Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 7, No. 2, 2019, pp. 152-162

A numerical method for partial fractional Fredholm integro-differential equations

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Abstract In this paper, an expansion method based on orthonormal polynomials is presented to find the numerical solution of partial fractional Fredholm integro-differential equations (PFFIDEs). A PFFIDE is converted to a system of linear algebraic equations, which is solved for the expansion coefficients of approximate solution based on orthonormal polynomials. An estimation error is discussed and some illustrative examples are given to demonstrate the validity and applicability of the proposed method.

Keywords. Partial integro-differential equation, Fractional operator, Expansion method.

2010 Mathematics Subject Classification. 65R20.

1. Introduction

Fractional ordinary and partial differential and integral equations have been applied for solving many practical problems which are modeled in various fields of science and engineering, such as, physics, electrochemistry, viscoelasticity, continuum and statistical mechanics [2, 7, 10, 16]. Since there are not simple analytical methods for solving these equations, many researchers proposed approximate methods for solving these kinds of equations, such as partial differential and integro-differential equations. For example, the Adomian decomposition, variational iteration, homotopy perturbation, fractional differential transform(FDTM), wavelet, block pulse operational matrix and Tau methods [4, 5, 8, 9, 11, 12, 15, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27] are some instances of given methods.
More recently, a combination of the spectral method with Bernstein operational matrix in [13] and a combination of the Ritz method with fractional operational matrix method in [14] have been proposed to solve two-dimensional fractional optimal control problems.
But partial fractional integro-differential equations have not solved by any numerical

Received: 23 September 2018 ; Accepted: 16 January 2019.
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method. So in this paper, we propose a numerical method to solve these kinds of equations. To this end, we consider a PFFIDE of the form

$$D_t^{\alpha} u(x,t) - \lambda \int_0^1 \int_0^1 k(x,t,y,z) u(y,z) dy dz = f(x,t), \\ (x,t) \in I = [0,1] \times [0,1], \tag{1.1}$$

with initial conditions

$$\frac{\partial^k}{\partial t^k} u(x,0) = h_k(x), \quad k = 0,1,...,m-1, \quad m-1 < \alpha \leq m, \quad m \in \mathbf{N}, \tag{1.2}$$

where D_t^{α} denotes the Caputo fractional operator of order α with respect to t, $f(x,t)$ and $k(x,t,y,z)$ are given continuous functions and $u(x,t)$ is a solution to be determined. In this paper, we apply an expansion method based on some orthonormal polynomials for problem (1.1)- (1.2). Here, we replace the differential and integral parts of a PFFIDE by their operational matrix representations and then convert it to a corresponding system of linear algebraic equations. In a similar manner, we transform the initial conditions to a linear algebraic system of equations. Finally, by combining these two linear systems of algebraic equations, we obtain a system of linear algebraic equations and solve it to obtain an approximate solution of the problem.

2. Basic definition of fractional calculus

In this section, we briefly present some definitions and results in fractional calculus from [1, 6] which are used in the paper.

Definition 2.1. The Riemann-Liouville fractional integral operator J^{α} of order α is given by

$$J^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau, \quad \alpha > 0,$$
$$J^0 u(t) = u(t).$$

Some important properties of the fractional integral operator are as follows

$$1)\ J^{\alpha} J^{\beta} u(t) = J^{\alpha+\beta} u(t), \tag{2.1}$$

$$2)\ J^{\alpha} J^{\beta} u(t) = J^{\beta} J^{\alpha} u(t), \tag{2.2}$$

$$3)\ J^{\alpha} t^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)} t^{\alpha+\nu}, \tag{2.3}$$

where $\alpha, \beta > 0$ and $\nu > -1$ are real numbers.

Definition 2.2. The partial Rimann-Liouville fractional integral operator with respect to t of order $\alpha > 0$ is defined as

$$J_t^{\alpha} f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(x,s) ds. \tag{2.4}$$

Definition 2.3. The Caputo partial fractional derivative with respect to t of order $\alpha > 0$ is defined by

$$\begin{aligned} D_t^{\alpha} f(x,t) &= J_t^{m-\alpha} D_t^m f(x,t) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} \frac{\partial^m f(x,s)}{\partial s^m} ds, \quad m-1 < \alpha \leq m, \end{aligned} \tag{2.5}$$

where m is an integer.

3. Main results

In this section, we give some theorems and lemmas to convert the differential and integral parts of PFFIDE to their matrix representations. Let $\{\phi_j(t)\}$ be a sequence of orthonormal polynomials. Then a function $u(t) \in L^2[0,1]$ can be expanded as

$$u(t) = \sum_{j=0}^{\infty} c_j \phi_j(t) = C\Phi X_t,$$

where $C = (c_0, c_1, ..., c_n, ...)$ with

$$c_j = \int_0^1 u(t)\phi_j(t)dt,$$

$X_t = (1, t, t^2, ..., t^n, ...)^T$ and Φ is a lower triangular matrix such that $\Phi X_t = (\phi_0(t),$ $\phi_1(t), ..., \phi_n(t), ...)^T$.
Now we recall the following lemma from [17].

Lemma 3.1. *Let $u(t) = CX_t$ be a polynomial. Then we have*

$$D^r u(t) = \frac{d^r}{dt^r} u(t) = C\eta^r X_t, \qquad r = 0, 1, 2, ..., \tag{3.1}$$

where $C = (c_0, c_1, ..., c_n, 0, 0, ...)$, $X_t = (1, t, t^2, ..., t^n, ...)^T$ and

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 3.2. *For $X_t = (1, t, t^2, ..., t^n, ...)^T$, we have*

$$J_t^{m-\alpha}(X_t) = GA\Phi X_t, \tag{3.2}$$

where G is a diagonal matrix with diagonal elements

$$G_{i,i} = \frac{\Gamma(i+1)}{\Gamma(m-\alpha+i+1)}, \quad i = 0, 1, 2, ..., \tag{3.3}$$

and $A = (a_{ij})_{i,j=0}^{\infty}$ with

$$a_{ij} = \int_0^1 t^{m-\alpha+i} \phi_j(t)dt, \quad i, j = 0, 1, 2, ... \tag{3.4}$$

Proof. By (2.3), we have

$$\begin{aligned} J_t^{m-\alpha}(X_t) &= (J_t^{m-\alpha}(1), J_t^{m-\alpha}(t), ..., J_t^{m-\alpha}(t^n), ...)^T \\ &= \left(\frac{\Gamma(1)t^{m-\alpha}}{\Gamma(m-\alpha+1)}, \frac{\Gamma(2)t^{m-\alpha+1}}{\Gamma(m-\alpha+2)}, ..., \frac{\Gamma(n+1)t^{m-\alpha+n}}{\Gamma(m-\alpha+n+1)}, ... \right) \\ &= G\Pi, \end{aligned} \tag{3.5}$$

where G is a diagonal matrix with elements (3.3) and

$$\Pi = (t^{m-\alpha}, t^{m-\alpha+1}, ..., t^{m-\alpha+n}, ...)^T.$$

But $t^{m-\alpha+i}$ can be written as:

$$t^{m-\alpha+i} = \sum_{j=0}^{\infty} a_{ij}\phi_j(t) = \mathbf{a}_i \Phi X_t, \qquad \mathbf{a}_i = (a_{i0}, a_{i1}, ..., a_{in}, ...),$$

where a_{ij} obtained by (3.4). Therefore, we have

$$\Pi = (\mathbf{a}_0\Phi X_t, \mathbf{a}_1\Phi X_t, ..., \mathbf{a}_n\Phi X_t, ...)^T = A\Phi X_t, \tag{3.6}$$

where

$$A = (\mathbf{a}_0, \mathbf{a}_1, ..., \mathbf{a}_n, ...)^T.$$

Hence, by substituting (3.6) into (3.5), (3.2) is obtained. □

Consider the approximate solution of the Eq.(1.1) in the form

$$u_n(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} u_{ij}\ \phi_i(x)\phi_j(t) = {X_x}^T \Phi^T U \Phi X_t, \tag{3.7}$$

where ${X_x}^T = (1, x, x^2, ..., x^n, ...)$ and

$$U = \begin{pmatrix} u_{00} & u_{01} & \cdots & u_{0n} & 0 & \cdots \\ u_{10} & u_{11} & \cdots & u_{1n} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ u_{n0} & u_{n1} & \cdots & u_{nn} & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with the unknown elements $u_{ij}, i, j = 0, \cdots, n$. By using (2.5), (3.1), (3.2) and (3.7), we obtain the matrix form

$$\begin{aligned} D_t^{\alpha} u_n(x,t) &= J_t^{m-\alpha} D_t^m ({X_x}^T \Phi^T U \Phi X_t) = J_t^{m-\alpha}({X_x}^T \Phi^T U \Phi \eta^m X_t) \\ &= {X_x}^T \Phi^T U \Phi \eta^m J_t^{m-\alpha}(X_t) = {X_x}^T \Phi^T U \Phi \eta^m G A \Phi X_t, \end{aligned} \tag{3.8}$$

for differential operator D_t^{α}. On the other hand, the function $f(x,t)$ can be expanded as

$$f(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} f_{ij}\phi_i(x)\phi_j(t) = {X_x}^T \Phi^T F \Phi X_t, \tag{3.9}$$

where $F = (f_{ij})_{i,j=0}^{n}$ with

$$f_{ij} = \int_0^1 \int_0^1 f(x,t)\phi_i(x)\phi_j(t)dxdt. \tag{3.10}$$

Theorem 3.3. *If* $k(x,t,y,z) = \sum_{s=0}^{n}\sum_{r=0}^{n}\sum_{j=0}^{n}\sum_{i=0}^{n} k_{ijrs}\phi_i(x)\phi_j(t)\phi_r(y)\phi_s(z)$, *then*

$$\int_0^1 \int_0^1 k(x,t,y,z)u_n(y,z)dydz = {X_x}^T\Phi^T\Lambda\Phi X_t, \tag{3.11}$$

where Λ *is an* $(N+1)\times(N+1)$ *matrix with the elements*

$$\Lambda_{ij} = \sum_{s=0}^{n}\sum_{r=0}^{n} k_{ijrs}u_{rs}, \qquad i,j = 0,1,...,n.$$

Proof. Since $u_n(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} u_{ij}\ \phi_i(x)\phi_j(t)$, then we have

$$\begin{aligned}
&\int_0^1 \int_0^1 k(x,t,y,z)u_n(y,z)dydz \\
&= \int_0^1 \int_0^1 \left(\sum_{s=0}^{n}\sum_{r=0}^{n}\sum_{j=0}^{n}\sum_{i=0}^{n} k_{ijrs}\phi_i(x)\phi_j(t)\phi_r(y)\phi_s(z)\right) \\
&\times \left(\sum_{l=0}^{n}\sum_{m=0}^{n} u_{lm}\phi_l(y)\phi_m(z)\right) dydz \\
&= \sum_{s=0}^{n}\sum_{r=0}^{n}\sum_{j=0}^{n}\sum_{i=0}^{n}\sum_{l=0}^{n}\sum_{m=0}^{n} k_{ijrs}u_{lm}\phi_i(x)\phi_j(t)\int_0^1 \phi_r(y)\phi_l(y)dy \int_0^1 \phi_s(z)\phi_m(z)dz \\
&= \sum_{s=0}^{n}\sum_{r=0}^{n}\sum_{j=0}^{n}\sum_{i=0}^{n}\sum_{l=0}^{n}\sum_{m=0}^{n} k_{ijrs}u_{lm}\phi_i(x)\phi_j(t)\delta_{rl}\delta_{sm} \\
&= \sum_{s=0}^{n}\sum_{r=0}^{n}\sum_{j=0}^{n}\sum_{i=0}^{n} k_{ijrs}u_{rs}\phi_i(x)\phi_j(t) = {X_x}^T\Phi^T\Lambda\Phi X_t.
\end{aligned}$$

□

By substituting from (3.8), (3.9) and (3.11) into Eq.(1.1) and noting that ${X_x}^T\Phi^T$ and ΦX_t are basis vectors, we obtain the operational matrix representation

$$U\Phi\eta^m GA - \Lambda = F, \tag{3.12}$$

which is a system of linear algebraic equations.
Similarly, we convert the conditions (1.2) to a system of linear algebraic equations. To this end, let us assume that

$$h_k(x) = \sum_{j=0}^{n} \overline{h}_{kj}\phi_j(x) = {X_x}^T\Phi^T\mathbf{h}_k, \quad k = 0,1,...,m-1, \tag{3.13}$$

where $\mathbf{h}_k = (\overline{h}_{k0}, \overline{h}_{k1}, ..., \overline{h}_{kn}, 0, 0, ...)^T$. Then by using (3.8), we have

$$\frac{\partial^k}{\partial t^k} u(x,t)|_{t=0} = {X_x}^T \Phi^T \eta^k \; U \Phi X_t|_{t=0}. \tag{3.14}$$

Substituting (3.13) and (3.14) into (1.2) yields

$$\eta^k \; U \Phi X_t|_{t=0} = \mathbf{h}_k, \quad k = 0, 1, ..., m-1. \tag{3.15}$$

By solving the system of equations (3.12) and (3.15) simultaneously and determining the unknown coefficients u_{ij}, the approximate solution $u_n(x,t)$ can be determined from (3.7).

Remark 3.4. By applying the integral operator (2.4) to both sides of (1.1), using the conditions (1.2) and some classical results from fractional calculus, we get:

$$\begin{aligned} u(x,t) = & \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(x,s) ds \\ & + \lambda \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_0^1 \int_0^1 k(x,s,y,z) u(y,z) dydzds, \end{aligned} \tag{3.16}$$

and by changing the order of integration in (3.16), we can derive

$$u(x,t) = g(x,t) + \lambda \int_0^1 \int_0^1 K(x,t,y,z) u(y,z) dydz, \tag{3.17}$$

where

$$g(x,t) = \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(x,s) ds,$$

and

$$K(x,t,y,z) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} k(x,s,y,z) ds.$$

Therefore, problem (1.1)-(1.2) is equivalent to the Fredholm integral equation (3.17). Since f and k are continuous functions on their corresponding domains, it follows that $g(x,t)$ and $K(x,t,y,z)$ are continuous. The convergence of the proposed method with Legendre polynomial bases for (3.17) was proved in [3], so the proposed method is converge.

4. Error Estimation

In this section, we present an estimation error for the approximate solution. To do this, we define the error function as

$$e_n(x,t) := u(x,t) - u_n(x,t), \tag{4.1}$$

where $u(x,t)$ and $u_n(x,t)$ are the exact and the approximate solutions, respectively. By replacing $u(x,t)$ by $u_n(x,t)$ in (1.1)-(1.2), a perturbed problem is obtained as

$$D_t^{\alpha} u_n(x,t) - \int_0^1 \int_0^1 k(x,t,y,z) u_n(y,z) dydz = f(x,t) + p(x,t), \tag{4.2}$$

with the initial conditions

$$\frac{\partial^k}{\partial t^k}u_n(x,0)=h_k(x),\quad k=0,1,...,m-1, \tag{4.3}$$

where the perturbation term p is computed from

$$p(x,t)=D_t^{\alpha}u_n(x,t)-\int_0^1\int_0^1 k(x,t,y,z)u_n(y,z)dydz-f(x,t). \tag{4.4}$$

By subtracting (4.2) from (1.1) and (4.3) from (1.2) and using (4.1), we obtain the following initial value problem for the error function $e_n(x,t)$

$$D_t^{\alpha}e_n(x,t)-\int_0^1\int_0^1 k(x,t,y,z)e_n(y,z)dydz=-p(x,t), \tag{4.5}$$

$$\frac{\partial^k}{\partial t^k}e_n(x,0)=0,\quad i=0,1,...,m-1. \tag{4.6}$$

Solving problem (4.5)-(4.6) by the same way as we did before for the solution of (1.1)-(1.2), an approximation $e_{nl}(x,t)$ of degree l is obtained for the exact error function $e_n(x,t)$. We should notice that, only the right-hand sides of equations (4.5)-(4.6) differ from (1.1) and (1.2).

5. Numerical examples

In this section, we use normalized Legendre polynomials as basis functions and solve some examples to illustrate efficiency and applicability of the proposed method.

Example 5.1. Consider the equation

$$D_t^{0.5}u(x,t)-\int_0^1\int_0^1 (y^2+tz)u(y,z)dydz=\frac{2x\sqrt{t}}{\Gamma(0.5)}-\frac{1}{3}t-\frac{13}{40},\ (x,t)\in I,$$

with the initial condition $u(x,0)=x^2$ and exact solution $u(x,t)=x^2+xt$. By applying the presented method, we obtain $u_2(x,t)=x^2+xt$ which is the exact solution.

Example 5.2. As the second example, consider the equation

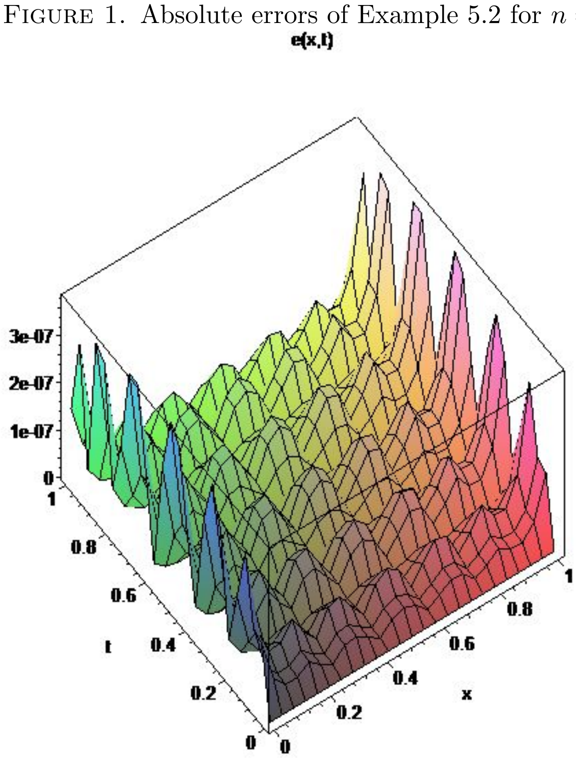
$$D_t^{0.75}u(x,t)-\int_0^1\int_0^1 (xz+tsin(y))u(y,z)dydz$$
$$=\frac{4\ t^{1/4}cos(x)}{\Gamma(0.25)}-\frac{1}{3}sin(1)\ x-\frac{1}{4}sin(1)^2\ t,\quad (x,t)\in I,$$

with the initial condition $u(x,0)=0$. The exact solution of this problem is $u(x,t)=t\ cos(x)$. Table 1 shows the absolute errors ($e(x,t)$) and the corresponding estimations ($e_n(x,t)$) at some selected points for $n=8$. Absolute errors plotted in Figure 1.

TABLE 1. Numerical results of Example (5.2).

(x,t)	$e(x,t)$	$e_n(x,t)$
(0,0.6)	0.6651e -7	0.1734e -7
(0.1,0.1)	0.4075e -8	0.4508e -8
(0.2,0.7)	0.3101e -7	0.1054e -7
(0.3,0.8)	0.3899e -7	0.1717e -7
(0.4,0.4)	0.3746e -8	0.3568e -8
(0.6,0.9)	0.8366e -8	0.3533e -8
(0.8,0.5)	0.8094e -7	0.1248e -7
(0.9,0.1)	0.4074e -8	0.4508e -8

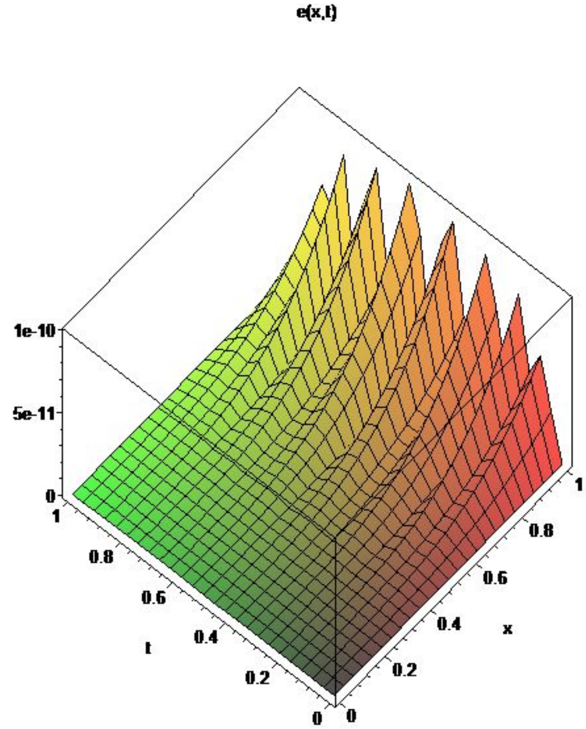
FIGURE 1. Absolute errors of Example 5.2 for $n = 8$.



Example 5.3. As the final example, we consider the equation

$$D_t^{\alpha} u(x,t) - \int_0^1 \int_0^1 u(y,z) dy dz = 2x^4 e^{2t} + \frac{1}{10} - \frac{1}{10} e^2, \quad (x,t) \in I, \quad 0 < \alpha \leq 1,$$

Figure 2. Absolute error for $n = 10$ and $\alpha = 1$ of Example 3.



subject to the initial condition $u(x, 0) = x^4$. For $\alpha = 1$, the exact solution is $u(x, t) = x^4 e^{2t}$. For $n = 10$ and different values of α approximate values of the solution are reported in Table 2. As shown in this table, the results of our method for $\alpha = 1$ (the only case that we know the exact solution) is in the best agreement with the exact values. For $\alpha = 1$ the plot of absolute error is shown in Figure 2.

6. Conclusion

In this paper, we presented an expansion method, based on orthonormal polynomials to solve PFFIDEs. We applied the presented method on three test problems and showed that the numerical results have high accuracy comparing with exact values. We concluded that the presented method is efficient and it can be developed for similar partial fractional Volterra integro-differential equations.

ACKNOWLEDGEMENTS

The authors would like to thank the editors and anonymous referees for their valuable comments and suggestions.

Table 2. Numerical results of Example (5.3) for various α.

(x,t)	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	Exact solution for $\alpha = 1$
(0.1,0.1)	0.111603	0.048368	0.015931	0.000122	0.000122
(0.2,0.2)	0.187484	0.087515	0.032154	0.002387	0.002387
(0.3,0.3)	0.263031	0.134484	0.058627	0.014759	0.014759
(0.4,0.4)	0.440334	0.214756	0.117241	0.056974	0.056974
(0.5,0.5)	0.571382	0.378956	0.252958	0.169893	0.169893
(0.6,0.6)	0.955369	0.716930	0.549075	0.430287	0.430287
(0.7,0.7)	1.697018	1.386118	1.152511	0.973653	0.973653
(0.8,0.8)	3.095794	2.658581	2.311338	2.028762	2.028762
(0.9,0.9)	5.599890	4.963044	4.427511	3.969174	3.969174
(1,1)	9.967113	8.993351	8.142245	7.389056	7.389056

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