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Factorization method for fractional Schrödinger equation in D-dimensional fractional space and homogeneous manifold $SL(2,c)/GL(1,c)$

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Abstract In this paper, we consider a D-dimensional fractional Schrödinger equation with a Coulomb potential. By using the associated Laguerre and Jacobi equations, we obtain the wave function and energy spectrum and this then enable us to separate this equation in terms of the radial and angular momentum parts respectively. Also, the associated Laguerre and Jacobi equations makes it possible to further factorize the D-dimensional fractional Schrödinger equation such that the resulting equations can be expressed in terms of the first order operators which are basically raising and lowering operators.

Keywords. Factorization method, Fractional Schrödinger equation, Laguerre equation, Jacobi equation.

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1. Introduction

The factorization method is an operational procedure to answer questions about eigenvalue problems. The idea of this method is to consider a pair of first-order differential equations which are equivalent to a given second-order differential equation with boundary conditions. The eigenvalues and a manufacturing process for the normalized eigenfunctions can be find by this method. The method also can handle perturbation problems [4]. This idea has been used for solving different type of problems [4, 6, 11, 15].

We are interested in using the factorization method to obtain the solution of the fractional Schrödinger equation in D-dimensional fractional space so that we can obtain the raising and lowering operators that are supersymmetric structures. The putative fractional dimension can be considered as an effective dimension of compactified higher dimensions or as a manifestation of a nontrivial microscopic lattice structure of space [13]. Consequently the study of the Schrödinger equation in a non integer dimensional space is still an open problem. We recall that the experimental measurement of dimension D of the real world is $D = (3 \pm 10^{-6})$ [14]. Thus, the fractional value of D is in agreement with the experimental physical observations. We highlight the application of the formalism in general relativity for example where the gravitational fields are understood to be geometric perturbations in 4D-dimensional space-time [9]. Another interesting example was provided by Zeilinger and Svozil [16] regarding the current discrepancy between the theoretical and experimental values of the anomalous magnetic moment of the electron cloud. Other applications of the suggested formalism from [14] are related to problems such as excitons [3], magnetoexcitons [10].

Different authors have solved the Schrödinger equation in different settings [1, 7, 8, 14]. For example, Stillinger [14] defined a generalization of the Laplace operator in a polar variable and then obtained fractional time-independent Schrödinger equation with a potential that depended only on a radial distance r. He then separated this equation into the radial and polar parts and then used the Gegenbauer polynomials to solve the polar part. Furthermore, Eid et al. [1] solved the Schrödinger equation in D-dimensional fractional space with a Coulomb potential for different values of b and then obtained an exact solution. In this paper, we solve the equation by using the factorization method and then show that we can use the method for $b = 3$ in D-dimensional fractional space where $a > 0$. Thus, we use the factorization method from [11] to find the solution of the Schrödinger equation with the non-central modified Kratzer potential. In this method, the second order differential equations transforms into product of two functions by using separation of variables. The methodology will discuss in detail.

The structure of our paper is as follows: In section 2, we introduce the corresponding fractional Schrödinger equation and then separate it to terms of radial and angular momentum part. In section 3, we do calculation for the angular part of the equation and we compare the resulting equation with the associated Jacobi equation to helps us obtain the angular part of the wave function. Finally, we make use of the Jacobi equation to get the first order operators. Then, we introduce the operators of the

$gl(2,c)$ Lie algebra on the group manifold $SL(2,c)$, these operators are of the form of the raising and lowering operators for the angular part of equation. In section 4, we introduce the radial part of equation and compare it with the associated Laguerre equation and obtain the radial part wave function. This comparing helps us to calculated the energy spectrum and also to arrange α and β with the corresponding b variable. In that case we obtain the first order operators which are the raising and lowering operators. In section 5, we present the conclusion and also highlight some open problem for future work.

2. Fractional Schrödinger equation

We start with the Schrödinger equation given by [14]

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{1}{r^{D-1}}\frac{\partial}{\partial r}(r^{D-1}\frac{\partial}{\partial r}) + \frac{l^2}{r^2}\right) - e^2\frac{K_b}{r^{b-2}}\right]\psi(r,\theta) = (E-E_g)\psi(r,\theta), \quad (2.1)$$

where l^2 corresponds to the angular momentum operator given by

$$l^2\psi(r,\theta) = \left[\frac{-\hbar^2}{\sin^{D-2}(\theta)}\frac{\partial}{\partial\theta}(\sin^{D-1}(\theta)\frac{\partial}{\partial\theta})\right]\psi(r,\theta) = l(l+D-2)\psi(r,\theta), \quad (2.2)$$

such that $D > 0$ is a non integer number and the dimension of a solid. The constant K_b is generally defined as

$$K_b = \frac{\Gamma(\frac{b}{2})}{2\Pi^{\frac{b}{2}}(b-2)\epsilon_0}, \qquad b > 2.$$

We substitute $\Psi(r,\theta) = R(r)\Theta(\theta)$ into the Schrödinger equation (2.1). It leads to the following equation with an independent variables:

$$rR'' + (D-1)R' + \left[\frac{2\mu r}{\hbar^2}\left((E-E_g) + e^2\frac{K_b}{r^{b-2}}\right) - \frac{l(l+D-2)}{r}\right]R = 0, \quad (2.3)$$

and similarly from Eq. (2.2) we obtain

$$\Theta''(\theta) + (D-2)\cot\theta\Theta'(\theta) + l(l+D-2)\Theta(\theta) = 0. \quad (2.4)$$

In the next sections, our goal is to solve Eqs. (2.3) and (2.4) within the framework of factorization method.

3. Angular part

Now, we solve the polar angle part of the Eq. (2.4). With the choice of the variable $x = -\cos\theta$ in Eq. (2.4), we obtain

$$(1-x^2)\Theta'' + (1-D)x\Theta' - l(l+D-2)\Theta = 0. \quad (3.1)$$

We substitute $\Theta(x) = U(x)P(x)$ into Eq. (3.1) and we get

$$(1-x^2)P''(x) + \left[2(1-x^2)\frac{U'}{U} + x(1-D)\right]P'(x) + \quad (3.2)$$

$$\left[(1-x^2)\frac{U''}{U} + x(1-D)\frac{U'}{U} - l(l+D-2)\right]P(x) = 0.$$

Now, we compare Eq. (3.2) with the associated Jacobi differential equation given as follows [2, 12]

$$(1-x^2)P''^{(\alpha,L)}_{n,m}(x) - [2L+(2\alpha-2L+2)x]\,P'^{(\alpha,L)}_{n,m}(x) + \left[n(2\alpha-2L+n+1) - \frac{m(2\alpha-2L+m+2Lx)}{1-x^2}\right] P^{(\alpha,L)}_{n,m}(x) = 0, \tag{3.3}$$

and we obtain the parameters,

$$L = 0, \qquad \alpha_\theta = \pm\frac{D-3}{2}, \tag{3.4}$$

also, the angular momentum operator is

$$l(l+D-2) = -n(n+1-2\alpha_\theta) - \frac{2\alpha_\theta+3-D}{2}. \tag{3.5}$$

By using the shape invariance in Eq. (3.3), we obtain the operators $J+(m)$ and $J-(m)$ on the homogeneous manifold $SL(2,c)/GL(1,c)$ as follows

$$J+(m) = \frac{\partial}{\partial\theta} - \frac{\imath}{\sin\theta}\frac{\partial}{\partial\phi} - \frac{2(m+\alpha_\theta)-1}{2\tan\theta}, \qquad J-(m) = -\frac{\partial}{\partial\theta} - \frac{\imath}{\sin\theta}\frac{\partial}{\partial\phi} - \frac{2(m+\alpha_\theta)-1}{2\tan\theta}, \tag{3.6}$$

such that

$$J+(m)J-(m)\psi_{n,0,m}(\theta,\phi,0) = E_{n,m}\psi_{n,0,m}(\theta,\phi,0), \qquad J-(m)J+(m)\psi_{n,0,m-1}(\theta,\phi,0) = E_{n,m}\psi_{n,0,m-1}(\theta,\phi,0), \tag{3.7}$$

where

$$\psi_{n,0,m}(\theta,\phi,0) = a_n(-1)^m\left[(1-x^2)^{-\frac{2m+\alpha_\theta-1}{4}}\left(\frac{d}{dx}\right)^{n-m}(1-x^2)^{n+\alpha_\theta}\right]_{x=-\cos\theta},$$

and the spectrum $E_{n,m}$ have been introduced as the following equation[2]

$$E_{n,m} = (n-m+1)(n+m+2\alpha_\theta).$$

By comparing Eq. (3.2) with the associated Jacobi differential equation in Eq. (3.3), we obtain

$$U(x) = N_0(1-x^2)^{\frac{2\alpha_\theta-D+3}{4}},$$

so, the solution of Eq. (2.4) is

$$\Theta(\theta) = N_0(\sin(\theta))^{\frac{2\alpha_\theta-D+3}{2}}\psi_{n,0,m}(\theta,\phi,0).$$

4. Radial Part

By the same procedure, we substitute $R(r) = U(r)L(r)$ in Eq. (2.3) and we obtain

$$rL'' + \left[2r\frac{U'}{U} + D - 1\right] L' \quad + \Big[\frac{rU''}{U} + (D-1)\frac{U'}{U} + \frac{2\mu r}{\hbar^2}\left((E - E_g) + e^2\frac{K_b}{r^{b-2}}\right) - \frac{l(l+D-2)}{r}\Big] L = 0. \tag{4.1}$$

In order to obtain the function $U(r)$, we compare Eq. (4.1) with the following associated Laguerre differential equation

$$rL''^{(\alpha,\beta)}_{n,m} + (1 + \alpha - \beta r)L'^{(\alpha,\beta)}_{n,m} + \left[(n - \frac{m}{2})\beta - \frac{m}{2}(\alpha + \frac{m}{2})\frac{1}{r}\right] L^{(\alpha,\beta)}_{n,m} = 0,$$

and we obtain

$$U(r) = N_0 e^{-\frac{\beta r}{2}} r^{1+\frac{\alpha-D}{2}}. \tag{4.2}$$

Therefore, from Eq. (4.2) we obtain the corresponding eigenfunction as

$$R(r) = N_0 e^{-\frac{\beta r}{2}} r^{1+\frac{\alpha-D}{2}} L^{(\alpha,\beta)}_{n,m}(r). \tag{4.3}$$

Here, by comparing Eq. (4.1) with the associated Laguerre differential equation with different b, we obtain α, β and E corresponding. If $b = 3$, then $E = E_g - \frac{\hbar^2\beta^2}{8\mu}$ so that

$$\beta = \frac{4\mu e^2 K_3}{\hbar^2(D + 1 - 2n + m)}, \tag{4.4}$$

and α are roots of equation

$$\alpha^2 + 2m\alpha + (m^2 - (D-2)^2 - 4l(l + D - 2)) = 0, \tag{4.5}$$

so

$$E = E_g - \frac{2\mu e^4 K_3^2}{\hbar^2} b_{n,m,D} \,, \tag{4.6}$$

where

$$b_{n,m,D} = \frac{1}{(D + 1 - 2n + m)^2}. \tag{4.7}$$

Similarly as in the previous section, the raising and lowering operators Laguerre equation in the Rodrigues representation are,

$$A^+_{n,m} = r\frac{d}{dr} - \beta r + n + \alpha - \frac{m}{2}, \qquad A^-_{n,m} = -r\frac{d}{dr} + n - \frac{m}{2},$$
$$L^{(\alpha,\beta)}_{n,m} = \frac{a_{n,m}(\alpha,\beta)}{r^{\alpha+\frac{m}{2}}e^{-\beta r}}\left(\frac{d}{dx}\right)^{n-m}(r^{\alpha+n}e^{-\beta r}),$$

where $a_{n,m}(\alpha, \beta)$ is the normalization coefficient. But, if $b \neq 3$ we cannot use the factorization method for the solution of Eq. (2.3), because if $b = 2$ then we have to solve the following system

$$\begin{cases} \frac{\beta}{2}(1 + \alpha) = (n - \frac{m}{2})\beta, \\ -1 + \frac{1}{4}(\alpha^2 - D^2 + 4D) - l(l + D - 2) = -\frac{m}{2}(\alpha + \frac{m}{2}). \end{cases}$$

Therefore, we obtain α and β as

$$\alpha = -m \pm (a - 2 + 2l),$$
$$\beta = 0,$$

which is in contradiction with the definition of the weight function $W(x)$ in [5]. So, this condition leads us to have $b = 3$. Finally, we can say just for the case of $b = 3$ and the mentioned α and β, we factorized the second order equation in terms of first order equation with respect to n and m. Also, from the normalization condition in quantum mechanics, one can obtain the normalization coefficient N_0 as

$$\int_0^\infty rR(r)R^*(r)dr = 1,$$
$$\mid N_0 \mid^2 \int_0^\infty rL^2_{n,m}{}^{(\alpha,\beta)}(r)U^2(r)d(r) = 1.$$

5. Conclusion

In this paper, we solved the D-dimensional fractional Schrödinger equation with a Coulomb potential. We compered the radial and the angular parts with Laguerre and associated Jacobi equation, respectively. These Laguerre and the associated Jacobi helped us to factorized the radial and angular part of the corresponding equations. Finally, we took advantage from these polynomials and obtained the wave function and energy spectrum. In future, it may be interesting to show that the corresponding raising and lowering operators for the radial and angular part of equations will be a form of generators algebra. So, it may be interesting to study the partner potential and supersymmetry generators. In that case, we can arrange the supercharges by using the raising and lowering operators.

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