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Stabilization of linear systems of delay differential equations by the delayed feedback method

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Abstract This paper consists of two folds. At first, we deal with the stability analysis of a linear system of delay differential equations. It is shown that the direct and cluster treatment methods are not applicable if there are some purely imaginary roots of the characteristic equation with multiplicity greater than one. To overcome the above difficulty, the system is decomposed into several subsystems. For the decomposition of a system, an invertible transformation is required to convert the matrices of the system into a block triangular (diagonal) form simultaneously. To achieve this goal, a necessary and sufficient condition is established. The second part concerns the stabilization of a linear system of delay differential equations using the delayed feedback method and design a controller for generating the desired response. More precisely, the unstable poles of the linear system of delay differential equations are moved to the left-half of the complex plane by the delayed feedback method. It is shown that the performance of the linear system of delay differential equations can be improved by applying the delayed feedback method.

Keywords. Linear time delay system, Stability analysis, Simultaneous block triangularization, Delayed feedback method, Stabilization, Controller design.

2010 Mathematics Subject Classification. 93C30, 05C50, 93D15, 93D20.

1. Introduction

Linear system of delay differential equations (LSDDE) appears naturally in many branches of science and engineering. A common way of describing of an LSDDE is the state-space representation which is frequently used in control theory. Unlike ordinary differential equations (ODEs), in DDEs the rate of change of an unsteady process not only depends on the current state but also depends on its history state.

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The stability analysis of an LSDDE is extremely important from both theoretical and practical points of view [5, 15, 24]. Due to the presence of the delayed term in an LSDDE, the corresponding characteristic equation is a quasi-polynomial instead of a polynomial. Therefore, the stability analysis of such systems becomes more complicated. Rekasius [21], Walton and Marshall [26] and Olgac and Sipahi [17] presented some methods for the stability analysis of linear systems of DDEs. However, these methods cannot be applied for an arbitrary LSDDE. More precisely, Mesbahi and Haeri [14] presented an example of an LSDDE, which its characteristic equation has a multiple root with multiplicity greater than one, that the method proposed by Olgac and Sipahi cannot be used to analyze the stability of the system. They removed the repeated roots by decomposing the original 4 by 4 system into two subsystems with 2 by 2 dimension to perform the stability analysis of the original system.

One of the most common types of a DDE is the retarded one. In a retarded LSDDE, the derivative term $\dot{x}$, does not depend on the delay. In this paper, we present some retarded linear systems of DDEs that confirm the cluster treatment method [17] is not applicable to the system's stability analysis. Furthermore, we show that the direct method [26] leads us to ambiguous results in detecting the number of unstable poles of the presented systems. Nevertheless, decomposing the original linear systems of DDEs into several subsystems enables one to handle the above difficulties. To decompose an LSDDE with a single delay and with two state matrices, we need to find an invertible transformation that simultaneously transforms both state matrices of the system into block triangular (diagonal) forms.

The simultaneous triangularization (diagonalization) of a set of matrices has attracted a great deal of attention recently because of applications in multidimensional systems [3], discrete time switching systems [7] and differential inclusions [16, 23]. The following two questions are crucial in simultaneous triangularization (diagonalization): 1- When two or more generally, a finite set of matrices can be transformed simultaneously into a block triangular (diagonal) form? 2- What kind of transformations can put the matrices into a block triangular (diagonal) form simultaneously? To answer the first question, one of the most famous classical theorems is McCoy's theorem [13] which states that the pair of matrices $\{A, B\}$ is triangularizable if and only if $p(A, B)(AB - BA)$ is nilpotent for every noncommutative polynomial p.

It is easy to show that every set of commutative matrices can be simultaneously transformed into an upper triangular form, but the converse is not true [6, 20]. Levitzky [20] proved that every semigroup of nilpotent matrices is triangularizable. For a semigroup of n by n matrices, say $\mathcal{F}$, over a field that contains all the eigenvalues of the members of $\mathcal{F}$ and whose characteristic is either zero or greater than $\frac{n}{2}$, Radjavi [19] proved that $\mathcal{F}$ is triangularizable if and only if trace is permutable on $\mathcal{F}$. Uhlig [25] proved that the finest simultaneous block diagonalization of nonsingular pair of real symmetric matrices A and B contains k blocks of dimensions $n_1, n_2, \ldots, n_k$ if and only if the real Jordan normal form of $A^{-1}B$ consists k Jordan blocks of dimensions $n_1, n_2, \ldots, n_k$. Laffey [10] showed that for every n by n matrices A and B $(n \leq 5)$ with the property that for all λ: $A^3 = B^3 = (A + \lambda B)^5 = 0$, the pair of matrices $\{A, B\}$ is triangularizable if and only if AB is nilpotent. Dubi [4] proposed an algorithm to construct a simultaneous triangularization of a set on N matrices in $\mathbb{C}^{n\times n}$.

His algorithm answers the first and second question for non-block triangularization and uses Shemesh's idea [22] to compute the invertible transformation. Kaczorek [8] proved that a set of N real matrices $\{A_1, A_2, \ldots, A_N\}$ is triangularizable if and only if there exists a full column rank matrix $J \in \mathbb{R}^{n\times n}$ such that $rank[J \quad A_iJ] = r$ for $i = 1, 2, \ldots, N$.

However, the presence of a time delay in a system can cause various complications, but it can be useful in some senses. Kwon et al. [9] obtained the state feedback tracking controller by the delayed feedback method. They show that the performance of a system can be improved by the delayed feedback method and also disturbance attenuation and robustness against parameters variation can be modified. As we know, a time delay can be a source for instability of an LSDDE. Nevertheless, Abdallah et al. [1] showed that some oscillatory systems can be stabilized by the delayed feedback method. Pyragas [18] applied the delayed feedback method to control chaos and also he employed this type of feedback to stabilize the unstable periodic orbits. Usually, one of the major goals in control theory is controlling an equilibrium solution or equivalently, the regulator problem. In fact, in a regulator problem, one needs to obtain an asymptotically stable steady state solution which attracts all nearby initial conditions. Dahms et al. [2] considered the extended time-delayed feedback method to control unstable steady states.

In this paper, we show that the block simultaneous triangularization of a finite set of square matrices is equivalent to the existence of a common invariant subspace for the matrices. In this direction, we prove a proposition which characterizes the invariant subspaces of a matrix by means of generalized eigenvectors [11]. Furthermore, we present some linear systems of DDEs that their stability analysis cannot be investigated by the direct and the cluster treatment methods. Also, we show that an unstable LSDDE can be stabilized by the delayed feedback method. In addition, we adopt this feedback for putting the poles of an LSDDE in suitable coordinates to generate a desired response for the system. More precisely, by the delayed feedback method, the settling time of the system can be remarkably reduced.

The remaining of the paper is organized as follows. In section 2, we introduce some required mathematical details and problem statement. Stability analysis of the linear systems of DDEs using decomposition of the matrices of the system into a block triangular (diagonal) form is considered in section 3. In section 4, stabilization of an unstable LSDDE by the delayed feedback method is discussed. Section 5 is devoted to design a controller via the delayed feedback method. Finally, conclusion is available in section 6.

2. Mathematical details and problem statement

2.1. **Definitions, lemmas and theorems.** In this section, first, we address some notations which are used throughout this paper and then we provide some definitions, theorems, and lemmas which are related to simultaneous block triangularization of a finite set of square matrices. Finally, the controllability theorem for the linear systems of DDEs is expressed.

$\mathbb{R}^n$ and $\mathbb{C}^n$ are real and complex Euclidean spaces respectively. $\mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) is the space of real (complex) square n by n matrices. Also, we denote crossing frequencies and their corresponding delays by ω_{ck} and τ_{kl} respectively, where $k = 1, 2, \dots, n$ and $l = 1, 2, \dots$. $F(s, \tau)$ denotes characteristic equation of an LSDDE.

Definition 2.1. Let $A_1, A_2, \dots, A_N$ be a set of matrices belong to $\mathbb{R}^{n\times n}$. This set of matrices are said to be simultaneously block triangularizable with dimension k if there exists an invertible transformation Q such that

$$QA_iQ^{-1} = \tilde{A}_i = \begin{bmatrix} \tilde{A}_{i1} & \tilde{A}_{i2} \\ 0 & \tilde{A}_{i4} \end{bmatrix}, \qquad i = 1, 2, \dots, N, \tag{2.1}$$

where $\tilde{A}_{i1} \in \mathbb{R}^{k\times k}$, $\tilde{A}_{i2} \in \mathbb{R}^{k\times(n-k)}$, $\tilde{A}_{i4} \in \mathbb{R}^{(n-k)\times(n-k)}$ and $1 \le k < n$.

Example 2.2. [8] Consider the following matrices

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 4 & 0 \\ 2 & 2 & 0 \end{bmatrix}.$$

If we put

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

then we have

$$QA_1Q^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}, \qquad QA_2Q^{-1} = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix}.$$

Clearly, in this example $k = 2$.

Definition 2.3. A vector subspace $V \subset \mathbb{R}^n$ is said to be $(A_1, A_2, \dots, A_N)$-invariant if $A_iv \in V$ for all $v \in V$ and $i = 1, 2, \dots, N$.

Definition 2.4 ([6])**.** Let A be a matrix that belongs to $\mathbb{R}^{n\times n}$. $x_0, x_1, \dots, x_k$ is called a Jordan chain of A corresponding to the eigenvalue λ_0 if $x_0 \neq 0$ and the following relations hold

$$\begin{aligned} Ax_0 &= \lambda_0 x_0, \\ Ax_1 \lambda_0 x_1 &= x_0, \\ Ax_2 - \lambda_0 x_2 &= x_1, \\ &\vdots \\ Ax_k - \lambda_0 x_k &= x_{k-1}. \end{aligned}$$

The first relation (together with $x_0 \neq 0$) confirms that x_0 is an eigenvector of A corresponding to the eigenvalue λ_0. The vectors $x_1, x_2, \dots, x_k$ are called generalized eigenvectors of A corresponding to the eigenvalue λ_0 and the eigenvector x_0.

Definition 2.5 ([11])**.** Let $A \in \mathbb{R}^{n\times n}$. A set of Jordan vectors for A is a set of linearly independent vectors in $\mathbb{C}^n$ made up of a union of Jordan chains.

Definition 2.6 ([5])**.** Consider the following system

$$\dot{x}(t) = \sum_{k=0}^{N} A_k x(t-\tau_k) + Bu(t), \tag{2.2}$$

where $x(t)$, $u(t) \in \mathbb{R}^n$, $A_0, A_1, \ldots, A_N, B \in \mathbb{R}^{n\times n}$ and $\tau_0 = 0$. The system is $\mathbb{R}^n$-controllable on $[t_0, t_1]$ if for all $x_0 \in \mathcal{C}(-\tau_N, 0)$ and $x_1 \in \mathbb{R}^n$ there exists a piecewise-continuous function $u(t) = u(t, x_0, x_1)$ such that the solution of the system (2.2) with the initial condition $x_{t_0} = x_0$ satisfies $x_{t_1} = x_1$.

In the following theorem, we propose a necessary and sufficient condition for a finite set of matrices to have the simultaneous block triangularization (diagonalization) property.

Theorem 2.7. *The matrices $A_1, A_2, \ldots, A_N \in \mathbb{R}^{n\times n}$ are simultaneously block triangularizable with dimension k if and only if there exists an $(A_1, A_2, \ldots, A_N)$-invariant k-dimensional subspace $W \subset \mathbb{R}^n$.*

Proof. For the sake of simplicity, we prove this theorem only for two matrices. Let $W \subset \mathbb{R}^n$ be an arbitrary k-dimensional vector subspace and

$$Q = [Q_k \quad Q_{n-k}] = [q_1 \cdots q_k \quad q_{k+1} \cdots q_n],$$

be a nonsingular matrix where the first k columns of it, i.e., $Q_k = [q_1 \cdots q_k]$; form a basis for the subspace W. By assuming

$$Q^{-1}A_iQ = \begin{bmatrix} \tilde{A}_{i1} & \tilde{A}_{i2} \\ \tilde{A}_{i3} & \tilde{A}_{i4} \end{bmatrix}, \qquad i = 1, 2, \tag{2.3}$$

we show that $\tilde{A}_{13} = \tilde{A}_{23} = 0$ if and only if the first k columns of Q, Q_k, form a basis for the subspace W. To do this end, let $[a_1^{i,1} \cdots a_k^{i,1}]$ be the k columns of $\tilde{A}_{i1}$ and $[a_1^{i,3} \cdots a_k^{i,3}]$ be the k columns of $\tilde{A}_{i3}$ for $i = 1, 2$, respectively. Since Q is a nonsingular matrix, for $i = 1, 2$, Eq. (2.3) gives

$$\begin{aligned} A_i[Q_k \quad Q_{n-k}] =& [Q_k \quad Q_{n-k}] \begin{bmatrix} \tilde{A}_{i1} & \tilde{A}_{i2} \\ \tilde{A}_{i3} & \tilde{A}_{i4} \end{bmatrix} \\ =& [Q_k\tilde{A}_{i1} + Q_{n-k}\tilde{A}_{i3} \quad Q_k\tilde{A}_{i2} + Q_{n-k}\tilde{A}_{i4}]. \end{aligned} \tag{2.4}$$

Therefore, by equating corresponding columns in (2.4), we obtain the following relations

$$A_iq_j = [Q_k\tilde{A}_{i1} + Q_{n-k}\tilde{A}_{i3}]_j, \quad j = 1, \ldots, k, \quad i = 1, 2.$$

So, for $j = 1, \ldots, k$ and $i = 1, 2$, we have

$$A_iq_j = Q_k a_j^{i,1} + Q_{n-k} a_j^{i,3}.$$

As the first k columns of Q are linearly independent, therefore $\tilde{A}_{13} = \tilde{A}_{23} = 0$, which means that for $i = 1, 2$, W is A_i-invariant. Clearly, if W is A_i-invariant, then $\tilde{A}_{13} = \tilde{A}_{23} = 0$. This completes the proof. □

Remark 2.8. It is clear that Q is not unique.

The following corollaries are immediate consequence of Theorem 2.7.

Corollary 2.9. *If $A_1, A_2, \ldots, A_N$ have k common eigenvectors, then they can be transformed simultaneously into a k-dimensional block triangular form.*

Corollary 2.10. *Let $n = 2$. $A_1, A_2, \ldots, A_N$ are transformed simultaneously into a block triangular form if and only if they have a common eigenvector.*

The following theorem is a consequence of Theorem 2.7.

Theorem 2.11. [8] *Let $A_1, A_2, \ldots, A_N$ be the square matrices in $\mathbb{R}^{n\times n}$. These matrices can be put simultaneously in the form (2.1) by means of transformation T, if and only if there exists a full column rank matrix $J \in \mathbb{R}^{n\times r}$ such that*

$$rank[J \quad A_iJ] = r, \ \ for\ i = 1, 2, \cdots, N.$$

The following proposition characterizes k-dimensional invariant subspaces of a square matrix A.

Proposition 2.12. *Let A be a real n by n matrix. A k-dimensional subspace $W \subset \mathbb{R}^n$ is A-invariant if and only if W has a basis consisting of a set of Jordan vectors for A.*

Proof. Assume W has a set of Jordan vectors, say $\{x_1, x_2, \ldots, x_k\}$, for A as a basis. By the assumption, $W = span < x_1, x_2, \ldots, x_k >$. Since $\{x_1, x_2, \ldots, x_k\}$ belongs to a Jordan chain, so by proposition 1.3.1 in Ref. [6], W is A-invariant. Conversely, let $X = [x_1, x_2, \ldots, x_k]$ be a n by k matrix whose columns form an arbitrary basis for W. Since W is A-invariant, there exists $G \in \mathbb{R}^{k\times k}$ such that $AX = XG$. The Jordan matrix decomposition of G can be written as $G = SJS^{-1}$ for some S, which leads us to $AXS = XSJ$ and therefore, we get $J = (XS)^{-1}A(XS)$. Here, the columns of the matrix XS form the Jordan vector for A. □

Now, we give the following theorem which is crucial for the controllability of linear systems of DDEs.

Theorem 2.13. [12] *If $(A_0 + A_1, B)$ is controllable, then the following system is controllable*

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau) + Bu(t),$$

where $A_0, A_1, B \in \mathbb{R}^{n\times n}$, $u \in \mathbb{R}^{n\times 1}$.

2.2. **Problem statement.** In this paper, first, we consider the stability analysis of the following linear systems of DDEs

$$\dot{x}(t) = A_1x(t) + A_2x(t-\tau), \tag{2.5}$$

where $A_1, A_2 \in \mathbb{R}^{n\times n}$ and $\tau > 0$ is time delay. We will show that the stability analysis of some systems like (2.5) required to decompose them into subsystems with lower dimension and then analyze the stability of each subsystem and finally stability of the whole system is achieved. Second, by the delayed feedback method, we attempt to stabilize the following system

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau) + Bu(t), \tag{2.6}$$

where $\tau > 0$ is a fixed delay and A_0, A_1, B, u are the same as defined in Theorem 2.13.

3. Stability analysis of linear systems of DDEs via decomposition

Here, we provide two examples of linear systems of DDEs that show the direct method cannot recognize the number of unstable poles. In addition, the cluster treatment method also fails to analyze the stability of the systems.

Example 3.1. Consider the following LSDDE

$$\dot{x}(t) = A_1x(t) + A_2x(t-\tau), \tag{3.1}$$

where

$$A_1 = \begin{bmatrix} 3.2423 & -1.4176 & -2.7298 & 4.6267 \\ -1.0366 & -0.9812 & -0.7598 & -3.2319 \\ 2.0250 & 0.8723 & 0.0129 & 4.0908 \\ -0.9802 & 1.5668 & 1.2885 & -1.2741 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1.4104 & 1.1252 & -0.1052 & 0.9652 \\ -0.2045 & -0.5965 & -0.2415 & 0.2683 \\ 0.4985 & 0.7644 & 0.1801 & 0.4498 \\ -0.3069 & 0.4843 & 0.4550 & 0.0060 \end{bmatrix}.$$

The characteristic equation of the LSDDE crosses the imaginary axis at $s = \pm j$, $s = \pm\sqrt{3}j$, and corresponding delays are $\tau_k = \pi + 2\pi k, \tau_k = \frac{2\pi k}{\sqrt{3}}$, $k = 0, 1, \ldots$, respectively. All roots of the characteristic equation are in the right half-plane for $\tau = 0$. Therefore, the system is unstable for $\tau = 0$. By using the direct method, the expression $\mathrm{sgn} W^{'}(\omega^2)$ is positive at $\omega = \sqrt{3}$ and zero at $\omega = 1$. Thus, the system is unstable for all τ. In other words, the direct method says all roots of the characteristic equation of the system (3.1) are in the right half-plane. If we want to apply the cluster treatment method, then we have $\frac{\partial F(s,\tau)}{\partial s} = 0$ and $\frac{\partial F(s,\tau)}{\partial \tau} = 0$ at crossing frequency $\omega = 1$. Hence, the root tendency cannot be determined.

Now, we decompose the system (3.1) as follows. A_1 and A_2 have a common invariant subspace with dimension 2. A basis for this subspace can be considered as

$$E = \begin{bmatrix} 0.3878 & 0.8143 \\ -0.2562 & -0.1180 \\ 0.5371 & 0.2878 \\ -0.2094 & -0.1772 \end{bmatrix}.$$

In fact, a linear combination of the columns of E forms a two-dimensional (A_1, A_2)-invariant subspace. One choice for the transformation T in (2.1) is

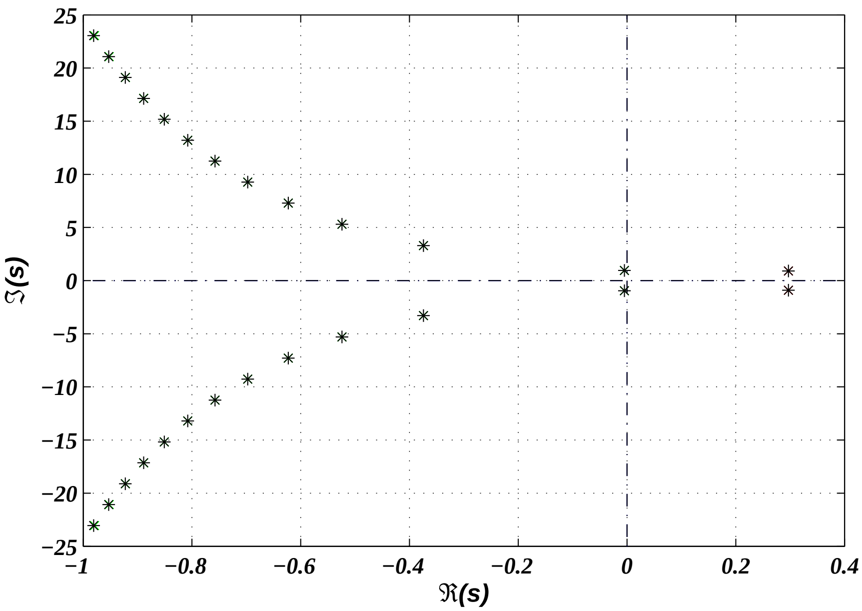
$$T = \begin{bmatrix} -1.0000 & 3.6667 & 4.3333 & 0 \\ 1.7321 & 1.6000 & 0 & 1.2500 \\ 0.2857 & 0 & 0.8333 & 2.6667 \\ 0 & 2.2500 & 1.3333 & 0.6667 \end{bmatrix}.$$

By applying this transformation to the system (3.1), we derive two subsystems which are

$$\dot{z_1}(t) = \bar{A}_{11}z_1(t) + \bar{B}_{11}z_1(t-\tau), \tag{3.2}$$

$$\dot{z_2}(t) = \bar{A}_{22}z_2(t) + \bar{B}_{22}z_2(t-\tau), \tag{3.3}$$

FIGURE 1. The roots of the characteristic equation of the subsystem (3.2) for $\tau = 3.2$.



where $x(t) = Tz(t)$ and

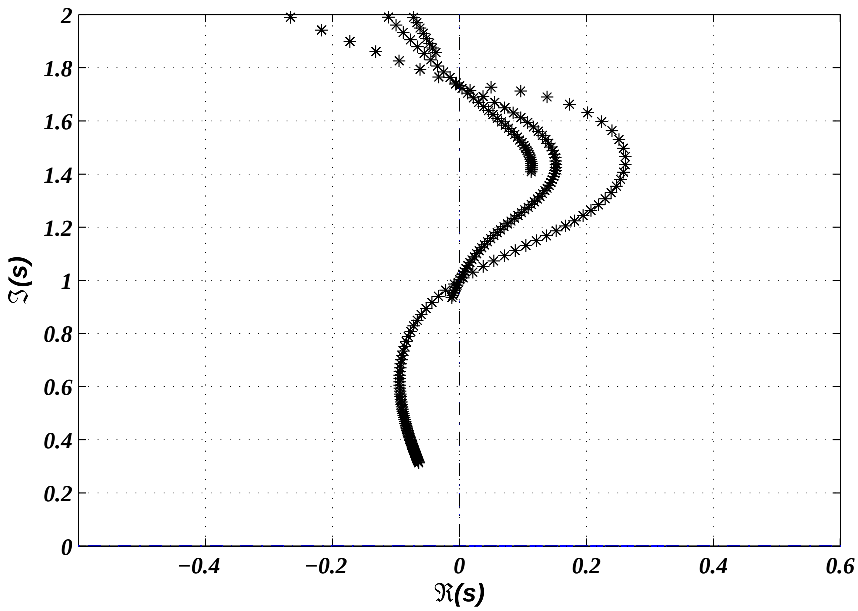
$$\bar{A}_{11} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad \bar{B}_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{A}_{22} = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix},$$
$$\bar{B}_{22} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Now, we employ the direct method to each of the subsystems (3.2) and (3.3) separately. Subsystem (3.2) has a crossing frequency at $s = \pm j$ and $\mathrm{sgn} W'(\omega^2)$ is zero at this frequency. Therefore, the subsystem is unstable for all τ. The characteristic equation of subsystem (3.3) crosses the imaginary axis at $s = \pm\sqrt{3}j$ and $s = \pm j$. The quantity $\mathrm{sgn} W'(\omega^2)$ is positive and negative at these frequencies, respectively. Therefore, this subsystem is stable for $\pi < \tau < \frac{2\pi}{\sqrt{3}}$. Thus, the system (3.1) has two stable poles for $\pi < \tau < \frac{2\pi}{\sqrt{3}}$. The roots of the characteristic equation of the subsystem (3.2) and the root locus of the subsystem (3.3) near the imaginary axis are plotted in Figure 1 and Figure 2, respectively.

Example 3.2. Let us analyze the stability of the following LSDDE

$$\dot{x}(t) = A_1 x(t) + A_2 x(t - \tau), \tag{3.4}$$

FIGURE 2. The root locus of the subsystem (3.3) near the imaginary axis for $0 < \tau < 4$.



where

$$A_1 = \begin{bmatrix} -14.6102 & -4.9441 & 11.3503 & -11.5177 & -11.9699 \\ -3.9437 & -1.0804 & 3.4948 & -3.3674 & -3.2193 \\ 6.4695 & 0.5153 & -4.1521 & 3.9784 & 5.0394 \\ 6.0633 & 2.1406 & -4.6372 & 5.0694 & 4.8474 \\ 20.3590 & 4.5468 & -15.5102 & 13.5751 & 16.7733 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -11.1098 & -3.6577 & -2.2712 & -13.4823 & -4.0327 \\ -3.1263 & -1.0354 & -0.6680 & -3.7568 & -1.1390 \\ 4.8695 & 1.7361 & 1.6197 & 5.1076 & 1.8581 \\ 4.4403 & 1.4397 & 0.8037 & 5.5222 & 1.5967 \\ 163449 & 5.4846 & 3.8268 & 19.2118 & 6.0034 \end{bmatrix}.$$

First, we decompose the system (3.4) into subsystems and then we apply the direct method to each of the subsystems to explore the stability analysis of the whole system. The columns of the full column rank matrix

$$E = \begin{bmatrix} 1.2775 & -1.3977 \\ 0.6036 & -0.4111 \\ -0.5536 & 0.9967 \\ -0.5480 & 0.4946 \\ -1.9230 & 2.3550 \end{bmatrix},$$

constitute a two-dimensional (A_1, A_2)-invariant subspace. We may choose the transformation T in (2.1) as

$$T = \begin{bmatrix} 0.1250 & 4.5000 & 0.6667 & 2.7500 & 0 \\ 1.4142 & 0.7500 & 1.6250 & 0 & 0.7071 \\ 3.0000 & 0.8571 & 0 & 4.4286 & 1.0000 \\ -1.0000 & 0 & 2.0000 & 2.6667 & -2.0000 \\ 0 & 1.7321 & -1.0000 & 1.4142 & 0.4286 \end{bmatrix}.$$

After applying this transformation to the system (3.4), we derive two subsystems as follows

$$\dot{z_1}(t) = \bar{A}_{11} z_1(t) + \bar{B}_{11} z_1(t-\tau), \tag{3.5}$$

$$\dot{z_2}(t) = \bar{A}_{22} z_2(t) + \bar{B}_{22} z_2(t-\tau), \tag{3.6}$$

where $x(t) = Tz(t)$ and

$$\bar{A}_{11} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad \bar{B}_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{A}_{22} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix},$$

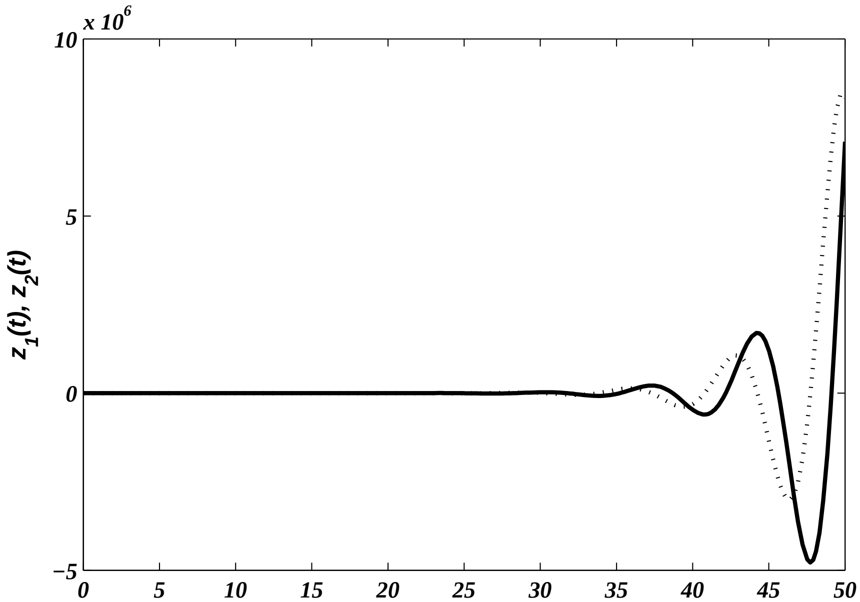
$$\bar{B}_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

By applying the direct method, the subsystem (3.5) is always unstable while the subsystem (3.6) has two stable poles for $3.1416 < \tau < 3.3077$. For this subsystem the crossing frequencies are $s = \pm j$ and $s = \pm\sqrt{1+\sqrt{2}}j$. However, the direct method confirms that all characteristic roots of the system (3.4) are in the right half-plane, but by decomposing the system, we find out that the system has two stable poles $3.1416 < \tau < 3.3077$. The time domain response of the subsystem (3.5) for the constant initial condition $z_1(t) = 1, -\tau \le t \le 0$, is sketched in Figure 3. Also, Figure 4 confirms that the subsystem (3.6) has two stable poles for $\tau = 3.2$.

4. Stabilization of unstable time delay systems by the delayed feedback method

As we said in the introduction, the delayed feedback method can be employed to stabilize an unstable LSDDE. However, there are infinitely many roots for the characteristic equation of a retarded DDE, but the number of unstable poles are finite [5]. By considering this issue, we are attempting to move the unstable poles of an LSDDE to the left-half complex plane by the delayed feedback method. Indeed, stabilization of an unstable LSDDE is possible, if the characteristic equation of the system crosses the imaginary axis. The larger crossing frequency of an LSDDE corresponds to one where the roots of the characteristic equation of the system cross from left to right (i.e., destabilizing) of the complex plane [26]. Therefore, to stabilize an unstable LSDDE by the delayed feedback method, the characteristic equation of the system must be crossed the imaginary axis at least two times. After finding the crossing frequencies (if they exist), we can compute the corresponding delays for each of the crossing

FIGURE 3. Time domain response of the subsystem (3.5). The solid line depicts $z_1(t)$ and dots display $z_2(t)$.



frequencies. Finally, the interval of time delay which the closed-loop system is stable can be obtained by the cluster treatment or direct methods.

As a concrete example, we consider the subsystem (3.2) that its stability analysis is done in the previous section. Let us consider the open-loop time delay system as

$$\dot{z}(t) = A_1 z(t) + A_2 z(t - 3.2), \tag{4.1}$$

where A_1 and A_2 are the same as $\tilde{A}_{11}$ and $\tilde{B}_{11}$, respectively. As we see earlier, the system (4.1) had two unstable poles.

Now, our goal is the stabilization of the following closed-loop system by the delayed feedback method

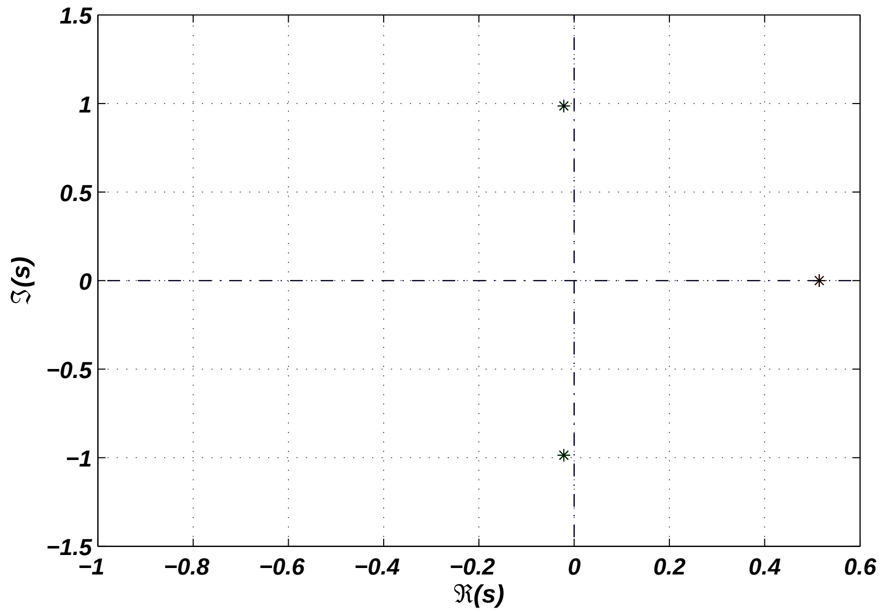
$$\dot{z}(t) = A_1 z(t) + A_2 z(t - 3.2) - BK(z(t) - z(t - \tau)), \tag{4.2}$$

where $B = [1 \quad 0]^T$ and $K = [k_1 \quad k_2]$. As we said before, the stabilization of the system (4.2) requires that the characteristic equation of the system (4.2) crosses the imaginary axis. Before going further, we propose the following lemma which gives us a necessary condition such that $s = \omega_c j$ to be a root of the characteristic equation of the system (4.2).

Lemma 4.1. *If $s = \omega_{c\beta}$ is a root of the characteristic equation of the system (4.2), where*

$$\omega_{c\beta} \in \{\omega \in \mathbb{R} : |\omega| \leq \beta \, for \; some \, \beta\},$$

FIGURE 4. The roots of the characteristic equation of the subsystem (3.6) for $\tau = 3.2$.



then the following relation holds

$$k_2 - |k_2| - \beta(1-\beta) \leq 1 - k_1 + |k_1|(3+\beta).$$

Proof. The proof is straightforward. By separating the real and imaginary parts of the characteristic equation of the system (4.2), we get the following relations for $s = j\omega$

$$1 + \cos(\omega(\tau+3.2))k_1 - k_1\omega\sin(\omega\tau) + (k_1+k_2)\cos(\omega\tau) - k_1\cos(3.2\omega) \\ -\omega\sin(3.2\omega) - \omega^2 - k_1 - k_2 = 0,$$

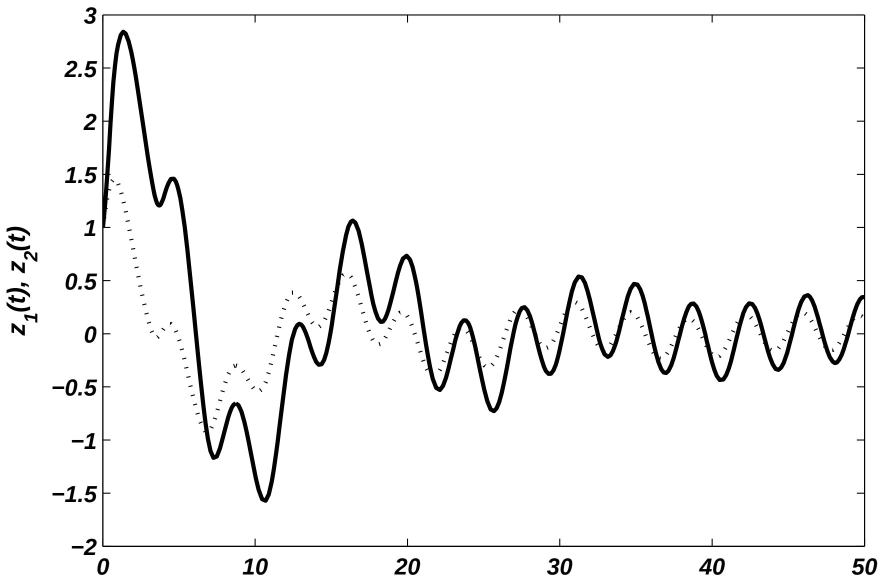
$$-\sin(\omega(\tau+3.2))k_1 - k_1\omega\cos(\omega\tau) - (k_1+k_2)\sin(\omega\tau) + k_1\sin(3.2\omega) \\ -\omega\cos(3.2\omega) + k_1(\omega-1) = 0.$$

After applying the triangle inequality and using the famous inequalities $|\sin(x)|$, $|\cos(x)| \leq 1$, the result follows immediately. □

Now, we return to the stabilization process. According to Lemma 4.1, if we choose $k_1 = 1$ and $k_2 = -5$, then there are two crossing frequencies. These frequencies and corresponding delays are

$$\omega_{c1} = \pm 1.6564,\ \tau_{11} = 0.4540,\ \tau_{12} = 4.2473, \ldots,$$
$$\omega_{c2} = \pm 3.5116,\ \tau_{21} = 0.9469,\ \tau_{22} = 2.7362, \ldots.$$

FIGURE 5. Time domain response of the system (4.2) for $\tau = 0.5$, $k_1 = 1$, $k_2 = -5$. The solid line depicts $z_1(t)$ and dots display $z_2(t)$.

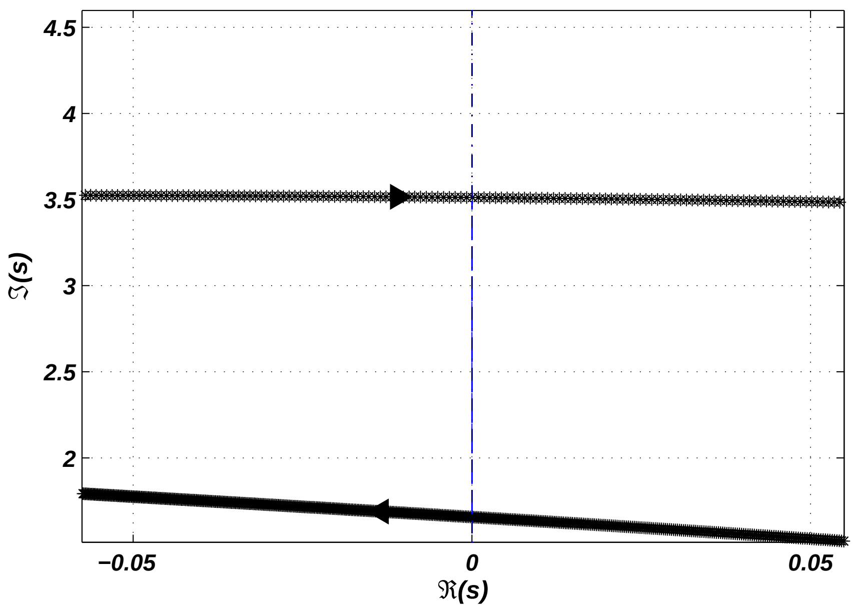


After obtaining the crossing frequencies, there are two scenarios which can be used to find the stability interval. The first one is the direct method. Using this method yields that the roots of the characteristic equation of the system move to the right half-plane at the larger crossing frequency and the next larger corresponds to stabilizing one. Therefore, the system (4.2) is stable for $0.4540 < \tau < 0.9469$. In the second scenario, we apply the cluster treatment method. The root tendency at the larger crossing frequency is 1 and it is -1 at the next one. Hence, the time delay system (4.2) is stable for $0.4540 < \tau < 0.9469$. In fact, these two approaches yield the same result. The time domain response and the root locus of the system (4.2) near the imaginary axis for $\tau = 0.5$, $k_1 = 1$ and $k_2 = -5$ are illustrated in Figure 5 and Figure 6, respectively.

5. Design controller via the delayed feedback method

In this section, we employ the delayed feedback method to produce a desired response for an LSDDE. As we know, the response of a stable linear time-invariant system extremely depends on its dominant poles. Since by the method of steps [5], an LSDDE can be converted to a linear system of ODEs, the dominant poles of an LSDDE determine the response of the system. By summarizing the above issues, the response of a stable LSDDE can be determined by its dominant poles. In general case, i.e., when the LSDDE has some unstable poles, the rightmost pole determine

FIGURE 6. The root locus of the system (4.2) near the imaginary axis for $0 < \tau < 2$.



the response. Here, we consider the subsystem (3.3) and we change the location of its dominant poles by the delayed feedback method.

Let

$$\dot{z}(t) = A_1 z(t) + A_2 z(t - 3.2), \tag{5.1}$$

where A_1 and A_2 are the same as $\tilde{A}_{11}$ and $\tilde{B}_{11}$, respectively. As we see in Figure 7, the settling time for this system is very high. To reduce the settling time of the system (5.1), we consider the following closed-loop system

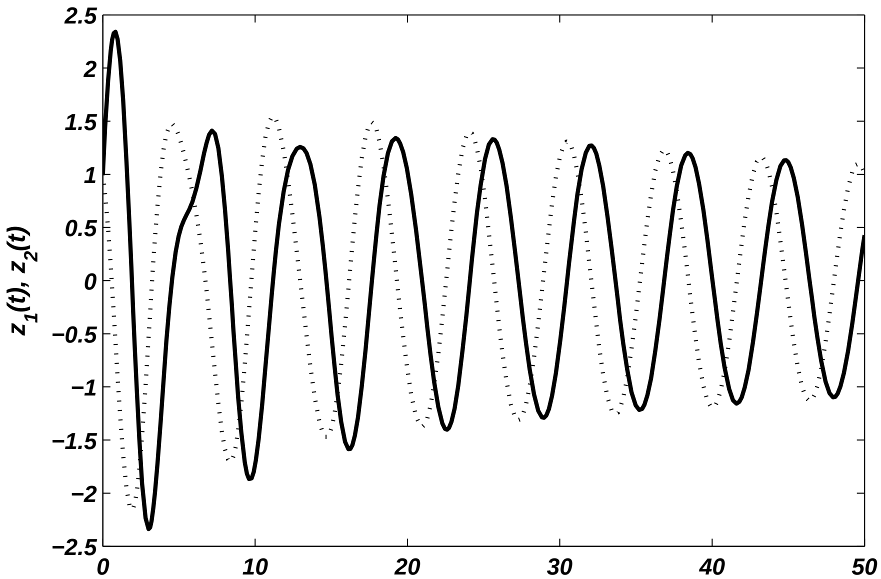
$$\dot{z}(t) = A_1 z(t) + A_2 z(t - 3.2) - BK(z(t) - z(t - \tau)), \tag{5.2}$$

where $B = [1 \quad 0]^T$ and $K = [k_1 \quad k_2]$. If we force $s = -0.3254 \pm 0.3254j$ to be poles of the system (5.2), the settling time is highly reduced. After doing this, we obtain two equations. Here, we have three parameters. We choose τ freely. If we put $\tau = 0.1$, then we have $k_1 = 40.5925$ and $k_2 = -105.0352$. The response of the system (5.2) is plotted in Figure 8.

6. Conclusion

In this paper, we showed that the stability of some linear systems of DDEs cannot be analyzed by the direct and cluster treatment methods. For analyzing the stability

FIGURE 7. Time domain response of the system (5.1). The solid line depicts $z_1(t)$ and dots display $z_2(t)$.



of such systems, we decomposed them into several subsystems by an invertible transformation. Furthermore, we discussed the problem of stabilization of an LSDDE by the delayed feedback method. Also, we improved the performance of an LSDDE by this type of feedback. Further studies may be considered to evaluate the stability and stabilization of neutral type systems.

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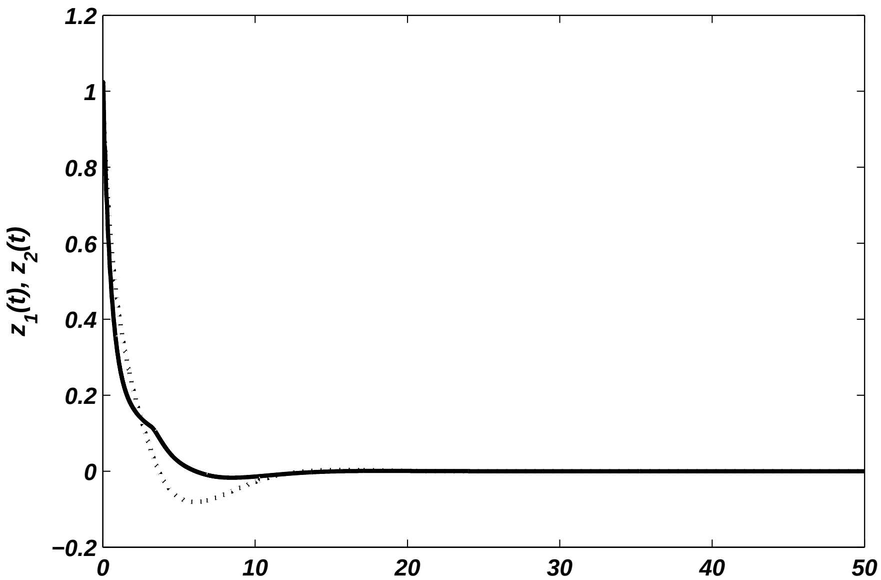
Figure 8. Time domain response of the system (5.2) for $\tau = 0.1$, $k_1 = 40.5925$ and $k_2 = -105.0352$. The solid line depicts $z_1(t)$ and dots display $z_2(t)$.



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