular initial and boundary value problems, Collocation method, Limit points.

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1. Introduction

To study the behavior of differential equations we need to find their exact solutions by the classical techniques such as trigonometry, calculus etc. By these solutions one can know the behavior of differential equation under the given different circumstances. The techniques used for calculating the exact solution are known as analytical methods. But this works only for simple equations. That is, differential equations with simple coefficients, for higher order differential equations with complex coefficients becomes very difficult to find exact solution. Therefore, we need numerical methods to solve such equations. Singular initial and boundary value problems for ordinary differential equations arises in many fields such as gas dynamics, atomic structures, chemical reactions and nuclear physics [11]. In many cases, extracting the analytical solutions for singular initial and boundrary value problems from analytical methods is

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not possible, in such cases collocation method is one of the most widely used numerical schemes to solve differential equations, though wavelet based collocation method will give high precision numerical solution [18, 19, 20, 27].
Wavelets are localized waves or small wave. Instead of oscillating forever, they drop to zero. Wavelets theory is a newly emerging area in mathematical research field. It has been applied in engineering disciplines such as, signal analysis for wave form representation and segmentations, time frequency analysis, harmonic analysis etc. Wavelets permit the accurate representation of a variety of functions and operators. Wavelets are assumed as basis functions $\psi_{i,j}(x)$ in continuous time. Basis is a set of functions which are linearly independent and these linearly independent functions can be used to produce all admissible functions say $f(t)$. It is represented in wavelet space as $f(t) = \sum_{i,j} a_{i,j}\psi_{i,j}(x)$. Special feature of the wavelet basis is that all functions $\psi_{i,j}(x)$ are constructed from a single function called mother wavelet $\psi(x)$ which is a small pulse. Usually set of linearly independent functions (basis) created by translation and dilation of mother wavelet.
Different methods were developed for the solution of singular initial and boundary value problems. Such as, Hermites wavelets method [2, 25], New ultraspherical wavelet collocation method [5], Differential transformation method [6], Wavelet analysis method [14], An efficient wavelet based spectral method [16], Wavelet Galerkin method [17], Haar wavelet collocation method [19, 20], Laguerre wavelet method [22], Legendre wavelet method [26], Adomian decomposition method [27], Lagurre wavelets mathod [30]. There are two different approaches for solving differential equations, one approach is based on converting differential equations into integral equations then eliminate the integral operator by operational matrix of integration [26, 29]. Another method is based on the operational matrix of derivative. Here, we solve the differential equations by converting into system of linear or nonlinear algebraic equations through either operational matrix of derivative method [14, 16, 30]or series approximation method [2, 5, 8, 9, 10, 23, 25].
In this paper, our effort is to bring the solutions of singular initial and boundary value problems under the Hermite space (The space generated by Hermite wavelet basis). But present method will give exact solutions of all second order linear singular problems, which are in the polynomial form of degree n for different conditions, hence we are concentrating to solve problems whose solutions are not in polynomial form of finite degree. i.e. we are expressing the solutions of singular initial and boundary value problems in terms of Hermite wavelet basis.
Let $(a, b) \subset R$ be an interval and $p(x), q(x), r(x, y) : (a, b) \to R$ be continuous real valued functions. Throughout this paper we consider the singular second order equations given by

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x, y), \qquad a < x < b, \tag{1.1}$$

subjected to following initial and boundary conditions:

$$Type\ I : y(a) = \alpha_1,\ y(b) = \beta_1,$$
$$Type\ II : y'(a) = \alpha_2,\ y(b) = \beta_2,$$
$$Type\ III : y(a) = \alpha_3,\ y'(b) = \beta_3,$$

$$Type\ IV : a_1y(a) + a_2y'(a) = \alpha_4,\ b_1y(b) + b_2y'(b) = \beta_4,$$

$$Type\ V : y(a) = \alpha_5,\ y'(a) = \beta_5$$

where a_i, b_i, $i = 1, 2$. α_j, β_j, $j = 1, 2, 3, 4, 5$ are known constants. In the proposed method, unknown function appearing in the differential equations is replaced by series expansions of polynomial basis of Hermite wavelets. After collocating the equation by suitable collocation points with the given conditions, we obtain a system of linear or nonlinear equations which can be solved using iterative methods to get the unknown coefficients. Hence the required solution is obtained by substituting these unknown coefficients in the unknown function.
The rest of this paper is organized as follows. In section 2 properties of Hermite wavelet is discussed. Section 3 presents function approximation and theorem on exact solution. Section 4 reveals that method of solution. In section 5 the numerical examples to test the efficiency of the proposed method. Results discussion and conclusion is drawn in section 6.

2. Hermite Wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called mother wavelet. When the dilation parameter a and translation parameter b varies continuously, we have the following family of continuous wavelets:

$$\psi_{a,b}(x) = |a|^{\frac{-1}{2}}\psi(\frac{x-b}{a}), \forall\ a, b \in R, a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k},\ b = nb_0a_0^{-k},\ a_0 > 1,\ b_0 > 0$. We have the following family of discrete wavelets

$$\psi_{k,n}(x) = |a|^{1/2}\psi(a_0^kx - nb_0), \forall a, b \in R, a \neq 0,$$

where $\psi_{k,n}$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$,then $\psi_{k,n}(x)$ forms an orthonormal basis. Hermite wavelets are defined as [2]

$$\psi_{n,m}(x) = \begin{cases} \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}H_m(2^kx - 2n + 1) & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}} \\ 0, & otherwise \end{cases} \tag{2.1}$$

where $m = 0, 1, \ldots, M-1$. Here $H_m(x)$ is Hermite polynomials of degree m with respect to weight function $W(x) = \sqrt{1-x^2}$ on the real line R and satisfies the following reccurence formula $H_0(x) = 1$, $H_1(x) = 2x$,

$$H_{m+2}(x) = 2xH_{m+1}(x) - 2(m+1)H_m(x) \tag{2.2}$$

where $m = 0, 1, 2, \ldots$.

3. Function Approximation and Convergence Analysis

We would like to bring a solution function $y(x)$ under Hermite space by approximating the $y(x)$ by elements of Hermite wavelet basis as follows:

$$y(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{n,m} \psi_{n,m}(x), \tag{3.1}$$

where $\psi_{n,m}(x)$ is given in Eq. (2.1).
We approximate $y(x)$ by truncating the series represented in Eq. (3.1) as,

$$y(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} \psi_{n,m}(x) = C^T \psi(x), \tag{3.2}$$

where C and $\psi(x)$ are $2^{k-1}M \times 1$ matrix,

$$C^T = [C_{1,0}, \ldots, C_{1,M-1}, C_{2,0}, \ldots, C_{2,M-1}, \ldots, C_{2^{k-1},0}, \ldots, C_{2^{k-1},M-1}], \tag{3.3}$$

$$\psi(x) = [\psi_{1,0}, \ldots, \psi_{1,M-1}, \psi_{2,0}, \ldots, \psi_{2,M-1}, \ldots, \psi_{2^{k-1},0}, \ldots, \psi_{2^{k-1},M-1}]^T. \tag{3.4}$$

Theorem 3.1. *Let R^n is polynomial space of degree $n+1$ over field R and $y : [a,b] \to R^n$ be the solution of arbitrary linear second order linear differential equation then the solution for such differential equation by present method is exact.*

Proof. Let R^n is polynomial space of degree $n+1$ over the field R and $y(x)$ be the solution of arbitrary second order differential equation of degree at most n. If the $y(x)$ be any polynomial of degree n with real coefficients, then there exist a subset $S = \{\psi_{i,0}, \psi_{i,1}, ..., \psi_{i,n}\}$of basis of $n+1$ dimensional Hermite space (space generated by basis of Hermite wavelets), where $\psi_{i,0}, \psi_{i,1}, ..., \psi_{i,n}$ are polynomials of degree $0, 1, 2, ..., n$ respectively. Let,

$$y(x) = \sum_{j=0}^{n} a_{i,j} \psi_{i,j}(x) \ \ for\ a\ fixed\ i,$$

which is a linear combination of elements of Hermite wavelet basis. By equating the coefficients of same degree x on both side, we get values of $a_{i,j}$. Hence $y(x)$ is approximated exactly as a linear combination of basis elements of Hermite wavelets.

Theorem 3.2. *A bounded continuous function y(x) in $H^2[0,1)$ defined on $[0,1)$ then the Hermite wavelets expansion of y(x) converges to it* [21].

Proof. Let y(x) a bounded real valued function on $[0,1)$ and $y(x)$ is approximated as follows, $y(x) = \sum C_{n,m} \psi_{n,m}(x)$ where, n, m are defined in section 2. Then Hermite wavelet coefficients of continuous function y(x) is defined as ($<,>$ represents inner product),

$$C_{n,m} = < y(x), \psi_{n,m}(x) >,$$
$$C_{n,m} = \int_0^1 y(x) \psi_{n,m}(x) dx,$$
$$C_{n,m} = \int_I y(x) \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} H_m(2^k x - 2n + 1) dx,$$

where, $I = [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}]$ and put $2^k x - 2n + 1 = z$.

We obtain:

$$C_{n,m} = \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} \int_{-1}^{1} y(\tfrac{z-1+2n}{2^k}) H_m(z) 2^{-k} dz,$$
$$C_{n,m} = \frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}} \int_{-1}^{1} y(\tfrac{z-1+2n}{2^k}) H_m(z) dz,$$

Using generalized mean value theorem for integrals,

$$C_{n,m} = \frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}} y(\tfrac{w-1+2n}{2^k}) \int_{-1}^{1} H_m(z) dz,$$

for some $w \in (-1, 1)$ and $H_m(z)$ is bounded in the given interval hence put

$$\int_{-1}^{1} H_m(z) dz = h,$$

$$|c_{n,m}| = |\frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}}||y(\tfrac{w-1+2n}{2^k})|h.$$

Since y is bounded. Therefore $\sum_{n,m=0}^{\infty} C_{n,m}$ is absolutely convergent. Hence the Hermite series expansion of y(x)convergence uniformly.

Theorem 3.3. *Suppose $y \in C^p[0,1)$ is an p times continuously differentiable function such that $y = \sum_{n=1}^{2^{k-1}} y_n(x)$ and $\{\psi_{n,m}\}$ be a sequence of Hermite wavelets, where $n = 1, ...2^{k-1}$ and $m = 0, ...M-1$, k is any positive integer. Let $Y_n = L(\{\psi_{n,m}\})$ be the linear space spanned by $\{\psi_{n,m}\}$. If $C_n^T H_n(x)$ is best approximation to y_n from Y_n then $C^T H(x)$ approximates y with following error bound $||y - C^T H(x)||_2 \leq \frac{K}{\sqrt{(2p+1)2^{(k-1)(p+\frac{1}{2})}}}$, where $K = max\ y^p(\zeta)$ $\forall \zeta \in [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}})$.*

Proof. The Taylor expansion for the function $y_n(x)$ is

$$\bar{y_n}(x) = y_n(\tfrac{n-1}{2^{k-1}}) + y_n^{'}(\tfrac{n-1}{2^{k-1}})\frac{(x-\frac{n-1}{2^{k-1}})}{1!} + ... + y_n^{p-1}(\tfrac{n-1}{2^{k-1}})\frac{(x-\frac{n-1}{2^{k-1}})^{p-1}}{(p-1)!}.$$

For which it is known that

$$|y_n(x) - \bar{y_n}(x)| \leq |y_n^p(\zeta)|\frac{(x-\frac{n-1}{2^{k-1}})^p}{(p)!}.$$

where $\zeta \in [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}]$. Since $C_n^T H_n(x)$ is the best approximation of y_n from Y_n and $\bar{y_n} \in Y_n$.

$$||y_n - C_n^T H_n(x)||_2^2 \leq ||y_n - \bar{y_n}||_2^2,$$

$$||y_n - C_n^T H_n(x)||_2^2 = \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} |y_n - \bar{y_n}|^2 dx,$$

$$||y_n - C_n^T H_n(x)||_2^2 \leq \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (|y_n^p(\zeta)|\frac{(x-\frac{n-1}{2^{k-1}})^p}{(p)!})^2 dx,$$

$$||y_n - C_n^T H_n(x)||_2^2 = (\frac{y^p(\zeta)}{p!})^2 \frac{1}{(2p+1)2^{(k-1)(2p+1)}},$$

put $K = \frac{y^p(\zeta)}{p!}$.

Now,

$$||y - C^T H(x)||_2^2 \leq \sum_{n-1}^{2^{k-1}} ||y_n - C_n^T H_n(x)||_2^2,$$

$$||y - C^T H(x)||_2^2 \leq \frac{K^2}{(2p+1)2^{(k-1)(2p+1)}},$$

$$||y - C^T H(x)||_2 \leq \frac{K}{\sqrt{2p-1}2^{(k-1)(p+\frac{1}{2})}}.$$

4. Method of solution

Solution of Eq. (1.1) can be expanded using basis elements of Hermite wavelets as follows:

$$y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{n,m}\psi_{n,m}(x),$$

where $\psi_{n,m}(x)$ is given in Eq. (2.1). We approximate $y(x)$ by truncated series as,

$$y(x) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = C^T\psi(x),$$

where C and $\psi(x)$ are $2^{k-1}M \times 1$ matrix,

$$C^T = [C_{1,0}, \ldots, C_{1,M-1}, C_{2,0}, \ldots, C_{2,M-1}, \ldots, C_{2^{k-1},0}, \ldots, C_{2^{k-1},M-1}],$$

$$\psi(x) = [\psi_{1,0}, \ldots, \psi_{1,M-1}, \psi_{2,0}, \ldots, \psi_{2,M-1}, \ldots, \psi_{2^{k-1},0}, \ldots, \psi_{2^{k-1},M-1}]^T.$$

Then $2^{k-1}M$ number of conditions required to determine $2^{k-1}M$ number of coefficients such as,

$$C_{1,0}, \ldots, C_{1,M-1}, C_{2,0}, \ldots, C_{2,M-1}, \ldots, C_{2^{k-1},0}, \ldots, C_{2^{k-1},M-1}.$$

Since two conditions are furnished by the initial or boundary conditions discussed in the section 1, we see that there should be $2^{k-1}M - 2$ extra conditions to recover the unknown coefficients $C_{n,m}$ substitute Eq. (3.2) in Eq. (1.1) we get,

$$\frac{d^2}{dx^2}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) + p(x)\frac{d}{dx}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x)$$
$$+ q(x)\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = f(x, \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x)). \tag{4.1}$$

Now collocate the Eq. (4.1) by limit points of the following sequence to get $2^{k-1}M-2$ number of equations,

$$x_i = \{\frac{1}{2}(1 + cos\frac{(i-1)\pi}{2^{k-1}M-1})\}\ where\ i =\ 2,\ 3,\ \ldots, \tag{4.2}$$

we get,

$$\frac{d^2}{dx^2}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x_i) + p(x_i)\frac{d}{dx}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x_i)$$
$$+ q(x_i)\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x_i) = f(x_i, \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x_i)). \tag{4.3}$$

From the given initial or boundary conditions and Eq. (4.3), we obtain $2^{k-1}M$ number of linear or nonlinear system of equations, by solving these equations, we get $2^{k-1}M$ unknown coefficients values, substituting these unknown coefficients values in Eq. (3.2), we get solution of Eq. (1.1).

5. Numerical Examples

Example 5.1. Lot and Mahdiani [13] used wavelet Galerkin method to solve boundary value problem with Dirichlet homogeneous boundary condition,

$$y''(x) - \pi^2 y(x) = -2\pi^2 sin(\pi x), \qquad 0 < x < 1, \tag{5.1}$$

subjected to the boundary conditions $y(1) = 0 = y(0)$, the exact solution is $y(x) = sin(\pi x)$. We solved Eq. (5.1) by the present method at $k = 1$ and $M = 5$. Figure 1 represents graphical interpretation of obtained numerical solution of above equation with exact solution $y(x) = sin(\pi x)$ and Table 1 represents the comparison between the absolute error (AE) of approximate solution, analytical solution and other existing methods.

Method of implementation for Example 5.1 at $k = 1$ and $M = 5$:
We approximate $y(x)$ by truncated series as,

$$y_{1,5}(x) \approx \sum_{m=0}^{4} C_{1,m}\psi_{1,m}(x) = C^T\psi(x), \tag{5.2}$$

where C and $\psi(x)$ are row vectors with appropriate dimensions,
$C^T = [C_{1,0}, C_{1,1}, C_{1,2}, C_{1,3}, C_{1,4}]$ and $\psi(x) = [\psi_{1,0}, \psi_{1,1}, \psi_{1,2}, \psi_{1,3}, \psi_{1,4}]^T$ respectively. Then we need five equations to find five unknowns, $C_{1,0}, C_{1,1}, C_{1,2}, C_{1,3}, C_{1,4}$. Since two conditions are furnished by the given boundary conditions, we see that, there should be three extra conditions to recover the unknown coefficients $C_{n,m}$. These conditions can be obtained by substituting Eq. (5.2) in Eq. (5.1) we get:

$$\frac{d^2}{dx^2}\sum_{m=0}^{4} C_{1,m}\psi_{1,m}(x) - \pi^2\sum_{m=0}^{4} C_{1,m}\psi_{1,m}(x) + 2\pi^2 sin(\pi x) = 0, \tag{5.3}$$

Now collocate Eq. (5.3) by limit points of the following sequence to get three equations other than equations obtained by given boundary conditions,

$$\{x_i\} = \{\frac{1}{2}(1 + cos\frac{(i-1)\pi}{4})\} \; where \; i = \; 2, \, 3, \, \dots,$$

we get,

$$\frac{d^2}{dx^2}\sum_{m=0}^{4} C_{1,m}\psi_{1,m}(x_i) - \pi^2\sum_{m=0}^{4} C_{1,m}\psi_{1,m}(x_i) + 2\pi^2 sin(\pi x_i) = 0, \tag{5.4}$$

from the given boundary conditions and Eq. (5.4) we obtain a system with five linear equations as follows,

$$1.1284\ C_{1,0} + 2.2568\,C_{1,1} + 2.2568\,C_{1,2} + 2.2568\,C_{1,3} + 2.2568\,C_{1,4} = 0,$$
$$1.1284\ C_{1,0} - 2.2568\,C_{1,1} + 2.2568\,C_{1,2} - 2.2568\,C_{1,3} + 2.2568\,C_{1,4} = 0,$$
$$-11.13\ C_{1,0} - 15.74\,C_{1,1} + 36.10\,C_{1,2} + 168.94\,C_{1,3} + 311.13\,C_{1,4} = -8.7645,$$

Table 1. Comparison of the absolute error (AE) of HWM (present method at $k = 1$, $M = 10$), wavelet Galerkin method by Coiflet [13] and Legendre wavelet Galerkin method (LWGM) [24] with exact solution for the Example 5.1.

x	Exact solution	AE in [13]	AE in [24]	AE by HWM
0.1	0.30913725197	1.5199×10^{-4}	3.5999×10^{-8}	5.1200×10^{-8}
0.2	0.58798983215	2.5825×10^{-4}	3.9999×10^{-8}	1.4719×10^{-8}
0.3	0.80923991060	2.8099×10^{-4}	6.0000×10^{-10}	2.6366×10^{-8}
0.4	0.95121269410	1.9751×10^{-4}	1.0999×10^{-9}	2.7786×10^{-8}
0.5	0.99999980013	4.0000×10^{-4}	5.8999×10^{-8}	2.6863×10^{-8}
0.6	0.95082179339	2.9448×10^{-4}	1.9200×10^{-8}	2.7786×10^{-8}
0.7	0.80849640381	5.4005×10^{-3}	3.7499×10^{-8}	2.6366×10^{-8}
0.8	0.58696655704	1.0297×10^{-3}	3.1899×10^{-8}	1.4719×10^{-8}
0.9	0.30793445381	1.3620×10^{-3}	5.8000×10^{-9}	5.1200×10^{-8}

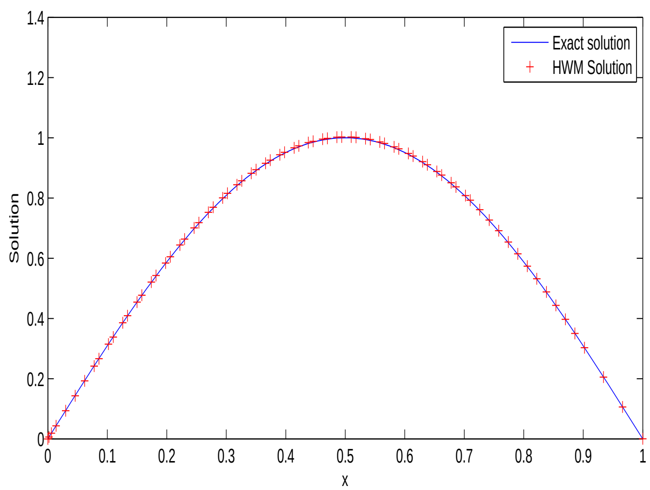
$$-11.1367\, C_{1,0} + 0\, C_{1,1} + 58.3814\, C_{1,2} + 0\, C_{1,3} - 166.7058\, C_{1,4} = -19.7392,$$

$$-11.13\, C_{1,0} + 15.74\, C_{1,1} + 36.10\, C_{1,2} - 168.94\, C_{1,3} + 311.13\, C_{1,4} = -8.7645,$$

by solving these equations, we get five unknown coefficients values as, $C_{1,0} = 0.4191$, $C_{1,1} = 0$, $C_{1,2} = -0.2222$, $C_{1,3} = 0$, $C_{1,4} = 0.0126$. substituting these unknown coefficients values in Eq. (5.2), we get approximate solution of Eq. (5.1) as,

$$y(x) = 3.6432\, x^4 - 7.2864\, x^3 + 0.5433\, x^2 + 3.0999\, x.$$

Figure 1. Physical interpretation of HWM solution with exact solution for Example 5.1 at $k = 1$ and $M = 10$.



Example 5.2. Consider the singular boundary value problem [14];

$$y''(x) + |4x-1|y' - 32 = 8|4x-1|(4x-1), \qquad 0 < x < 1,$$

subjected to the boundary conditions, $y(0) = 1$, $y(1) = 9$. the exact solution is $y(x) = (4x-1)^2$, by applying the technique described in section 4 at $M = 5$ and $k = 1$, we have a linear system of five equations. solving this system we obtain $c_0 = 2.65868077$, $c_1 = 1.77245385$, $c_2 = 8.86226925$, $c_3 = 0$, $c_4 = 0$. Thus the corresponding solution is $y(x) = C^T\psi(x) = (4x-1)^2$.

Example 5.3. Consider the following nonlinear boundary value problem [17];

$$y''(x) + (1+\frac{r}{x})y'(x) = \frac{5x^3(5x^5e^y - x - r - 4)}{4+x^5}, \tag{5.5}$$

subjected to the boundary conditions, $y(1) + 5y'(1) = \ln(\frac{1}{5}) - 5$, $y'(0) = 0$, the exact solution is $y(x) = ln(\frac{1}{4+x^5})$. We solve this equation by the present method with $k = 1$ and $M = 10$. Table 2 represents absolute error obtained by the approximate solution with analytical solution for different values of r. The numerical solution of Eq. (5.5) is presented in the Figure 2 at $k = 1$ and $M = 10$ with the exact solution and Figure 3 represents absolute error obtained by present method with exact solution for *r=0.25* and *0.75*. This shows that, as decreasing the values of r we get more accuracy.

Method of implementation for $k = 1$, $M = 10$ and $r = 0.25$:

We approximate $y(x)$ by truncated series as,

$$y_{1,9}(x) \approx \sum_{m=0}^{9} C_{1,m}\psi_{1,m}(x) = C^T\psi(x), \tag{5.6}$$

where C and $\psi(x)$ are 10×1 matrix,

$$C^T = [C_{1,0}, C_{1,1}, C_{1,2}, C_{1,3}, C_{1,4}, C_{1,5}, C_{1,6}, C_{1,7}, C_{1,8}, C_{1,9}],$$

$$\psi(x) = [\psi_{1,0}, \psi_{1,1}, \psi_{1,2}, \psi_{1,3}, \psi_{1,4}, \psi_{1,5}, \psi_{1,6}, \psi_{1,7}, \psi_{1,8}, \psi_{1,9}]^T,$$

Then we need ten equations to find ten unknowns,

$$C_{1,0}, C_{1,1}, C_{1,2}, C_{1,3}, C_{1,4}, C_{1,5}, C_{1,6}, C_{1,7}, C_{1,8}, C_{1,9}.$$

Since two conditions are furnished by the given boundary conditions, we see that there should be eight extra conditions to recover the unknown coefficients $C_{n,m}$. These conditions can be obtained by substituting Eq. (5.6) in Eq. (5.5), we get,

$$\begin{aligned} &\tfrac{d^2}{dx^2}\textstyle\sum_{m=0}^{9} C_{1,m}\psi_{1,m}(x) + (1+\tfrac{r}{x})\tfrac{d}{dx}\sum_{m=0}^{9} C_{1,m}\psi_{1,m}(x) \\ = &\ \tfrac{5x^3(5x^5 e^{(\sum_{m=0}^{9} C_{1,m}\psi_{1,m}(x))} - x - r - 4)}{4+x^5}. \end{aligned} \tag{5.7}$$

Now collocate Eq. (5.7) by limit points of the following sequence to get eight equations other than equations obtained by given boundary conditions,

$$\{x_i\} = \{\frac{1}{2}(1 + cos\frac{(i-1)\pi}{9})\} \ where \ i = 2, 3, \ldots,$$

we get

$$\begin{aligned}&\frac{d^2}{dx^2}\sum_{m=0}^{9}C_{1,m}\psi_{1,m}(x_i)+(1+\frac{r}{x_i})\frac{d}{dx}\sum_{m=0}^{9}C_{1,m}\psi_{1,m}(x_i)\\&=\frac{5x_i^3(5x_i^5e^{(\sum_{m=0}^{9}C_{1,m}\psi_{1,m}(x_i))}-x_i-r-4)}{4+x_i^5},\end{aligned}\tag{5.8}$$

from the given boundary conditions and Eq. (5.8), we obtain nonlinear system having ten equations as follows,

$0C_{1,0}+4.5135C_{1,1}-18.0541C_{1,2}+40.6217C_{1,3}-72.2163C_{1,4}+112.8379C_{1,5}$
$-162.4866C_{1,6}+221.1623C_{1,7}-288.8651C_{1,8}+365.5949C_{1,9}=0,$

$1.1284C_{1,0}+24.82C_{1,1}+92.52C_{1,2}+205.36C_{1,3}+363.33C_{1,4}+566.44C_{1,5}+814.68C_{1,6}+$
$1.11\times10^3C_{1,7}+1.4466\times10^3C_{1,8}+1.8302\times10^3C_{1,9}=-6.6094,$

$0\,C_{1,0}+5.6770\,C_{1,1}+57.4466\,C_{1,2}+246.7071\,C_{1,3}+686.1751\,C_{1,4}+1.4607\times10^3C_{1,5}+$
$2.5770\times10^3C_{1,6}+3.9253\times10^3C_{1,7}+5.2665\times10^3C_{1,8}+6.2507\times10^3C_{1,9}=4.0281\mathrm{e}^{a_1}+$
$4.9009,$

$a_1=(1.1284C_{1,0}+2.1207C_{1,1}+1.7288C_{1,2}+1.1284C_{1,3}+0.3919C_{1,4}-0.3919C_{1,5}-$
$1.1284C_{1,6}-1.7288C_{1,7}-2.1207C_{1,8}-2.2568C_{1,9}),$

$0C_{1,0}+5.7914C_{1,1}+53.8539C_{1,2}+189.3707C_{1,3}+376.4325C_{1,4}+453.3241C_{1,5}+$
$211.1524C_{1,6}-427.5014C_{1,7}-1.2514\times10^3C_{1,8}-1.7697\times10^3C_{1,9}=2.0368\mathrm{e}^{a_2}+3.8950,$

$a_2=1.1284C_{1,0}+1.7288C_{1,1}+0.3919C_{1,2}-1.1284C_{1,3}-2.1207C_{1,4}+453.3241C_{1,5}$
$+211.1524C_{1,6}-427.5014C_{1,7}+1.7288C_{1,8}+2.2568C_{1,9},$

$0C_{1,0}+6.0180C_{1,1}+48.1442C_{1,2}+108.3244C_{1,3}+48.1442C_{1,4}-210.6308C_{1,5}$
$-433.2976C_{1,6}-210.6308C_{1,7}+481.4418C_{1,8}+974.9196C_{1,9}=0.5907\mathrm{e}^{a_3}+2.4891,$

$a_3=(1.1284C_{1,0}+1.1284C_{1,1}-1.1284C_{1,2}+-2.2568C_{1,3}-1.1284C_{1,4}+1.1284C_{1,5}$
$+2.2568C_{1,6}+1.1284C_{1,7}-1.1284C_{1,8}-2.2568C_{1,9}),$

$0C_{1,0}+6.4364\,C_{1,1}+40.5788\,C_{1,2}+20.6405\,C_{1,3}-135.1057\,C_{1,4}-151.9728\,C_{1,5}+$
$210.0267\,C_{1,6}+408.9957\,C_{1,7}-167.8620\,C_{1,8}-753.9233\,C_{1,9}=0.0864\mathrm{e}^{a_4}+1.2009,$

$a_4=1.1284C_{1,0}+0.3919C_{1,1}-2.1207C_{1,2}-1.1284C_{1,3}+1.7288C_{1,4}+1.7288C_{1,5}$
$-1.1284C_{1,6}-2.1207C_{1,7}+0.3919C_{1,8}+2.2568C_{1,9},$

$0C_{1,0}+7.2445\,C_{1,1}+31.0762\,C_{1,2}-56.7328\,C_{1,3}-99.3874\,C_{1,4}+196.6206\,C_{1,5}$
$+137.8421\,C_{1,6}-442.2549\,C_{1,7}-58.4150\,C_{1,8}+753.9233\,C_{1,9}=0.0053\mathrm{e}^{a_5}+0.4099,$

$a_5 = (1.1284C_{1,0} - 0.3919C_{1,1} - 2.1207C_{1,2} + 1.1284C_{1,3} - 99.3874C_{1,4} + 196.6206C_{1,5} + 137.8421C_{1,6} - 442.2549C_{1,7} - 58.4150C_{1,8} - 2.2568C_{1,9}$,

$0\,C_{1,0} + 9.0270\,C_{1,1} + 18.0541\,C_{1,2} - 108.3244\,C_{1,3} + 108.3244\,C_{1,4} + 135.4055\,C_{1,5} - 433.2976\,C_{1,6} + 315.9462\,C_{1,7} + 361.0813\,C_{1,8} - 974.9196\,C_{1,9} = 9.5344 \times 10^{-5} \mathrm{e}^{a_6} + 0.0879$,

$a_6 = 1.1284C_{1,0} - 1.1284C_{1,1} - 1.1284C_{1,2} + 2.2568C_{1,3} - 1.1284C_{1,4} - 1.1284C_{1,5} + 2.2568C_{1,6} - 1.1284C_{1,7} - 1.1284C_{1,8} + 2.2568C_{1,9}$,

$0C_{1,0} + 14.1596C_{1,1} - 7.2794C_{1,2} - 108.7310C_{1,3} + 333.9697C_{1,4} - 506.4026C_{1,5} + 372.4317C_{1,6} + 213.5346C_{1,7} - 1.0917 \times 10^3 C_{1,8} + 1.7697 \times 10^3 C_{1,9} = 2.1913 \times 10^{-7} \mathrm{e}^{a_7} + 0.0087$,

$a_7 = (1.1284C_{1,0} - 1.7288C_{1,1} + 0.3919C_{1,2} + 2.2568C_{1,3} - 2.1207C_{1,4} + 2.1207C_{1,5} - 1.1284C_{1,6} - 0.3919C_{1,7} + 1.7288C_{1,8} - 2.2568C_{1,9})$,

$0C_{1,0} + 41.9344C_{1,1} - 121.5138C_{1,2} + 114.9620C_{1,3} + 137.8086C_{1,4} - 775.2706C_{1,5} + 1.8536 \times 10^3 C_{1,6} - 3.2989 \times 10^3 C_{1,7} + 4.8856 \times 10^3 C_{1,8} - 6.2507 \times 10^3 C_{1,9} = 4.2717 \times 10^{-12} \mathrm{e}^{a_8} + 1.4669 \times 10^{-4}$,

$a_8 = 1.1284C_{1,0} - 2.1207C_{1,1} + 0.3919C_{1,2} + 1.7288C_{1,3} + 0.3919C_{1,4} + 0.3919C_{1,5} - 1.1284C_{1,6} + 1.7288C_{1,7} - 2.1207C_{1,8} + 2.2568C_{1,9}$,
by solving these equations, we get ten unknown coefficient values as follows,

$$C = \begin{bmatrix} C_{1,0} = -1.27882513320300916204 4 \\ C_{1,1} = -0.04153758705937662492 1 \\ C_{1,2} = -0.02307621979038960458 6 \\ C_{1,3} = -0.00800229697612622262 0 \\ C_{1,4} = -0.00133385222318859134 2 \\ C_{1,5} = 0.00008972373448222186 6 \\ C_{1,6} = 0.00008098845883170097 3 \\ C_{1,7} = 0.00001220846875197187 7 \\ C_{1,8} = -0.00000072753033343762 3 \\ C_{1,9} = -0.00000074741533782954 1 \end{bmatrix},$$

substituting these unknown coefficients values in the Eq. (5.6), we get approximate solution of Eq. (5.5) as,
$y(x) = -0.2211\,x^9 + 0.9411\,x^8 - 1.4245\,x^7 + 1.1208\,x^6 - 0.7486\,x^5 + 0.1243\,x^4 - 0.0158\,x^3 + 0.0008\,x^2 - 3.4976 \times 10^{-42}\,x - 1.3863$.

Example 5.4. Consider the following nonlinear Lane-Emden equation [30];

$$y''(x) + (\frac{6}{x})y'(x) + 14y(x) + 4y(x)ln(y(x)) = 0, \tag{5.9}$$

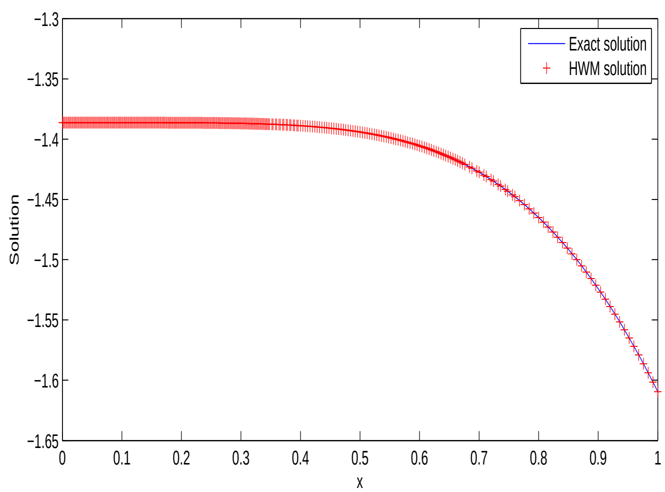
subjected to the boundary conditions

$$y(0) = 1,\ y(1) = \frac{1}{e},$$

TABLE 2. AE of the HWM solution ($k = 1$, $M = 10$) with exact solution for the Example 5.3.

x	Exact solution	AE in HWM at r=.25	AE in HWM at r=.75.
0.1	-1.386296861116766	3.9995×10^{-5}	3.9936×10^{-5}
0.2	-1.386374357920061	4.0302×10^{-5}	4.0452×10^{-5}
0.3	-1.386901676666466	3.8897×10^{-5}	3.9245×10^{-5}
0.4	-1.388851089901581	3.9408×10^{-5}	4.0067×10^{-5}
0.5	-1.394076501561946	4.0281×10^{-5}	4.1077×10^{-5}
0.6	-1.405547818041777	3.8905×10^{-5}	3.9730×10^{-5}
0.7	-1.427453098935757	3.8339×10^{-5}	3.9319×10^{-5}
0.8	-1.465031601657275	3.9700×10^{-5}	4.0857×10^{-5}
0.9	-1.523986772187307	3.8995×10^{-5}	4.0162×10^{-5}

FIGURE 2. Graphical representation of HWM solution and exact solution for Example 5.3 at $k = 1$, $M = 10$ and $r = 0.25$.



the exact solution is $y(x) = e^{-x^2}$. We solve this equation by the present method with $k = 1$ and $M = 10$. The numerical solutions of Eq. (5.9) are presented and compared with the exact solution and other existing method solutions in Table 3. Figure 4 represents the physical interpretation of approximate solution with analytical solution of Eq. (5.9).

Example 5.5. Consider the scalar problem discussed in [1],

$$y''(x) + (\frac{2}{x})y'(x) = -n^2 \cos(nx) - \frac{2}{x} n \sin(nx), \tag{5.10}$$

FIGURE 3. Graphical representation of absolute error obtained by HWM solution with exact solution for Example 5.3 at $k = 1$, $M = 10$ for different values of r.

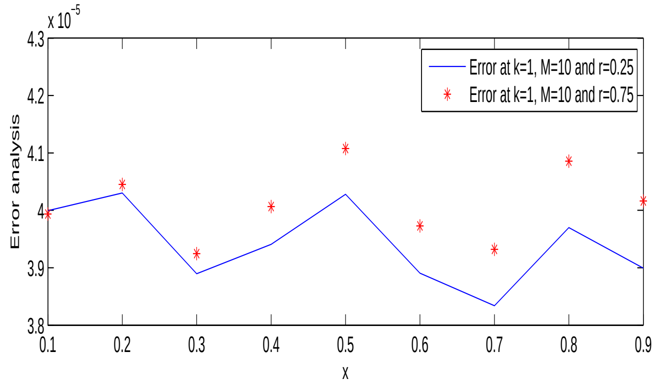


TABLE 3. AE of HWM (at $k = 1$, $M = 10$) and Laguerre wavelet method (LWM) [30] for different values of k and M with exact solution for the Example 5.4.

x	**Exact solution**	**AE in [30] at k=3, M=5**	**AE in [30] at k=2, M=6**	**AE in HWM**
0.1	0.99004983	4.1584×10^{-7}	3.9235×10^{-8}	5.7696×10^{-9}
0.2	0.96078943	4.1079×10^{-7}	3.5719×10^{-8}	1.7572×10^{-8}
0.3	0.91393118	1.3871×10^{-7}	3.4602×10^{-8}	4.4588×10^{-8}
0.4	0.85214378	2.0338×10^{-8}	2.8729×10^{-8}	9.2022×10^{-10}
0.5	0.77880078	5.2489×10^{-8}	2.8886×10^{-8}	1.1813×10^{-8}
0.6	0.69767632	4.2192×10^{-8}	2.7567×10^{-8}	3.9258×10^{-8}
0.7	0.61262639	3.3676×10^{-8}	1.5804×10^{-8}	2.9427×10^{-8}
0.8	0.52729242	2.0661×10^{-8}	9.9659×10^{-9}	2.4367×10^{-8}
0.9	0.44485806	7.6164×10^{-9}	1.0989×10^{-8}	2.0828×10^{-8}

subjected to the initial conditions,

$$y(0) = 2, \;\; y'(0) = 0,$$

the exact solution is $y(x) = 1 + \cos(nx)$. Singularity point of this problem is $x = 0$. The Hermite wavelet method (HWM) is employed to solve it with different values of M by fixing $k = 1$ and $n = 3$. It can be seen from the Table 4, increasing M gives rise to better approximate solutions in the vicinity of a singular point. The multiresolution analysis of wavelets makes us able to increase the degree of Hermite polynomials M to improve the accuracy of solutions. Figure 5 represents physical interpretation of HWM solution with Exact solution of Eq. (5.10).

FIGURE 4. Physical interpretation of HWM solution with exact solution for Example 5.4 at $k = 1$ and $M = 10$.

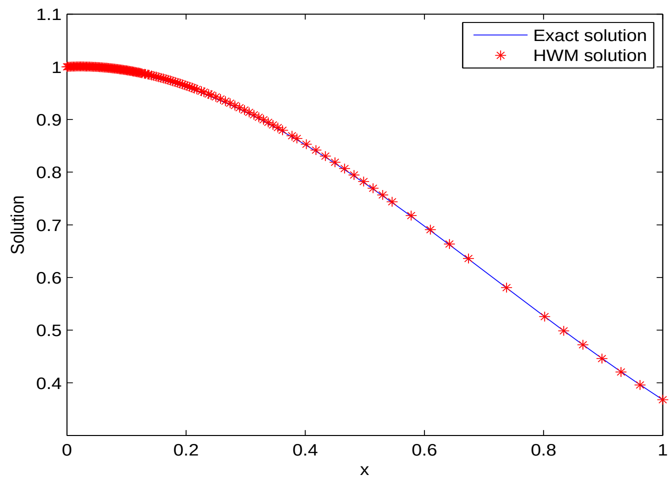


TABLE 4. AE of the HWM solution with exact solution for the Example 5.5.

x	Exact solution	AE in HWM at k=1,M=5	AE in HWM at k=1,M=6
0.1	1.955336489125606	2.8829×10^{-3}	1.9364×10^{-4}
0.2	1.825335614909678	5.3545×10^{-3}	1.3101×10^{-4}
0.3	1.621609968270664	2.5862×10^{-3}	2.3644×10^{-4}
0.4	1.362357754476673	5.2520×10^{-3}	5.5722×10^{-4}
0.5	1.070737201667703	1.4618×10^{-2}	6.3018×10^{-4}
0.6	0.772797905306913	2.0665×10^{-2}	5.1178×10^{-4}
0.7	0.495153895400143	1.9569×10^{-2}	3.6521×10^{-4}
0.8	0.262606284458754	1.0909×10^{-2}	2.5895×10^{-4}
0.9	0.095927857982939	1.2406×10^{-4}	1.2737×10^{-4}

Example 5.6. Consider the following nonlinear Lane-Emden equation [30];

$$y''(x) + \frac{2}{x}y'(x) + 4(2e^{y(x)} + e^{\frac{y(x)}{2}}) = 0, \qquad 0 < x < 1, \tag{5.11}$$

subjected to the initial conditions

$$y(0) = 0, \; y'(0) = 0,$$

the exact solution is $y(x) = -2ln(1 + x^2)$. We solve this equation by the present method with $k = 1$ and $M = 10$. Table 5 represents the comparison between the absolute error of approximate solution, analytical solution and other existing methods.

FIGURE 5. Graphical representation of HWM solution and exact solution for Example 5.5 at $k = 1$ and $M = 10$.

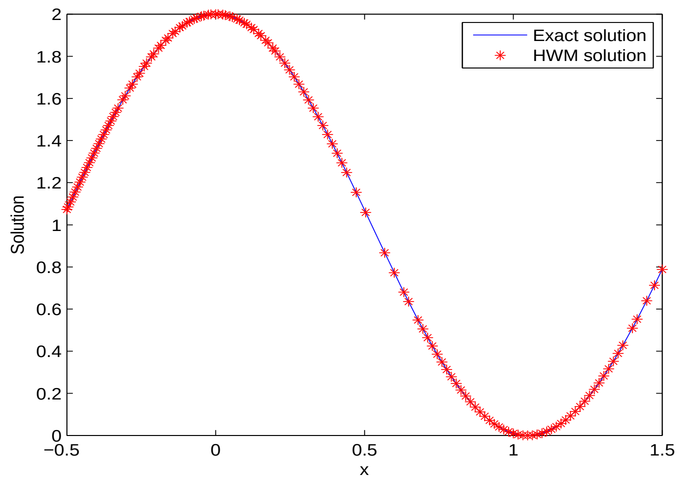


TABLE 5. AE of HWM, BPOM and Laguerre wavelet method (LWM) for different values of k and M with exact solution for the Example 5.6.

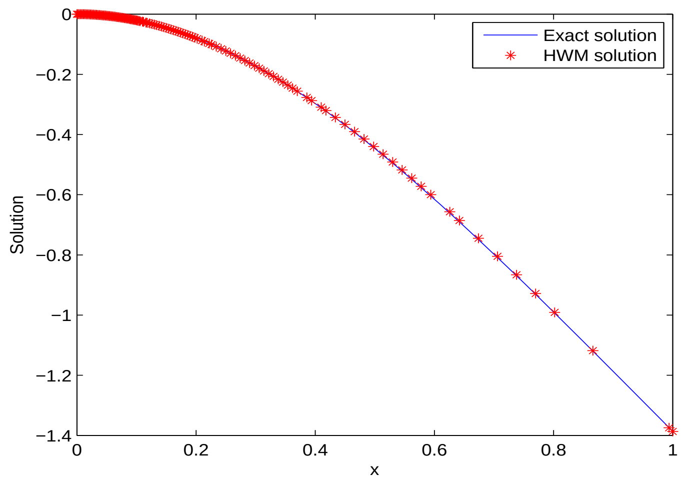
x	Exact solution	AE by BPOM at k=1, M=5 [4].	AE by LWM at k=2, M=7 [30].	AE by LWM at k=3,M=7 [30].	AE by HWM at k=1, M=10.
0.1	-0.01990066	2.0×10^{-5}	9.0×10^{-9}	4.5×10^{-12}	3.6×10^{-11}
0.2	-0.07844142	2.4×10^{-5}	1.6×10^{-9}	2.4×10^{-11}	3.4×10^{-11}
0.3	-0.17235532	2.4×10^{-5}	6.3×10^{-9}	5.3×10^{-10}	3.6×10^{-11}
0.4	-0.29684001	1.8×10^{-5}	9.3×10^{-9}	1.1×10^{-9}	3.4×10^{-11}
0.5	-0.44628710	2.4×10^{-5}	1.7×10^{-10}	1.4×10^{-9}	3.2×10^{-11}
0.6	-0.61496939	2.9×10^{-5}	1.7×10^{-7}	1.7×10^{-9}	3.3×10^{-11}
0.7	-0.79755223	2.5×10^{-4}	2.9×10^{-7}	1.8×10^{-9}	3.2×10^{-11}
0.8	-0.98939248	1.4×10^{-4}	3.6×10^{-7}	1.6×10^{-9}	2.9×10^{-11}
0.9	-1.18665369	8.6×10^{-3}	4.1×10^{-7}	1.3×10^{-9}	2.8×10^{-11}

Figure 6 represents physical interpretation of approximate and exact solution for the Eq. (5.11).

Example 5.7. Consider the initial value problem [12],

$$y''(x) - 2e^{y} = 0, \qquad 0 < x < 1, \tag{5.12}$$

FIGURE 6. Graphical representation of HWM solution and exact solution for Example 5.6 at $k = 1$ and $M = 10$.



subjected to the initial conditions

$$y(0) = 0, \ \ y'(0) = 0,$$

the exact solution is $y(x) = -2\ln(\cos x)$. Here, solving Eq. (5.12) using Hermite wavelets collocation method and comparison between the Numerical solution, exact solution and other existed method solutions can be observed in Table 6. Figure 7 represents the comparison between the approximate solution and analytical solution graphically.

Example 5.8. Consider the oxygen diffusion problem [30],

$$y''(x) + \frac{2}{x}y'(x) = \frac{0.76129y(x)}{y(x) + 0.03119}, \qquad 0 < x < 1, \tag{5.13}$$

subjected to the boundary conditions

$$y'(0) = 0, \ \ 5y(1) + y'(1) = 5,$$

where exact solution is unknown. Now we solve this equation by present method with $k = 1$ and different values of M. These solutions are in good agreement with the method in [11] and this results are tabulated in Table 7. As increasing M gives rise to better approximate solution, it can be seen from Table 7. Figure 8 shows absolute error between solution obtained by present method and method in [30].

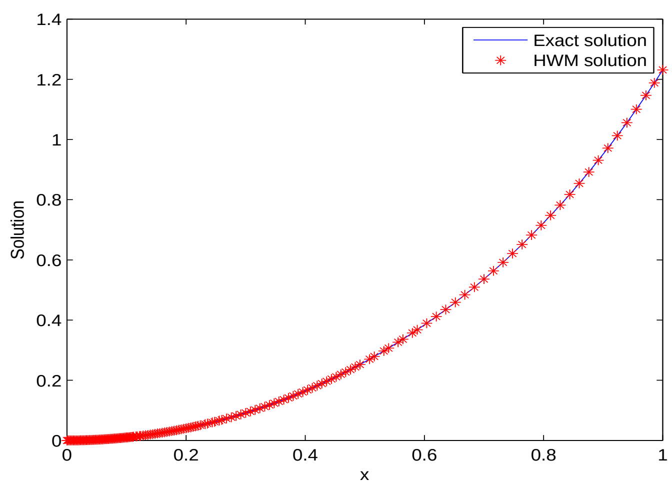
Example 5.9. Consider the initial value problem [7],

$$y'(x) + xe^{y} = 0, \qquad 0 < x < 1, \tag{5.14}$$

TABLE 6. AE of HWM, PIA and Legendre wavelet method (LWM) with exact solution for the Example 5.7.

x	Exact solution	AE in PIA(1,3) algorithm [12]	AE in Legendre wavelet [1]	AE in HWM at k=1,M=10
0.1	0.010016711246471	6.71×10^{-6}	9.00×10^{-8}	9.88×10^{-7}
0.2	0.040269546104817	9.55×10^{-6}	1.50×10^{-7}	1.41×10^{-6}
0.3	0.091383311852116	3.11×10^{-6}	6.14×10^{-7}	3.15×10^{-6}
0.4	0.164458038150111	8.04×10^{-6}	8.88×10^{-6}	3.70×10^{-6}
0.5	0.261168480887445	8.48×10^{-6}	5.67×10^{-5}	3.96×10^{-6}
0.6	0.383930338838875	2.03×10^{-5}	2.55×10^{-4}	6.10×10^{-6}
0.7	0.536171515135862	7.15×10^{-5}	9.24×10^{-4}	7.84×10^{-6}
0.8	0.722781493622688	2.91×10^{-4}	2.90×10^{-3}	8.02×10^{-6}
0.9	0.950884887171629	1.05×10^{-3}	7.90×10^{-3}	1.10×10^{-5}

FIGURE 7. Graphical representation of HWM solution and exact solution for Example 5.7 at $k = 1$ and $M = 10$.



subjected to the initial condition

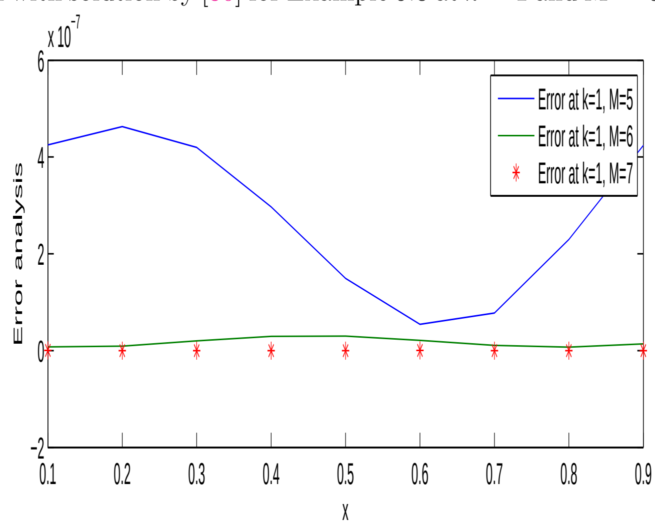
$$y(0) = 0,$$

the exact solution is $y(x) = -\ln(1 + \frac{x^2}{2})$. Here, we solved Eq. (5.14) using Hermite wavelets collocation method and Adomian decomposition method (ADM). Comparison between the Numerical, exact and Adomian decomposition method solutions can

Table 7. Numerical comparison between the HWM solution with the method in [11] for the Example 5.8.

X	Method in [30]	Present method at k=1, M=5	Present method at k=1, M=6	Present method at k=1, M=7
0.1	0.8297060924	0.8297056673	0.8297060849	0.8297060920
0.2	0.8333747335	0.8333742707	0.8333747242	0.8333747334
0.3	0.8394899139	0.8394894941	0.8394898940	0.8394899138
0.4	0.8480527849	0.8480524878	0.8480527556	0.8480527850
0.5	0.8590649271	0.8590647777	0.8590648972	0.8590649272
0.6	0.8725283199	0.8725282654	0.8725282986	0.8725283197
0.7	0.8884453056	0.8884452280	0.8884452950	0.8884453055
0.8	0.9068185480	0.9068183182	0.9068185410	0.9068185481
0.9	0.9276509883	0.9276505645	0.9276509744	0.9276509882

Figure 8. Graphical representation of absolute error by HWM solution with solution by [30] for Example 5.8 at $k = 1$ and $M = 5, 6, 7$.



be observed in Table 8. Figure 9 represents the comparison between the approximate solutions of HWM, ADM and analytical solution graphically. On solving above equation by ADM we obtain:

$$y_1(x) = \frac{-x^2}{2},$$
$$y_2(x) = \frac{x^4}{8},$$
$$y_3(x) = -\frac{x^6}{24},$$
$$\cdots$$

Table 8. AE of HWM and ADM with exact solution for the Example 5.9.

x	Exact solution	AE by ADM Y_4	AE by ADM Y_5	AE in HWM at k=1, M=10
0.1	-0.004987541511039	6.22×10^{-13}	2.69×10^{-15}	9.56×10^{-11}
0.2	-0.019802627296180	6.29×10^{-10}	1.04×10^{-11}	1.31×10^{-5}
0.3	-0.044016885416774	3.55×10^{-8}	1.33×10^{-9}	2.17×10^{-9}
0.4	-0.076961041136128	6.14×10^{-7}	4.08×10^{-8}	3.20×10^{-9}
0.5	-0.117783035656383	5.52×10^{-6}	5.74×10^{-7}	1.34×10^{-8}
0.6	-0.165514438477573	3.28×10^{-5}	4.91×10^{-6}	1.17×10^{-9}
0.7	-0.219135529916671	1.46×10^{-4}	2.98×10^{-5}	1.04×10^{-9}
0.8	-0.277631736598280	5.30×10^{-4}	1.40×10^{-4}	3.75×10^{-8}
0.9	-0.340037302785709	1.63×10^{-3}	5.46×10^{-4}	5.47×10^{-8}

$$y_n(x) = (-1)^n \frac{(x^2)^n}{n2^n}.$$

Then solution by ADM is,

$$y(x) = Y_3(x) = \frac{-x^2}{2} + \frac{x^4}{8} - \frac{x^6}{24},$$
$$y(x) = Y_5(x) = \frac{-x^2}{2} + \frac{x^4}{8} - \frac{x^6}{24} + \frac{x^8}{64} - \frac{x^{10}}{160}.$$

Figure 9. Graphical comparison between the HWM (at $k = 1$ and $M = 10$), ADM Y_5 with exact solution for Example 5.9.

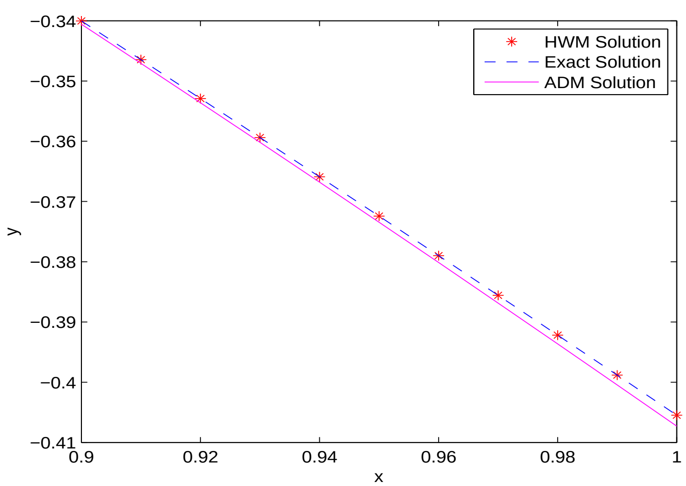


FIGURE 10. Graphical representation of AE by HWM (at $k = 1$ and $M = 10$), ADM Y_3, Y_5 with exact solution for Example 5.9.

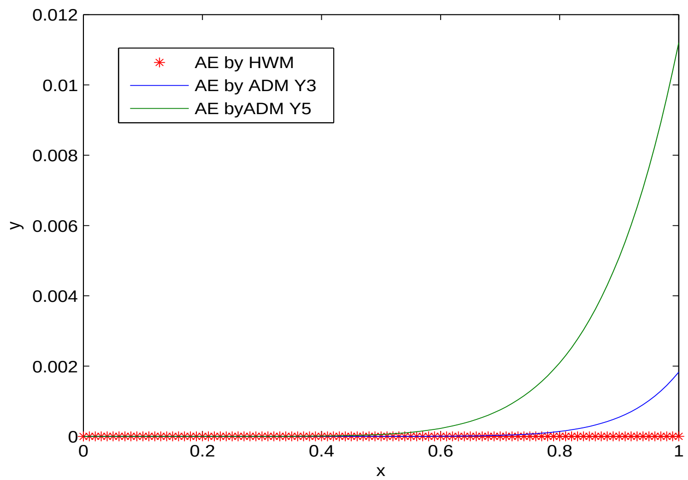


TABLE 9. Comparison between the HWM and HM with exact solution for the Example 5.10.

x	Exact solution	HWM solution	HM solution [3]
0.125	0.0081	0.0082	0.0088
0.375	0.0791	0.0792	0.0810
0.625	0.2361	0.2362	0.2399
0.875	0.4952	0.4953	0.4992

Example 5.10. Consider the initial value problem [3],

$$y'(x) + y(x) = e^x, \qquad 0 < x < 1, \tag{5.15}$$

subjected to the initial condition

$$y(0) = 0, \;\; y'(0) = 0,$$

the exact solution is $y(x) = \frac{1}{2}(e^x - sinx - cosx)$. Here, we solved Eq. (5.15) using Hermite wavelets collocation method and Haar wavelet method (HM). Comparison between the numerical, exact and adomian decomposition method solutions can be observed in Table 9.

6. Conclusion

In this paper, we introduced Hermite wavelet method (HWM) for solving linear and nonlinear singular initial and boundary value problems for different physical conditions. The proposed scheme is tested on some illustrative examples and the results are presented in Tables and Figures in comparison with the exact solutions. HWM solutions are quite satisfactory in comparison with the existing numerical solutions available in the literature [1, 3, 4, 12, 13, 24, 30]. This scheme is easy to implement with computer programs and it can be extend for higher order with slight modification. Proposed Theorem 3.1 reveals that the present method will contribute exact solution for differential equations, whose solutions are in the form of polynomials of finite degree. This is important for the development of new research in the field of numerical analysis and beneficial for new researchers. Also, Theorems 3.2 and 3.3 on uniform convergence and error analysis.

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