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The numerical values of the nodal points for the Sturm-Liouville equation with one turning point

Abdol Ali Neamaty*
Department of Mathematics,
University of Mazandaran, Babolsar, Iran.
E-mail: namaty@umz.ac.ir

Najibeh Yousefi
Department of Mathematics,
University of Mazandaran, Babolsar, Iran.
E-mail: nah.yosofy@gmail.com

Abdol Hadi Dabbaghian
Department of Mathematics,
Islamic Azad University, Neka Branch, Neka, Iran.
E-mail: a.dabbaghian@iauneka.ac.ir

Abstract An inverse nodal problem has first been studied for the Sturm-Liouville equation with one turning point. The asymptotic representation of the corresponding eigenfunctions of the eigenvalues has been investigated and an asymptotic of the nodal points is obtained. For this problem, we give a reconstruction formula for the potential function. Furthermore, numerical examples have been established and results have been illustrated in tables and graphics.

Keywords. Turning point, Inverse nodal problem, Nodal Points, Eigenvalues, Eigenfunctions.

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1. Introduction

In the literature review of mathematics, a large number of research studies has been devoted entirely or partially to the study of the Sturm-Liouville equation i.e.,

$$y''(x) + (\lambda\phi^2(x) - q(x))y = 0, \tag{1.1}$$

where $\lambda = \rho^2$ and the real valued functions ϕ^2 and q are said to be the coefficients of the problem, ϕ^2 is the weight and q is the potential function. The zeros of ϕ^2 are called turning points of (1.1). Differential equations with turning points play an important role in various areas of mathematics and other branches of natural sciences. For example, in elasticity, optics, geophysics(see [8, 11, 18] and the references therein).

Inverse spectral problems consist in recovering operators from their spectral characteristics. The first spectral problem was given by Ambarzumyan [4]. Since 1945, various forms of the inverse problems have been considered by several authors (see, for

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* Corresponding author.

example, [5, 10, 13, 15, 16, 17, 20, 23]). In particular, in later years, these problems were studied for Sturm-Liouville operators with turning points (refer to [3, 7, 9, 21] and the references therein).
Recently, some researchers have paid attention to a new class of inverse problem. This is the so-called inverse nodal problem. Inverse nodal problems consist in recovering operators from given nodes (zeros) of their eigenfunctions.
It seems that J.R. Mclaughlin [19] to be the first one who considered this sort of inverse problem, in 1988. Later on, these results expanded to some problems with different conditions. For example, X.F. Yang got the uniqueness for general boundary conditions using the same method as Mclaughlin (see [24]). Besides, several authors have studied inverse nodal problems for different operators (see [6, 12, 22] and other works).
In the references cited above, the inverse nodal problems were studied for second-order differential equations without turning points. In this work, we consider the Dirichlet problem

$$y''(x) + (\lambda x - q(x))y = 0, \qquad -1 \leq x \leq 1, \tag{1.2}$$

$$y(-1) = 0 = y(s), \tag{1.3}$$

with variable x on $(-1, s)$, $s \in [-1, 1]$ is fixed, $q(x)$ is a continuous function in the interval [-1,1] and λ is a real parameter.
The purpose of this paper is to present a method for solving the inverse nodal problem related to (1.2) where there is a turning point in [-1,1].
The paper is organized as follows. In the next section, we investigate the asymptotic behavior of the eigenvalues and the eigenfunctions and derive a detailed asymptotic formulas for the nodal points. In section 3, we give a reconstruction formula for the potential function q. In section 4, we have considered the eigenfunctions and nodal parameters in the numerical examples.

2. Asymptotic of the nodal points

Let $C(x, \lambda)$ is a solution for equation(1.2) with the initial conditions

$$C(-1, \lambda) = 0, \qquad \frac{\partial C}{\partial x}(-1, \lambda) = 1. \tag{2.1}$$

The function $C(s, \lambda)$ has a zero set for each s, say $\{\lambda_n(s)\}$, so that $C(s, \lambda_n(s)) = 0$, which corresponds to eigenvalues of the Dirichlet problem for equation (1.2) on the closed interval $[-1, s]$. Note that $\lambda_n(s) \neq 0$ for any s by Sturm's comparison theorem since we assume that $0 \leq q(x)$.
It is known that for a non-negative continuous function $q(x)$, the eigenvalues of the Dirichlet problem for (1.2) on $[-1, s]$, are real. Therefore, the eigenfunctions are real-valued.

The function $C(x,\lambda)$ satisfies the integral equations

$$C(x,\lambda)=\begin{cases}\frac{1}{\sqrt{\lambda}}(-x)^{-\frac{1}{4}}\sinh p(x)\sqrt{\lambda} \\ +\frac{1}{\sqrt{\lambda}}\int_{-1}^{x}(xt)^{-\frac{1}{4}}\sinh\sqrt{\lambda}(p(x)-p(t))q(t)C(t,\lambda)dt, & -1\le x<0,\\ \frac{x^{-\frac{1}{4}}}{\sqrt{\lambda}}\{e^{\frac{2}{3}\sqrt{\lambda}}\cos(\frac{2}{3}x^{\frac{3}{2}}\sqrt{\lambda}-\frac{\pi}{4})+e^{-\frac{2}{3}\sqrt{\lambda}}\frac{1}{2}\sin(\frac{2}{3}x^{\frac{3}{2}}\sqrt{\lambda}-\frac{\pi}{4})\} \\ +\frac{1}{\sqrt{\lambda}}\int_{0}^{x}(xt)^{-\frac{1}{4}}\sin\sqrt{\lambda}(f(x)-f(t))q(t)C(t,\lambda)dt, & x>0,\end{cases} \tag{2.2}$$

where $f(x)=\int_0^x\sqrt{\nu}d\nu$ and $p(x)=\int_{-1}^{x}\sqrt{-\nu}d\nu$.
In [2], it was shown that

$$C(x,\lambda)=\begin{cases}\frac{1}{\sqrt{\lambda}}(-x)^{-\frac{1}{4}}\sinh(p(x)\sqrt{\lambda})+O(\frac{1}{\lambda}\exp(|\sqrt{\lambda}|x)), & -1\le x<0,\\ \frac{x^{-\frac{1}{4}}}{\sqrt{\lambda}}\{e^{\frac{2}{3}\sqrt{\lambda}}\cos(\frac{2}{3}x^{\frac{3}{2}}\sqrt{\lambda}-\frac{\pi}{4})+e^{-\frac{2}{3}\sqrt{\lambda}}\frac{1}{2}\sin(\frac{2}{3}x^{\frac{3}{2}}\sqrt{\lambda}-\frac{\pi}{4})\} \\ +O(\frac{1}{\lambda}\exp(|\sqrt{\lambda}|x)),\ x>0.\end{cases} \tag{2.3}$$

The Dirichlet problem corresponding to equation (1.2) on $[-1,s]$, where $s<0$ is fixed, has an infinite number of negative eigenvalues $\{\lambda_n^{(1)}(s)\}$. The asymptotic distribution of each function $\lambda_n^{(1)}(s)$ is of the form

$$\sqrt{-\lambda_n^{(1)}(s)}=\frac{n\pi}{\int_{-1}^{s}\sqrt{-t}dt}+O(\frac{1}{n}),\qquad s<0. \tag{2.4}$$

For $s\in(0,1]$, fixed, the Dirichlet problem of (1.2) on $[-1,s]$ has an infinite number of positive and negative eigenvalues which we denote by $\{\lambda_n^{(2)}(s)\}$, $\{\lambda_n^{(3)}(s)\}$, respectively. The positive eigenvalues $\lambda_n^{(2)}(s)$ admit the asymptotic representation

$$\sqrt{\lambda_n^{(2)}(s)}=\frac{n\pi-\frac{\pi}{4}}{\int_0^s\sqrt{t}dt}+\frac{1}{2n\pi}T_1+O(\frac{1}{n^2}), \tag{2.5}$$

where

$$T_1=\frac{5}{72\int_0^s\sqrt{t}dt}+\frac{1}{2}\int_0^s\frac{q(t)}{\sqrt{t}}dt.$$

Similarly, the negative eigenvalues, $\lambda_n^{(3)}(s)$, admit the asymptotic representation of the form

$$\sqrt{-\lambda_n^{(3)}(s)}=\frac{n\pi-\frac{\pi}{4}}{\int_{-1}^{0}\sqrt{-t}dt}+\frac{1}{2n\pi}T_2+O(\frac{1}{n^2}), \tag{2.6}$$

where

$$T_2=\frac{5}{72\int_{-1}^{0}\sqrt{-t}dt}+\frac{1}{2}\int_{-1}^{0}\frac{q(t)}{\sqrt{-t}}dt.$$

For more details see [1].
The following theorem gives asymptotic representation for the eigenfunctions of the Dirichlet problem of the equation (1.2).

Theorem 2.1. *([2,Theorem 1]) Let $C(x,\lambda)$ be the solution of the Dirichlet problem (1.2) and (1.3) with variable x on $(-1,s)$, for fixed s, which satisfies the initial condition (2.1). Then*
a) For $s\in[-1,0)$ fixed, the corresponding eigenfunctions of the negative eigenvalues $\lambda_n^{(1)}(s)$, has the asymptotic representation

$$C(x,\lambda_n^{(1)}(s))=\frac{p(s)}{(-x)^{\frac{1}{4}}n\pi}\sin\frac{n\pi p(x)}{p(s)}+O(\frac{1}{n^2}). \tag{2.7}$$

b) For $s\in(0,1]$ fixed, the corresponding eigenfunctions of the positive eigenvalues, $\lambda_n^{(2)}(s)$, admit the asymptotic representation

$$C(x,\lambda_n^{(2)}(s))=\frac{e^{\frac{2}{3}(\frac{n\pi-\frac{\pi}{4}}{f(s)})}\cos[f(x)(\frac{n\pi-\frac{\pi}{4}}{f(s)})-\frac{\pi}{4}]}{x^{\frac{1}{4}}(\frac{n\pi-\frac{\pi}{4}}{f(s)})}+O(\frac{1}{n^2}). \tag{2.8}$$

c) For $s\in(0,1]$ fixed, the corresponding eigenfunctions of the negative eigenvalues, $\lambda_n^{(3)}(s)$, admit the asymptotic representation

$$C(x,\lambda_n^{(3)}(s))=\frac{2e^{(n\pi-\frac{\pi}{4})i}\cos[x^{\frac{3}{2}}(n\pi-\frac{\pi}{4})i-\frac{\pi}{4}]}{3x^{\frac{1}{4}}(n\pi-\frac{\pi}{4})i}+O(\frac{1}{n^2}).$$

Suppose $x_j^{(i)n}$, is the jth nodal point of the eigenfunction $C(x,\lambda_n^{(i)})$, $i\in\{1,2\}$. In other words, $C(x_j^{(i)n},\lambda_n^{(i)})=0$. Denote $X^{(i)}=\{x_j^{(i)n}\}_{n\geq1,j=\overline{1,n}}$. $X^{(i)}$ is called the set of nodal points.
Let $I_j^{(1)n}=[x_j^{(1)n},x_{j+1}^{(1)n}]$ be the j-th nodal domain of the n-th eigenfunction and let $l_j^{(1)n}=|I_j^{(1)n}|=x_{j+1}^{(1)n}-x_j^{(1)n}$ be the nodal length. We also define the function $j_n(x)$ to be the largest index j such that $-1\leq x_j^{(1)n}\leq x$.
Here we give asymptotic formulas of the nodal points for the problem (1.2)-(1.3).

Theorem 2.2. *We obtain the asymptotic formulae of the nodal points for the eigenfunction $C(x,\lambda_n^{(1)})$:*

$$x_j^{(1)n}=-1+\frac{jp(s)}{n}+O(\frac{1}{n^2}),\quad x<0,$$

as $n\to\infty$ uniformly in j.

Proof. For a fixed n, using (2.7), we arrive at

$$\sqrt{-\lambda_n^{(1)}(s)}C(x,\lambda_n^{(1)}(s))=(-x)^{-\frac{1}{4}}\sin\frac{n\pi}{p(s)}p(x)+\epsilon_n(x),$$

where $\epsilon_n(x)=O(\frac{1}{n})$. From

$$0=(-x)^{-\frac{1}{4}}\sin\frac{n\pi}{p(s)}p(x)+\epsilon_n(x),$$

we obtain

$$\frac{p(x)n\pi}{p(s)}=j\pi+(-1)^{j+1}\arcsin\epsilon_n(x),$$

which is equivalent for large n to

$$p(x) = X_n^j(p(x)) := \frac{jp(s)}{n} + \epsilon_n^j(x), \qquad j = \overline{1,n}, \tag{2.9}$$

where $\epsilon_n^j(x) = p(s)(-1)^{j+1}\frac{\arcsin \epsilon_n(x)}{n\pi}$. For $n \to \infty$:

$$\epsilon_n^j(x) = O(\frac{1}{n^2}), \tag{2.10}$$

uniformly in j. Consider the equation (2.9) on $\mathbb{R}$. According to (2.10) and since there exists $\theta \in (x_1, x_2)$ such that

$$X_n^j(p(x_1)) - X_n^j(p(x_2)) = (\epsilon_n^j(x))'(\theta)(x_1 - x_2),$$

there exists N_0 such that for all $n > N_0$ the function $X_n^j(p(x))$ is a contracting mapping in $\mathbb{R}$ for $j = \overline{1,n}$. Let $n > N_0$. Thus, for each $j = \overline{1,n}$ the equation (2.9) has a unique solution in $\mathbb{R}$, which we denote by $p(x_j^{(1)n})$. From (2.9) it follows that

$$\int_{-1}^{x_j^{(1)n}} \sqrt{-\upsilon}d\upsilon = p(x_j^{(1)n}) = \frac{jp(s)}{n} + O(\frac{1}{n^2}), \qquad n \to \infty$$

uniformly with respect to j. Solving the integral, we have

$$x_j^{(1)n} = -1 + \frac{jp(s)}{n} + O(\frac{1}{n^2}).$$

Hence, the function $C(x, \lambda_n^{(1)}(s))$ has exactly $n-1$ zeros inside the interval (-1,0), namely: $-1 < x_1^{(1)n} < ... < x_{n-1}^{(1)n} < 0$. $\diamond$

Theorem 2.3. *We obtain the asymptotic formulae of the nodal points for the eigenfunction $C(x, \lambda_n^{(2)})$:*

$$x_j^{(2)n} = [\frac{3(j-\frac{1}{2})}{2(n-\frac{1}{4})}f(s)]^{\frac{2}{3}} + O(\frac{1}{n^2}), \quad x > 0,$$

as $n \to \infty$ uniformly in j.

Proof. For a fixed n, using (2.8), we arrive at

$$\sqrt{\lambda_n^{(2)}(s)}C(x, \lambda_n^{(2)}(s)) = \frac{e^{\frac{2}{3}(\frac{n\pi-\frac{\pi}{4}}{f(s)})}\cos[f(x)(\frac{n\pi-\frac{\pi}{4}}{f(s)}) - \frac{\pi}{4}]}{x^{\frac{1}{4}}} + O(\frac{1}{n}).$$

From

$$0 = \frac{e^{\frac{2}{3}(\frac{n\pi-\frac{\pi}{4}}{f(s)})}\cos[f(x)(\frac{n\pi-\frac{\pi}{4}}{f(s)}) - \frac{\pi}{4}]}{x^{\frac{1}{4}}} + O(\frac{1}{n}),$$

and Taylor's expansion of $\arccos(t)$, we obtain the following formula as $n \to \infty$ uniformly in j:

$$\int_0^{x_j^{(2)n}} \sqrt{\upsilon}d\upsilon = f(x_j^{(2)n}) = \frac{j-\frac{1}{2}}{n-\frac{1}{4}}f(s) + O(\frac{1}{n^2}).$$

Solving the integral, we have

$$x_j^{(2)n} = [\frac{3(j-\frac{1}{2})}{2(n-\frac{1}{4})} f(s)]^{\frac{2}{3}} + O(\frac{1}{n^2}).$$

Remark 2.4. For $s \in (0,1]$ fixed, the corresponding eigenfunctions of the negative eigenvalues $\lambda_n^{(3)}(s)$ have no zeros.

For solving the inverse nodal problem we need a more detailed asymptotic formulas of the nodal points. Hence, we state a theorem which gives more precise asymptotic approximation for the eigenfunctions $C(x, \lambda_n^{(i)})$ where $i \in \{1,2,3\}$.

Theorem 2.5. *a) For $s \in [-1,0)$ fixed, the corresponding eigenfunctions of the negative eigenvalues, $\lambda_n^{(1)}(s)$, have the asymptotic representation:*

$$C(x, \lambda_n^{(1)}(s)) = \frac{p(s)(-x)^{-\frac{1}{4}}}{n\pi} \sin \frac{n\pi p(x)}{p(s)}$$
$$- \frac{p^2(s)(-x)^{-\frac{1}{4}}}{n^2\pi^2} \cos \frac{n\pi p(x)}{p(s)} \int_{-1}^{x} (-t)^{-\frac{1}{2}} q(t) \sin^2 \frac{n\pi p(t)}{p(s)} dt + O(\frac{1}{n^3}). \quad (2.11)$$

b) For $s \in (0,1]$ fixed, the corresponding eigenfunctions of the positive eigenvalues, $\lambda_n^{(2)}(s)$, admit the asymptotic representation:

$$C(x, \lambda_n^{(2)}(s)) = \frac{x^{-\frac{1}{4}} e^{\frac{2}{3}(\frac{n\pi-\frac{\pi}{4}}{f(s)})} \cos[f(x)(\frac{n\pi-\frac{\pi}{4}}{f(s)}) - \frac{\pi}{4}]}{(\frac{n\pi-\frac{\pi}{4}}{f(s)})} + \frac{x^{-\frac{1}{4}} e^{\frac{2}{3}(\frac{n\pi-\frac{\pi}{4}}{f(s)})}}{(\frac{n\pi-\frac{\pi}{4}}{f(s)})^2}$$
$$\times [\sin[f(x)(\frac{n\pi-\frac{\pi}{4}}{f(s)}) - \frac{\pi}{4}] \int_0^x t^{-\frac{1}{2}} q(t) \cos^2[f(t)(\frac{n\pi-\frac{\pi}{4}}{f(s)}) - \frac{\pi}{4}] dt] + O(\frac{1}{n^3}). \quad (2.12)$$

c) For $s \in (0,1]$ fixed, the corresponding eigenfunctions of the negative eigenvalues, $\lambda_n^{(3)}(s)$, admit asymptotic representation,

$$C(x, \lambda_n^{(3)}(s)) = \frac{2x^{-\frac{1}{4}} e^{(n\pi-\frac{\pi}{4})i} \cos[x^{\frac{3}{2}}(n\pi-\frac{\pi}{4})i - \frac{\pi}{4}]}{3(n\pi-\frac{\pi}{4})i} + \frac{x^{-\frac{1}{4}} e^{(n\pi-\frac{\pi}{4})i}}{(\frac{n\pi-\frac{\pi}{4}}{\frac{2}{3}})^2}$$
$$\times [\sin[x^{\frac{3}{2}}(n\pi-\frac{\pi}{4})i - \frac{\pi}{4}] \int_0^x t^{-\frac{1}{2}} q(t) \cos^2[t^{\frac{3}{2}}(n\pi-\frac{\pi}{4})i - \frac{\pi}{4}] dt] + O(\frac{1}{n^3}). \quad (2.13)$$

Proof. a) In this case the eigenvalues are negative. Substituting the asymptotic form (2.4) in (2.2) and noting that $\sqrt{\lambda} = i\sqrt{-\lambda_n^{(1)}(s)}$ we can get

$$C(x, \lambda_n^{(1)}(s)) = \frac{1}{i\sqrt{-\lambda_n^{(1)}(s)}} (-x)^{-\frac{1}{4}} \sinh(ip(x)\sqrt{-\lambda_n^{(1)}(s)})$$
$$+ \frac{1}{i\sqrt{-\lambda_n^{(1)}(s)}} \int_{-1}^{x} (xt)^{-\frac{1}{4}} \sinh(i(p(x)-p(t))\sqrt{-\lambda_n^{(1)}(s)}) q(t) C(t, \lambda_n^{(1)}(s)) dt.$$

Moreover, substituting the asymptotic form of $C(x,\lambda)$ from (2.3), we calculate

$$C(x,\lambda_n^{(1)}(s)) = \frac{(-x)^{-\frac{1}{4}}}{\frac{n\pi}{p(s)}+O(\frac{1}{n})}[\sin\frac{p(x)n\pi}{p(s)}\cos O(\frac{1}{n})+\cos\frac{p(x)n\pi}{p(s)}\sin O(\frac{1}{n})]$$

$$-\frac{(-x)^{-\frac{1}{4}}}{(\frac{n\pi}{p(s)}+O(\frac{1}{n}))^2}(\int_{-1}^{x}(-t)^{-\frac{1}{2}}q(t)[\sin\frac{p(t)n\pi}{p(s)}\cos O(\frac{1}{n})+\cos\frac{p(t)n\pi}{p(s)}\sin O(\frac{1}{n})]^2dt)$$

$$\times[\cos\frac{p(x)n\pi}{p(s)}\cos O(\frac{1}{n})-\sin\frac{p(x)n\pi}{p(s)}\sin O(\frac{1}{n})]+O(\frac{1}{n^3}).$$

Using the following facts for large n:

$$\cos O(\frac{1}{n}) = 1+O(\frac{1}{n^2}),\quad \sin O(\frac{1}{n}) = O(\frac{1}{n}),$$

we get the result.
Similarly by inserting the asymptotic formulae (2.5) and (2.6) into (2.2) we get the results (b) and (c).$\diamond$

Theorem 2.6. *We obtain the asymptotic formulae of the nodal points for the eigenfunction $C(x,\lambda_n^{(1)})$ as follows*

$$x_j^{(1)n} = -1+\frac{jp(s)}{n}+\frac{p^2(s)}{n^2\pi^2}\int_{-1}^{x_j^{(1)n}}\frac{q(t)}{(-t)^{\frac{1}{2}}}\sin^2\frac{n\pi p(t)}{p(s)}dt+O(\frac{1}{n^3}),\quad x<0, \tag{2.14}$$

as $n\to\infty$ uniformly in j. Hence, the nodal length is

$$l_j^{(1)n} = \frac{p(s)}{n}+\frac{p^2(s)}{n^2\pi^2}\int_{x_j^{(1)n}}^{x_{j+1}^{(1)n}}\frac{q(t)}{(-t)^{\frac{1}{2}}}\sin^2\frac{n\pi p(t)}{p(s)}dtdt+O(\frac{1}{n^3}). \tag{2.15}$$

Proof. For a fixed n, using (2.11), we arrive at

$$\sqrt{-\lambda_n^{(1)}(s)}C(x,\lambda_n^{(1)}(s)) = (-x)^{-\frac{1}{4}}\sin\sqrt{-\lambda_n^{(1)}(s)}p(x)$$

$$-\frac{(-x)^{-\frac{1}{4}}}{\sqrt{-\lambda_n^{(1)}(s)}}\cos\sqrt{-\lambda_n^{(1)}(s)}p(x)\int_{-1}^{x}(-t)^{-\frac{1}{2}}q(t)\sin^2\frac{n\pi p(t)}{p(s)}dt++O(\frac{1}{n^2}).$$

From

$$0 = (-x)^{-\frac{1}{4}}\sin\sqrt{-\lambda_n^{(1)}(s)}p(x)$$

$$-\frac{(-x)^{-\frac{1}{4}}}{\sqrt{-\lambda_n^{(1)}(s)}}\cos\sqrt{-\lambda_n^{(1)}(s)}p(x)\int_{-1}^{x}(-t)^{-\frac{1}{2}}q(t)\sin^2\frac{n\pi p(t)}{p(s)}dt+O(\frac{1}{n^2}),$$

we obtain

$$\tan\sqrt{-\lambda_n^{(1)}(s)}p(x) = \frac{\int_{-1}^{x}(-t)^{-\frac{1}{2}}q(t)\sin^2\frac{n\pi p(t)}{p(s)}dt}{\sqrt{-\lambda_n^{(1)}(s)}}+O(\frac{1}{n^2}).$$

Using Taylor's expansion of the arctangent function, we obtain the following formulae, as $n \to \infty$ uniformly in $j \in N$

$$\frac{n\pi}{p(s)}p(x_j^{(1)n}) = j\pi + \frac{p(s)}{n\pi}\int_{-1}^{x_j^{(1)n}} (-t)^{-\frac{1}{2}}q(t)\sin^2\frac{n\pi p(t)}{p(s)}dt + O(\frac{1}{n^2}),$$

which implies

$$\int_{-1}^{x_j^{(1)n}} \sqrt{-\nu}d\nu = p(x_j^{(1)n}) = \frac{jp(s)}{n} + \frac{p^2(s)}{n^2\pi^2}\int_{-1}^{x_j^{(1)n}} (-t)^{-\frac{1}{2}}q(t)\sin^2\frac{n\pi p(t)}{p(s)}dt + O(\frac{1}{n^3}).$$

Solving the integral, we have

$$x_j^{(1)n} = -1 + \frac{jp(s)}{n} + \frac{p^2(s)}{n^2\pi^2}\int_{-1}^{x_j^{(1)n}} \frac{q(t)}{(-t)^{\frac{1}{2}}}\sin^2\frac{n\pi p(t)}{p(s)}dt + O(\frac{1}{n^3}), \quad x<0.$$

The nodal length is

$$l_j^{(1)n} = x_{j+1}^{(1)n} - x_j^{(1)n} = \frac{p(s)}{n} + \frac{p^2(s)}{n^2\pi^2}\int_{x_j^{(1)n}}^{x_{j+1}^{(1)n}} \frac{q(t)}{(-t)^{\frac{1}{2}}}\sin^2\frac{n\pi p(t)}{p(s)}dt + O(\frac{1}{n^3}).\diamond$$

Corollary 2.7. *From Theorem 2.6 it follows that the set $X^{(1)} = \{x_j^{(1)n}\}$ is dense in [-1,0).*

Theorem 2.8. *We obtain the asymptotic formula of the nodal points for the eigenfunction $C(x, \lambda_n^{(2)})$ as follows:*

$$x_j^{(2)n} = [\frac{3}{2}\frac{(j-\frac{1}{4})f(s)}{n-\frac{1}{4}}]^{\frac{2}{3}} - \frac{1}{\sqrt[3]{[\frac{3}{2}\frac{(j-\frac{1}{4})f(s)}{n-\frac{1}{4}}]}}$$

$$\times[\frac{f^2(s)}{(n\pi-\frac{\pi}{4})^2}\int_0^{x_j^{(2)n}} \frac{q(t)}{t^{\frac{1}{2}}}\cos^2[f(t)(\frac{n\pi-\frac{\pi}{4}}{f(s)}) - \frac{\pi}{4}]dt + O(\frac{1}{n^2})], x>0 \qquad (2.16)$$

as $n \to \infty$ uniformly in j.

Proof. For a fixed n, using (2.12), we arrive at

$$\sqrt{\lambda_n^{(2)}(s)}C(x,\lambda_n^{(2)}(s)) = x^{-\frac{1}{4}}e^{\frac{2}{3}(\frac{n\pi-\frac{\pi}{4}}{f(s)})}\cos[f(x)(\frac{n\pi-\frac{\pi}{4}}{f(s)}) - \frac{\pi}{4}]$$

$$+\frac{x^{-\frac{1}{4}}e^{\frac{2}{3}(\frac{n\pi-\frac{\pi}{4}}{f(s)})}}{\sqrt{\lambda_n^{(2)}(s)}}\sin[f(x)(\frac{n\pi-\frac{\pi}{4}}{f(s)}) - \frac{\pi}{4}]\int_0^x t^{-\frac{1}{2}}q(t)\cos^2[f(t)(\frac{n\pi-\frac{\pi}{4}}{f(s)}) - \frac{\pi}{4}]dt + O(\frac{1}{n^2}).$$

From

$$0 = x^{-\frac{1}{4}}e^{\frac{2}{3}(\frac{n\pi-\frac{\pi}{4}}{f(s)})}\cos[f(x)(\frac{n\pi-\frac{\pi}{4}}{f(s)}) - \frac{\pi}{4}]$$

$$+\frac{x^{-\frac{1}{4}}e^{\frac{2}{3}(\frac{n\pi-\frac{\pi}{4}}{f(s)})}}{\sqrt{\lambda_n^{(2)}(s)}}\sin[f(x)(\frac{n\pi-\frac{\pi}{4}}{f(s)}) - \frac{\pi}{4}]\int_0^x t^{-\frac{1}{2}}q(t)\cos^2[f(t)(\frac{n\pi-\frac{\pi}{4}}{f(s)}) - \frac{\pi}{4}]dt + O(\frac{1}{n^2}),$$

we obtain

$$\cot(f(x)(\frac{n\pi-\frac{\pi}{4}}{f(s)})-\frac{\pi}{4}) = -\frac{f(s)}{n\pi-\frac{\pi}{4}}\int_0^x t^{-\frac{1}{2}}q(t)\cos^2[f(t)(\frac{n\pi-\frac{\pi}{4}}{f(s)})-\frac{\pi}{4}]dt + O(\frac{1}{n^2}).$$

Using Taylor's expansion of arccos(t), we obtain the following formula, as $n\to\infty$ uniformly in $j\in N$

$$f(x_j^{(2)n})(\frac{n\pi-\frac{\pi}{4}}{f(s)})-\frac{\pi}{4} = (j-\frac{1}{2})\pi - \frac{f(s)}{n\pi-\frac{\pi}{4}}\int_0^{x_j^{(2)n}} t^{-\frac{1}{2}}q(t)\cos^2[f(t)(\frac{n\pi-\frac{\pi}{4}}{f(s)})-\frac{\pi}{4}]dt + O(\frac{1}{n^2}),$$

which implies

$$\int_0^{x_j^{(2)n}}\sqrt{\upsilon}d\upsilon = f(x_j^{(2)n}) = \frac{j-\frac{1}{4}}{n-\frac{1}{4}}f(s) - \frac{f(s)^2}{(n\pi-\frac{\pi}{4})^2}\int_0^{x_j^{(2)n}} t^{-\frac{1}{2}}q(t)\cos^2[f(t)(\frac{n\pi-\frac{\pi}{4}}{f(s)})-\frac{\pi}{4}]dt + O(\frac{1}{n^3}).$$

Solving the integral, we have

$$x_j^{(2)n} = [\frac{3}{2}\frac{(j-\frac{1}{4})f(s)}{n-\frac{1}{4}}]^{\frac{2}{3}} - \frac{1}{\sqrt[3]{[\frac{3}{2}\frac{(j-\frac{1}{4})f(s)}{n-\frac{1}{4}}]}} \times [\frac{f^2(s)}{(n\pi-\frac{\pi}{4})^2}\int_0^{x_j^{(2)n}}\frac{q(t)}{t^{\frac{1}{2}}}\cos^2[f(t)(\frac{n\pi-\frac{\pi}{4}}{f(s)})-\frac{\pi}{4}]dt + O(\frac{1}{n^2})].\diamond \tag{2.17}$$

Corollary 2.9. *From theorem 2.8 it follows that the set* $X^{(2)} = \{x_j^{(2)n}\}$ *is dense in (0,1].*

3. Reconstruction of the potential function

We consider the following inverse nodal problem.

Problem. Fix $i\in\{1,2\}$. From given nodal points set $X^{(i)}$ which is dense in (-1,1), how to find the potential q?

Theorem 3.1. *Assume that* $q\in L^1[-1,0]$*, then*

$$q(x) = (-x)^{\frac{1}{2}}\lim_{n\to\infty}2(\rho_n^{(1)})^2(\frac{\rho_n^{(1)}l_j^{(1)n}}{\pi}-1), \tag{3.1}$$

for almost everywhere $x\in(-1,0)$*, with* $j=j_n(x)$*.*

Proof. We take $\rho_n^{(1)} = \sqrt{-\lambda_n^{(1)}}$. By (2.15), we have

$$l_j^{(1)n} = \frac{\pi}{\rho_n^{(1)}} + \frac{1}{(\rho_n^{(1)})^2}\int_{x_j^{(1)n}}^{x_{j+1}^{(1)n}}\frac{q(t)}{(-t)^{\frac{1}{2}}}\sin^2\frac{n\pi p(t)}{p(s)}dt + O(\frac{1}{n^3})$$

$$= \frac{\pi}{\rho_n^{(1)}} + \frac{1}{2(\rho_n^{(1)})^2}\int_{x_j^{(1)n}}^{x_{j+1}^{(1)n}}\frac{q(t)}{(-t)^{\frac{1}{2}}}dt - \frac{1}{2(\rho_n^{(1)})^2}\int_{x_j^{(1)n}}^{x_{j+1}^{(1)n}}\frac{q(t)}{(-t)^{\frac{1}{2}}}\cos(2p(t)\rho_n^{(1)})dt + O(\frac{1}{n^3}),$$

and

$$2(\rho_n^{(1)})^2(\frac{\rho_n^{(1)} l_j^{(1)n}}{\pi} - 1) = \frac{\rho_n^{(1)}}{\pi} \int_{x_j^{(1)n}}^{x_{j+1}^{(1)n}} \frac{q(t)}{(-t)^{\frac{1}{2}}} dt - \frac{\rho_n^{(1)}}{\pi} \int_{x_j^{(1)n}}^{x_{j+1}^{(1)n}} \frac{q(t)}{(-t)^{\frac{1}{2}}} \cos(2p(t)\rho_n^{(1)})dt + O(1)$$

$$= \frac{\rho_n^{(1)}}{\pi} \int_{x_j^{(1)n}}^{x_{j+1}^{(1)n}} \frac{q(t)}{(-t)^{\frac{1}{2}}} dt - H_n(x) + O(1),$$

where

$$H_n(x) = \frac{\rho_n^{(1)}}{\pi} \int_{x_j^{(1)n}}^{x_{j+1}^{(1)n}} \frac{q(t)}{(-t)^{\frac{1}{2}}} \cos(2p(t)\rho_n^{(1)})dt.$$

By the proof of [14, Lemma 3.2], the sequence of functions $H_n(x)$ tends to zero for almost every $x \in (-1, 0)$. Therefore,

$$\lim_{n\to\infty} 2(\rho_n^{(1)})^2(\frac{\rho_n^{(1)} l_j^{(1)n}}{\pi} - 1) = q(x)(-x)^{-\frac{1}{2}}.$$

Hence the proof of (3.1) is complete. ⋄

Theorem 3.2. *Assume that* $q \in L^1(0, 1)$*, then*

$$q(x) = \frac{2}{j} x^{\frac{1}{2}} \lim_{n\to\infty} (\rho_n^{(2)})^2 [\sqrt[3]{\frac{3(\rho_n^{(2)})^2 (j - \frac{1}{4})}{2\pi^2}} x_j^{(2)n} - \frac{3}{2}(j - \frac{1}{4})], \tag{3.2}$$

for almost everywhere $x \in (0, 1)$.

Proof. We take $\rho_n^{(2)} = \sqrt{\lambda_n^{(2)}}$. By (2.16), we have

$$x_j^{(2)n} \sqrt[3]{\frac{3\pi(j - \frac{1}{4})}{2\rho_n^{(2)}}} - (\frac{3\pi(j - \frac{1}{4})}{2\rho_n^{(2)}}) = \frac{1}{2(\rho_n^{(2)})^2} \int_0^{x_j^{(2)n}} \frac{q(t)}{(t)^{\frac{1}{2}}} dt$$

$$+ \frac{1}{2(\rho_n^{(2)})^2} \int_0^{x_j^{(2)n}} \frac{q(t)}{(t)^{\frac{1}{2}}} \cos 2(f(t)\rho_n^{(2)} - \frac{\pi}{4})dt + O(\frac{1}{n^3}).$$

By the same way as in the proof of Theorem 3.1, we obtain the following representation

$$2 \lim_{n\to\infty} (\rho_n^{(2)})^2 [\sqrt[3]{\frac{3(\rho_n^{(2)})^2 (j - \frac{1}{4})}{2\pi^2}} x_j^{(2)n} - \frac{3}{2}(j - \frac{1}{4})] = jq(x)x^{-\frac{1}{2}}.$$

Hence, the proof of (3.2) is complete.
⋄

4. Numerical Examples

In this section, we have considered the numerical examples about eigenfunction and nodal parameters for illustrating the theoretical results of the previous sections.

Example 1. We consider Eq.(1.2) for the special case of $q(x) = x$, $s = -0.1$ and $x \in (-1, 0)$.

TABLE 1. Detailed results for the nodal points of $x_j^{(1)n}$ where $j=\overline{1,8}$ and $n=\overline{1,8}$.

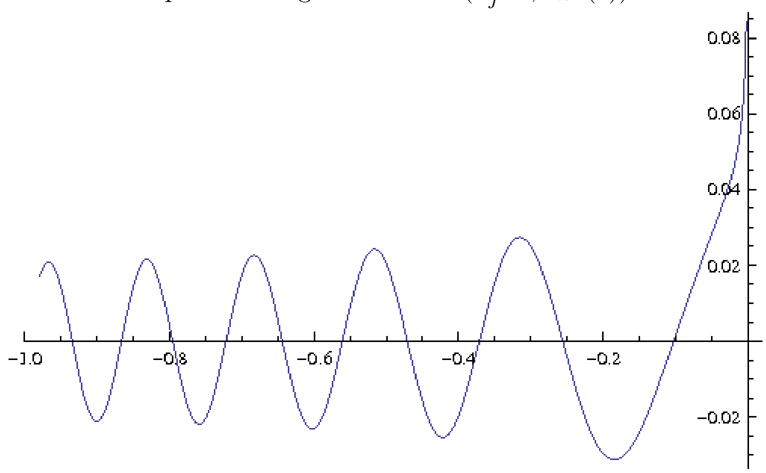
$x_j^{(1)n}$	j=1	j=2	j=3	j=4	j=5	j=6	j=7	j=8
n=1	-0.137721							
n=2	-0.645494	-0.112124						
n=3	-0.771919	-0.502777	-0.106618					
n=4	-0.831649	-0.644065	-0.423245	-0.104486				
n=5	-0.866544	-0.721873	-0.560833	-0.371741	-0.103512			
n=6	-0.889444	-0.771531	-0.643778	-0.501718	-0.335368	-0.103019		
n=7	-0.905631	-0.806062	-0.699902	-0.584949	-0.457277	-0.308165	-0.102717	
n=8	-0.917682	-0.831491	-0.740572	-0.643669	-0.538812	-0.422489	-0.286958	-0.102505

We can obtain the eigenfunction of this problem as

$$C(x_j^{(1)n},\lambda_n^{(1)}(s)) = \frac{p(s)(-x_j^{(1)n})^{-\frac{1}{4}}}{n\pi}\sin\frac{n\pi p(x_j^{(1)n})}{p(s)} - \frac{p(s)(-x_j^{(1)n})^{-\frac{1}{4}}}{n\pi}\cos\frac{n\pi p(x_j^{(1)n})}{p(s)}\int_{-1}^{x_j^{(1)n}}\frac{t}{(-t)^{\frac{1}{2}}}\sin^2\frac{n\pi p(t)}{p(s)}dt + O(\frac{1}{n^3}).$$

In Figure 1, we illustrate graph of the eigenfunction $C(x_j^{(1)n},\lambda_n^{(1)}(s))$ where $n=8$, $-1\le x<0$.
Using (2.14), we obtain

FIGURE 1. Graph of the eigenfunction $C(x_j^{(1)n},\lambda_n^{(1)}(s))$ where $n=8$.



$$x_j^{(1)n} = -1 + \frac{jp(s)}{n} + \frac{p^2(s)}{n^2\pi^2}\int_{-1}^{x_j^{(1)n}}\frac{t}{(-t)^{\frac{1}{2}}}\sin^2\frac{n\pi p(t)}{p(s)}dt + O(\frac{1}{n^3}), \quad x<0.$$

Example 2. We consider Eq.(1.2) for the special case of $q(x)=x$, $s=0.1$ and $x\in(0,1)$.

Table 2. Detailed results for the nodal points of $x_j^{(2)n}$ where $j = \overline{1, 8}$ and $n = \overline{1, 8}$.

$x_j^{(2)n}$	j=1	j=2	j=3	j=4	j=5	j=6	j=7	j=8
n=1	0.967396							
n=2	0.565054	0.994646						
n=3	0.419539	0.738236	0.997899					
n=4	0.341553	0.600941	0.812288	0.998887				
n=5	0.291896	0.513548	0.69415	0.853605	0.999312			
n=6	0.257053	0.452236	0.611272	0.751686	0.879995	0.999533		
n=7	0.231028	0.406443	0.549374	0.675568	0.790883	0.898316	0.999663	
n=8	0.210721	0.370714	0.501079	0.616179	0.721355	0.819343	0.91178	0.999745

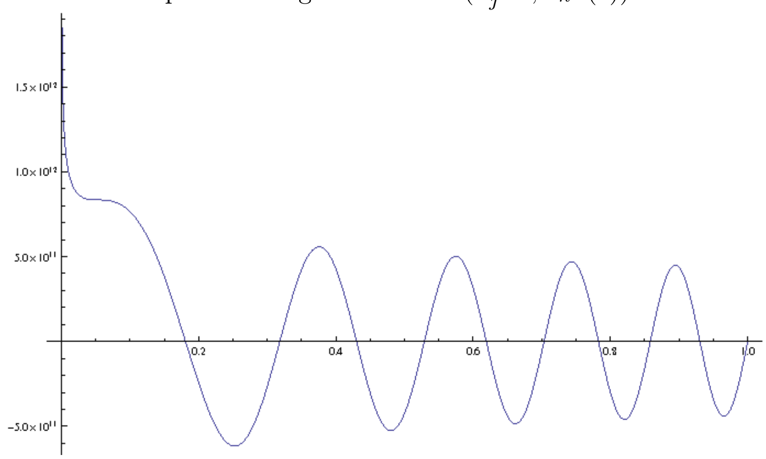
We can obtain the eigenfunction of this problem as

$$C(x, \lambda_n^{(2)}(s)) = \frac{x^{-\frac{1}{4}} e^{\frac{2}{3}(\frac{n\pi - \frac{\pi}{4}}{f(s)})} \cos[f(x)(\frac{n\pi-\frac{\pi}{4}}{f(s)}) - \frac{\pi}{4}]}{(\frac{n\pi-\frac{\pi}{4}}{f(s)})}$$

$$+ \frac{x^{-\frac{1}{4}} e^{\frac{2}{3}(\frac{n\pi - \frac{\pi}{4}}{f(s)})}}{\frac{n\pi-\frac{\pi}{4}}{f(s)}} \sin[f(x)(\frac{n\pi - \frac{\pi}{4}}{f(s)}) - \frac{\pi}{4}] \int_0^x t^{\frac{1}{2}} \cos^2[f(t)(\frac{n\pi - \frac{\pi}{4}}{f(s)}) - \frac{\pi}{4}]dt + O(\frac{1}{n^3}). \quad (4.1)$$

In Figure 2, we illustrate graph of the eigenfunction $C(x_j^{(2)n}, \lambda_n^{(2)}(s))$ where $n = 8$, $0 \leq x < 1$. Using (2.16), we obtain

Figure 2. Graph of the eigenfunction $C(x_j^{(1)n}, \lambda_n^{(2)}(s))$ where $n = 8$.



$$x_j^{(2)n} = [\frac{3}{2}\frac{(j-\frac{1}{4})f(s)}{n-\frac{1}{4}}]^{\frac{2}{3}} - \frac{1}{\sqrt[3]{[\frac{3}{2}\frac{(j-\frac{1}{4})f(s)}{n-\frac{1}{4}}]}}$$

$$\times [\frac{f^2(s)}{(n\pi - \frac{\pi}{4})^2} \int_0^{x_j^{(2)n}} \frac{t}{t^{\frac{1}{2}}} \cos^2[f(t)(\frac{n\pi - \frac{\pi}{4}}{f(s)}) - \frac{\pi}{4}]dt + O(\frac{1}{n^2})]. \quad (4.2)$$

5. Conclusions

In this study, the nodal points were used to solve the inverse Sturm-Liouville problem with a turning point. In the numerical examples, we have shown that the obtained nodes correspond to the roots of eigenfunction. Then we calculated the solution of inverse problem by using the nodal points.

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