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L^p-existence of mild solutions of fractional differential equations in Banach space

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Abstract We study the existence of mild solutions for semilinear fractional differential equations with nonlocal initial conditions in $L^p([0,1],E)$, where E is a separable Banach space. The main ingredients used in the proof of our results are measure of noncompactness, Darbo and Schauder fixed point theorems. Finally, an application is proved to illustrate the results of this work.

Keywords. Semilinear integro-differential fractional equations, Measure of noncompactness, Mild solutions, Schauder fixed point theorem, Darbo fixed point theorem.

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1. Introduction

Our aim in this paper is to discuss the existence and the uniqueness of the mild solution for fractional semilinear differential equation with local conditions :

$$\begin{cases} \frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f(t, x(t)), \quad t \in I, \\ \\ x(0) = g(x), \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $I = [0,1]$. $A : D(A) \subset E \to E$ is a infinitesimal generator of a strongly continuous semigroup of a bounded linear operator (i.e C_0-semigroup)$U(t)$ in a Banach space X and $f : [0,1] \times X \to X$ and $g : L^p([0,1]; X)$ are given X-valued functions.

The theory of fractional differential equations has attracted much authors due to their applications in valuable to various fields of science and engineering, see for example [3, 6, 7, 8, 9]. In this work, we shall improve some results and extend them to the $L^p([0,1]; X)$-case. By using some conditions on the functions g and f and by employing L^p-density results, we prove different existence results for problem (1.1). Our essential tools are the Darbo and Schauder fixed point theorems in Banach space. The article is organized as follows. Section 2 contains some preliminaries about fractional calculus and the Hausdorff's measure of noncompactness. In section 3 the existence result is given. For illustration, a partial integral differential system is worked out in section 4.

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2. Preliminaries tools

Let $(X, \|.\|)$ be a real Banach space. Denote by $C(I;X)$ the space of X-valued continuous functions on I equipped with the norm

$$|x| = \sup\{\|x(t)\| \ : \ t \in I\}$$

and by $L^p(I,X)$ the space of X-valued measurable functions on $[0,1]$ with $\int_0^1 \|x(t)\|^p dt < \infty$, provided with the norm $\|x(t)\|_p = \left(\int_0^1 \|x(t)\|^p dt\right)^{1/p}$. Next, we recall the following known definitions from the theory of fractional calculus. For more details, see [1].

Definition 2.1. The Riemann-Liouville fractional integral of $u : I \to X$ of order $\alpha \in (0,\infty)$ is defined by

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds.$$

Now let Φ_α be the Mainardi function:

$$\Phi_\alpha(z) = \sum_{n=0}^{+\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)},$$

then

1. $\Phi_\alpha(t) \geq 0$, for all $t > 0$
2. $\displaystyle\int_0^{+\infty} \Phi_\alpha(t)dt = 1$
3. $\displaystyle\int_0^{+\infty} t^\eta \Phi_\alpha(t)dt = \frac{\Gamma(1+\eta)}{\Gamma(1+\alpha\eta)}$, $\eta \in [0,1]$.

For the details we refer to [9]. We set

$$S_\alpha(t) = \int_0^{+\infty} \Phi_\alpha(s)U(st^\alpha)ds \tag{2.1}$$

and

$$P_\alpha(t) = \int_0^{+\infty} \alpha s \Phi_\alpha(s)U(st^\alpha)ds. \tag{2.2}$$

In what follows, we consider the C_0-semigroup $\{U(t)\}_{t>0}$ generated by A which is continuous and satisfies

$$\exists\ M > 0 \text{such that } M \ = \sup\{U(t) : \ t \geq 0\} < +\infty.$$

Before we proceed further, we give the some lemmas relative to operators S_α and P_α.

Lemma 2.2 ([9])**.** Let S_α and P_α be the operators defined respectively by (2.1) and (2.2). Then

i. $\|S_\alpha(t)x\| \leq M\|x\|$; $\|P_\alpha(t)x\| \leq \frac{\alpha M}{\Gamma(\alpha+1)}\|x\|$, for all $x \in E$ and $t \geq 0$.

ii. The operators $S_\alpha(t)(t \geq 0)$ and $P_\alpha(t)(t \geq 0)$ are strongly continuous.

Lemma 2.3. The operator I^α is continuous and bounded from $L^p([0,1],X)$ into itself and for every $f \in L^p([0,1],X)$

$$\|I^\alpha f\|_p \leq \frac{1}{\Gamma(\alpha+1)}\|f\|_p$$

Definition 2.4 ([9])**.** Let S_α and P_α be the operators defined respectively by (2.1) and (2.2). Then a continuous function $x : \mathbb{R}_+ \to E$ satisfying for any $t \geq 0$ the equation

$$x(t) = S_\alpha(t)g(x) + \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s,x(s))ds. \tag{2.3}$$

is called a mild solution of the equation (1.1)

Definition 2.5. (Nemytskii operator) Let $\Omega \subset \mathbb{R}$ be an open set and $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a given function. The Nemytskii's operator associated with f assigns to each function $x : \Omega \longrightarrow \mathbb{R}$, the function $N_f : \Omega \longrightarrow \mathbb{R}$, defined by

$$N_f x(t) = f(t,x(t)) \quad (x \in \Omega).$$

Theorem 2.6. ([4]) Let G be a measurable subset of $\mathbb{R}$ and let $f : G \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the Carathéodory's conditions i.e.

(i) $f(\cdot,x) : G \to \mathbb{R}$ is measurable for each $x \in \mathbb{R}$,

(ii) $f(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for a.e. $t \in G$.

Let $1 \leq p, r < \infty$, $a \in L^r(G)$, and assume that

$$|f(t,x)| \leq c|x|^{p/r} + a(t), \quad a.e.\ t \in G, \quad x \in \mathbb{R}. \tag{2.4}$$

Then the operator defined by

$$N_f x(t) = f(t,x(t)), \quad a.e.\ t \in G, \quad x \in L^p(G)$$

is bounded and continuous from $L^p([a,b])$ into $L^r([a,b])$.

We recall the Hausdorff measure of noncompactness $\mu(\cdot)$ defined on bounded subsets B of Banach space Y by

$$\mu(B) = \inf\{\varepsilon > 0; B \text{ has a finite } \varepsilon\text{-net in } Y\}$$

Some basic properties of $\mu(\cdot)$ are presented in the following lemma.

Lemma 2.7 ([2])**.** Let Y be a real Banach space and $B, C \subseteq Y$ be bounded, the following properties are satisfied:

(1) B is precompact if and only if $\mu(B) = 0$,
(2) $\mu(B) = \mu(\overline{B}) = \mu(\text{conv}\, B)$, where $\overline{B}$ and conv B mean the closure and convex hull of B respectively;
(3) $\mu(B) \leq \mu(C)$ when $B \subseteq C$;
(4) $\mu(B+C) \leq \mu(B) + \mu(C)$ where $B + C = \{x + y : \ x \in B, \ y \in C\}$;
(5) $\mu(B \cup C) \leq \max\{\mu(B), \mu(C)\}$;
(6) $\mu(\lambda B) = |\lambda|\mu(B)$ for any $\lambda \in \mathbb{R}$;

(7) If $\{W_n\}_{n=1}^{\infty}$ is a decreasing sequence of bounded closed nonempty subsets of Y and $\lim_{n\to\infty} \mu(W_n) = 0$, then $\cap_{n=1}^{\infty} W_n$ is nonempty and compact in Y.

Definition 2.8. The map $Q : W \subseteq Y \to Y$ is said to be a μ_Y-contraction if there exists a positive constant $k < 1$ such that $\mu(Q(C)) \leq k\mu(C)$ for any bounded closed subset $C \subseteq W$ where Y is a Banach space.

Lemma 2.9. ([2]). Let X be a real Banach space and $\Omega \subset X$ is bounded, closed and convex subset. Let further $F : \Omega \to \Omega$ be a continuous μ-contraction. Then, F has at least one fixed point in Ω.

We call $B \subset L^1((0,1);X)$ uniformly integrable if there exists $\eta \in L^1((0,1);\mathbb{R}_+)$ such that $\|u(t)\| \leq \eta(t)$ for all $u \in B$ and a.e. $t \in (0,1)$.

Lemma 2.10. ([5]) If $\{u_n\}_{n\geq 1} \subset L^1((0,1);X)$ be a uniformly integrable sequence, then $t \mapsto \mu(\{u_n(t)\}_{n\geq 1})$ is measurable and

$$\mu(\{\int_0^t u_n(s)ds\}_{n\geq 1}) \leq \int_0^t \mu(\{u_n(s)\}_{n\geq 1})ds. \tag{2.5}$$

Next we will give an equality for the Hausdorff measure of noncompactness in $L^p([0,1],X), p \geq 1$, which will be denoted by μ_p. This measure wil be useful in the next section.

Theorem 2.11. ([10]) Let $B \subset C([0,1],X)$ be bounded and equicontinuous on $(0,1]$. Then

$$\mu_p(B) = \Big(\int_0^1 (\mu(B(t)))^p dt\Big)^{1/p} \tag{2.6}$$

where $B(t) = \{u(t) \ : \ u \in B\} \subset X$.

3. Main results

In this section we use the measure of noncompactness of $L^p([0,1],X)$ to consider nonlocal problems (1.1) when g is continuous in the norm of $L^p([0,1],X)$. Suppose That X is a separable Banach space. In addition, we suppose that $\dfrac{1}{\alpha} < p < \dfrac{1}{1-\alpha}$. In what follows, we denote by $T_{\alpha,p} = (\dfrac{p-1}{\alpha p - 1})^{\frac{p-1}{p}}$ and $K_{\alpha,p} = \Big(\dfrac{1}{1+p(\alpha-1)}\Big)^{\frac{1}{p}}$. We will require the following assumptions.

(H_A) The C_0-semigroup $U(t)$ generated by A is equicontinuous,

(H_g) (i) $g : L^p([0,1],X) \to X$ is continuous,

(ii) there exist $c, d \geq 0$ such that $\|g(x)\| \leq a\|x\|_p + b$ for any $x \in L^p([0,1],X)$,

(iii) there exists $k_1 \geq 0$ such that for any $B \subset C([0,1];X)$ which is bounded and equicontinuous on $(0;1]$:

$$\mu(S_\alpha(t)g(B)) \leq k_1\mu_p(B) \ \text{ for a.e. } \ t \in [0,1];$$

(H_f) (i) the function $f : I[0,1] \times X \to X$ satisfies the Carathéodory conditions, i.e., $f(.,x)$ is measurable for all $x \in X$ and $f(t,.)$ is continuous for a.e. $t \in [0,1]$;

(ii) there exist $b \geq 0$ and $a \in L^{\frac{p}{p-1}}([0,1], \mathbb{R}_+)$ such that $\|f(t,x)\| \leq a_2(t) + b_2\|x\|^{p-1}$, for every $x \in X$ and for a.e. $t \in [0,1]$,

(iii) there exists $k_2 : \mathbb{R}_+ \to \mathbb{R}_+$ measurable and essentially bounded function on compact intervals of $\mathbb{R}_+$ such that for any $B \subset C([0,1]; X)$ which is bounded and equicontinuous on $(0,1]$

$$\mu(P_\alpha f(t,B)) \leq k_2(t)\mu(B) \text{ for a.e. } t \in [0,1];$$

(H) $k_1 + \dfrac{\alpha M T_{\alpha,p}}{\Gamma(\alpha+1)} \sup_{t\in[0,1]} k_2(t) < 1.$

Lemma 3.1. If the semigroup $P_\alpha(t)$ is equicontinuous and $p > \dfrac{1}{\alpha}$. Then, for any $B \subset L^1([0,T], X)$ which is uniformly integrable, then the set

$$V = \{v : v(t) = \int_0^t \frac{P_\alpha(t-s)}{(t-s)^{1-\alpha}} u(s)ds \text{ for some } u \in B\} \subset C([0,1], X)$$

is equicontinuous.

Proof. We need only to prove that V is equicontinuous at $t > 0$. Let $t > 0$ be fixed, $u \in B$ be such that

$$\|u(s)\| \leq \eta(s) \text{ a.e } s \in (0,1),$$

with $\eta \in L^p([0,1], \mathbb{R}_+)$. Then for any $\epsilon > 0$ there exists $\delta \in (0,t)$ such that

$$\int_J \eta^p(s)ds < \epsilon$$

$$\begin{aligned}
&\|v(t+h) - v(t)\| = \\
&\|\int_0^{t+h} (t+h-s)^{\alpha-1} S_\alpha(t+h-s)u(s)ds - \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)u(s)ds\| \\
&\leq \|\int_0^t ((t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}) S_\alpha(t+h-s)u(s)ds\| \\
&+ \|\int_0^t (t-s)^{\alpha-1}(S_\alpha(t+h-s) - S_\alpha(t-s))u(s)ds\| \\
&+ \|\int_t^{t+h} (t+h-s)^{\alpha-1} S_\alpha(t+h-s)u(s)ds\| \\
&\leq I_1 + I_2 + I_3,
\end{aligned}$$

where

$$I_1 = \|\int_0^t ((t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}) S_\alpha(t+h-s)u(s)ds\|,$$

$$I_2 = \|\int_t^{t+h} (t+h-s)^{\alpha-1} S_\alpha(t+h-s)u(s)ds\|,$$

$$I_3 = \|\int_0^t (t-s)^{\alpha-1}(S_\alpha(t+h-s) - S_\alpha(t-s))u(s)ds\|.$$

According to the lemma 2.3, we get,

$$
\begin{aligned}
I_1 &= \| \int_0^t ((t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}) S_\alpha(t+h-s) u(s) ds \| \\
&\leq \frac{\alpha M}{\Gamma(\alpha+1)} \int_0^t |(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}| \eta(s) ds \\
&\leq \frac{\alpha M}{\Gamma(\alpha+1)} \Big[\int_0^{t-\delta} |(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}| \eta(s) ds + \int_{t-\delta}^t (t-s)^{\alpha-1} \eta(s) ds \Big].
\end{aligned}
$$

According to lemma 2.3 and the assumption on η we have

$$
\lim_{h \to 0} \frac{\alpha M}{\Gamma(\alpha+1)} \int_0^{t-\delta} |(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}| \eta(s) ds = 0.
$$

For I_2, one has

$$
\begin{aligned}
I_2 &= \| \int_{t+h}^t (t+h-s)^{\alpha-1} S_\alpha(t+h-s) u(s) ds \| \\
&\leq \frac{\alpha M}{\Gamma(\alpha+1)} \int_{t+h}^t (t+h-s)^{\alpha-1} \eta(s) ds.
\end{aligned}
$$

Thus, one can see that I_2 tends to 0 as $h \to 0$.
As to I_3, one gets

$$
\begin{aligned}
I_3 &= \| \int_0^t (t-s)^{\alpha-1} [S_\alpha(t+h-s) - S_\alpha(t-s)] u(s) ds \| \\
&\leq \| \int_0^{t-\delta} (t-s)^{\alpha-1} [S_\alpha(t+h-s) - S_\alpha(t-s)] u(s) ds \| \\
&+ \| \int_{t-\delta}^t (t-s)^{\alpha-1} [S_\alpha(t+h-s) - S_\alpha(t-s)] u(s) ds \| \\
&\leq \int_0^{t-\delta} (t-s)^{\alpha-1} \| S_\alpha(t+h-s) - S_\alpha(t-s) \| \eta(s) ds \\
&+ \frac{2\alpha M}{\Gamma(\alpha+1)} \int_{t-\delta}^t (t-s)^{\alpha-1} \eta(s) ds \\
&\leq \| S_\alpha(\delta+h) - S_\alpha(\delta) \| \int_0^{t-\delta} (t-s)^{\alpha-1} \| S_\alpha(t-\delta-s) \| \eta(s) ds \\
&+ \frac{2\alpha M}{\Gamma(\alpha+1)} \int_{t-\delta}^t (t-s)^{\alpha-1} \eta(s) ds.
\end{aligned}
$$

Then, by application of the Holder inequality we get

$$\lim_{h\to 0^+}\|v(t+h)-v(t)\| \le \frac{3M}{\Gamma(\alpha)}\int_{t-\delta}^{t}(t-s)^{\alpha-1}\eta(s)ds$$
$$\le \frac{3\alpha M}{\Gamma(\alpha+1)}\Big(\int_{t-\delta}^{t}(t-s)^{(\alpha-1)\frac{p}{p-1}}ds\Big)^{\frac{p-1}{p}}\Big(\int_{t-\delta}^{t}\eta^p(s)ds\Big)^{\frac{1}{p}}$$
$$\le \frac{3\alpha M}{\Gamma(\alpha+1)}(\frac{p-1}{\alpha p-1})^{\frac{p-1}{p}}\delta^{\alpha-\frac{1}{p}}\epsilon^{\frac{1}{p}},$$

uniformly for $v\in V$. So V is equicontinuous at $t>0$ as ϵ is arbitrary and independent of v. □

Now we are ready to give the first existence result for problem (1.1) which is formulated in the following theorem.

Theorem 3.2. If the assumptions H_f, H_g and H_A are satisfied and the following inequality

$$cR+d+\frac{\alpha}{\Gamma(\alpha+1)}K_{\alpha,p}\Big(\|a\|_{\frac{p}{p-1}}+bR^{p-1}\Big)\le\frac{R}{M} \tag{3.1}$$

has at least one solution R_0, then the equation (1.1) has at least one mild solution.

Proof. We define the operator $F: L^p(0,1;X)\to L^p(0,1;X)$ by

$$(Fx)(t)=P_\alpha(t)g(x)+\int_0^t\frac{S_\alpha(t-s)}{(t-s)^{1-\alpha}}f(s,x(s))ds,\quad t\in[0,1]. \tag{3.2}$$

for all $x\in L^p([0,1],X)$. To prove the continuity of F , let $x\in L^p([a,b])$ and let $(x_n)_n$ be a sequence in $L^p([a,b])$ converging to x.

$$\|Fx_n(t)-Fx(t)\|$$
$$\le\|P_\alpha(t)[g(x_n)-g(x)]\|+\|\textstyle\int_0^1\frac{S_\alpha}{(t-s)^{1-\alpha}}[f(s,x_n(s))-f(s,x(s))]\|\,ds$$
$$\le M\|g(x_n)-g(x)\|+M\textstyle\int_0^1\frac{\|f(s,x_n(s))-f(s,x(s))\|}{(t-s)^{1-\alpha}}ds.$$

From the hypothesis $(H_f)(i)$, $(H_g)(i)$ and by using the properties of Hölder's inequality, we conclude that F is continuous from $L^p([0,1],X)$ to itself.

Next, let R_0 the solution of inequality (3.1) and define the set

$$\Omega=\{x\in L^p([0,1],X)\ :\|x(t)\|\ \le R_0\ \text{ for a.e. } t\in[0,1]\}.$$

Then $\Omega\subset L^p([0,1],X)$ is uniformly integrable, closed and convex in $L^p([0,1],X)$. For any $x\in\Omega$, we get

$$\|(Fx)(t)\|\le\|P_\alpha(t)g(x)\|+\|\int_0^t\frac{S_\alpha(t-s)}{(t-s)^{1-\alpha}}f(s,x(s))ds\|$$
$$\le M\|g(x)\|+\frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t\frac{a(s)+b\|x(s)\|^{p-1}}{(t-s)^{1-\alpha}}ds.$$

By the Holder inequality one has,

$$\begin{aligned}\|(Fx)(t)\| &\leq M\Big[c\|x\|_p + d\Big] + \frac{\alpha M}{\Gamma(\alpha+1)}K_{\alpha,p}\Big(\|a\|_{\frac{p}{p-1}} + b\|x\|_p^{p-1}\Big)\\ &\leq M\Big[cR_0 + d + \frac{\alpha}{\Gamma(\alpha+1)}K_{\alpha,p}\Big(\|a\|_{\frac{p}{p-1}} + bR_0^{p-1}\Big)\Big]\\ &\leq R_0\end{aligned}$$

Consequently $F(\Omega) \subseteq \Omega$.
Since $F(\Omega) \subset C([0,1],X)$ is bounded and equicontinuous on $(0,1]$, so is $conv(F\Omega)$. As X is separable, then from Lemma 2.10 and Theorem 2.11 for any $U \subset conv(F\Omega)$ we have

$$\begin{aligned}\mu((FU)(t)) &\leq \mu(P_\alpha(t)g(U) + \int_0^t \mu(\frac{S_\alpha(t-s)}{(t-s)^{1-\alpha}}f(s,U(s))ds\\ &\leq k_1\mu_p(U) + \int_0^t \frac{k_2(s)}{(t-s)^{1-\alpha}}\mu(U(s))ds\\ &\leq k_1\mu_p(U) + \frac{\widetilde{k}_2(t)}{\Gamma(\alpha)}\int_0^t \frac{\mu(U(s))}{(t-s)^{1-\alpha}}ds,\end{aligned}$$

for a.e. $t \in [0,1]$. By Theorem 2.11 and the Holder inequality, we get

$$\mu_p(FU) \leq \Big(k_1 + \frac{\widetilde{k}_2}{\Gamma(\alpha+1)}T_{\alpha,p}\Big)\mu_p(U). \tag{3.3}$$

Notice that, the inequality (3.3) may not remain valid in the case of $U \subset \Omega$ as Ω is not equicontinuous on (01]. Then we must look for another closed convex and bounded subset of $L^p([0,1],X)$ such that F is a μ_p-contraction on it. For this, let $V = L^p - \overline{conv}(FU)$, where $L^p - \overline{conv}$ means closure of convex hull in $L^p([0,1],X)$. Since $F\Omega \subset \Omega$ and Ω is closed and convex in $L^p([0,1],X)$, then $FV \subset V$. Now, we will show that V satisfy to the inequality (3.3). For this, let $B = V \cap L^p - conv(F\Omega)$, then $B \subset conv(F\Omega)$ and $V = L^p - cl(B)$ where $L^p - cl$ means closure in $L^p([0,1],X)$. From (3.3) we get

$$\begin{aligned}\mu_p(FV) &\leq \mu_p(L^p - cl(B)) = \mu_p(B)\\ &\leq \Big(k_1 + \frac{\widetilde{k}_2}{\Gamma(\alpha+1)}T_{\alpha,p}\Big)\mu_p(V).\end{aligned} \tag{3.4}$$

So $F : V \to V$ is a continuous and μ_p-contraction. This completes the proof.

Let us now formulate another existence result under the following assumptions:

- (H'_g) $g : L^p([0,1],X) \to X$ is continuous and compact.
- (H'_f) (i) the function $f : [0,1] \times X \to X$ satisfies the Caratheodory condition, i.e., $f(.,x)$ is measurable for all $x \in X$ and $f(t,.)$ is continuous for a.e. $t \in [0,1]$;
  (ii) there exists $m \in L^1([0,1],\mathbb{R}_+)$ and an increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|f(s,x(s)\| \leq m(s)\phi(\|x(s)\|)$ for a.e. $t \in [0,1]$ and $x \in X$,

(iii) there exists $k\ :\mathbb{R}_+ \to \mathbb{R}_+$ measurable and essentially bounded function on compact intervals of $\mathbb{R}_+$ such that for any $B \subset C([0,1];X)$ which is bounded and equicontinuous on $(0,1]$

$$\mu(P_\alpha f(t,B)) \leq k(t)\mu(B) \ \text{ for a.e. } \ t \in [0,1],$$

(H') $\dfrac{MT_{\alpha,p}}{\Gamma(\alpha+1)} \sup\limits_{t\in[0,1]} k(t) < 1.$

Theorem 3.3. If the assumptions (H_A)-(H') are satisfies and the following inequality

$$\sup_{x\in\Omega} \|g(x)\| + \frac{\alpha}{\Gamma(\alpha+1)}\phi(R) \sup_{t\in[0,1]} \int_0^t \frac{m(s)}{(t-s)^{1-\alpha}} ds \leq \frac{R}{M} \tag{3.5}$$

hold, then the equation (1.1) has at least one mild solution.

Without the hypothesis $(H_f)(ii)$ operator F defined above, may not be continuous from $L^p([0,1];X)$ to itself, since the Nemytskii operator may fail to be continuous under the growth condition $(H'_f)(ii)$. So we use the fixed point theorem on $C([0,1];X)$ rather than on $L^p([0,1],X)$. Now, we prove that operator F is continuous in Ω. To do this, let us fix $x \in \Omega$ and take arbitrary sequence $(x_n) \in \Omega$ such that xn converge to x in $C([0,1];X)$. Next, by Lebesgue dominated convergence theorem and lemma 2.3 we derive that $\lim\limits_{n\to\infty} \|Fx_n - F_x\| = 0$. This fact proves that F is continuous on $C([0,1];X)$. Now, we consider the sequence of sets $\{\Omega\}_{n=0}^{\infty}$ defined by induction as follows $\Omega_0 = \Omega$ and for $n = 0,1,...$ $\Omega_{n+1} = F(\Omega_{n+1})$. The hypothesis (H_A) and Lemma 3.1 imply that $F(\Omega_n) \subset C([0,1];X)$ is bounded and equicontinuous on $(0;1]$. Furthermore, it is easy to see that $\{\Omega\}_{n=0}^{\infty}$ is nondecreasing. Since $F(\Omega_n) \subset C([0,1b];X)$ is bounded and equicontinuous on $(0,1]$, so is $conv(F\Omega)$, where $conv$ means convex hull. As X is separable, from Lemma 2.3, Lemma 2.10 and Theorem 2.11 for any $B \subset conv(F\Omega)$ we have

$$\begin{aligned}
&\mu(\Omega_n(t)) = \mu((F\Omega_n)(t)) \leq \int_0^t \mu(\frac{S_\alpha(t-s)}{(t-s)^{1-\alpha}} f(s,\Omega_n(s))ds \\
&\leq \int_0^t \frac{k(s)}{(t-s)^{1-\alpha}} \mu(\Omega_n(s))ds \\
&\leq \frac{\widetilde{k}}{\Gamma(\alpha)} \int_0^t \frac{\mu(\Omega_n(s))}{(t-s)^{1-\alpha}} ds,
\end{aligned} \tag{3.6}$$

where $\widetilde{k} = \sup\{k(t) : 0 \leq t \leq 1\}$, obviously, $\widetilde{k}$ is nondecreasing. for a.e. $t \in [0,1]$. By Theorem 2.11 and lemma 2.7 we get

$$\mu_p(\Omega_{n+1}) \leq \Big(\frac{\widetilde{k}}{\Gamma(\alpha+1)} T_{\alpha,p}\Big)\mu_p(\Omega_n), \tag{3.7}$$

Now, we define the set $\widehat{\Omega}_n = L^p - \overline{conv}(\Omega_n)$ for $n = 1, 2, ...$ Since $\Omega_{n+1} \subset \Omega_n$ then $\widehat{\Omega}_{n+1} \subset \widehat{\Omega}_n$. Furthermore

$$\mu_p(\widehat{\Omega}_{n+1}) = \mu_p(\Omega_{n+1}) \leq \Big(\frac{\widetilde{k}\alpha}{\Gamma(\alpha+1)} T_{\alpha,p}\Big)\mu_p(\Omega_n), \tag{3.8}$$

for $n = 1, 2, ...$The lemma 2.7 show that $\widehat{\Omega} = \cap_{n=0}^{\infty} \widehat{\Omega}_n$ is nonempty, convex and compact in $L^p([0,T], E)$ and $F\widehat{\Omega} \subset \widehat{\Omega}$.
Let $Q = \overline{conv}(F\widehat{\Omega})$. Since

$$Q = \overline{conv}(F\widehat{\Omega}) \subset \overline{conv}(\widehat{\Omega}) \subset L^p - \overline{conv}(\widehat{\Omega}) = \widehat{\Omega},$$

then $Q \subset C([O,T], E)$ and $FQ \subset Q$. Next, we will show that Q is compact. First, by the hypothesis (H_A) and lemma 3.1, $F\widehat{\Omega}$ is equicontinuous on $[0,1]$, as g is continuous and $\widehat{\Omega} \subset L^p([0,1], X)$ is compact. Furthermore,

$$\mu((F\widehat{\Omega})(t)) \leq \mu\Big(P_\alpha(t)g(\widehat{\Omega})\Big) + \int_0^t \mu(\frac{S_\alpha(t-s)}{(t-s)^{1-\alpha}} f(s, \widehat{\Omega}(s))ds = 0$$

for any $t \in [0,1]$. Hence $F\widehat{\Omega} \subset C([0,1], X)$ is precompact, and hence so is $Q \subset C([0,1]; X)$. The proof is complete by Schauder's fixed point theorem.

4. Application

In this section, we discuss the existence of random solutions for the following semilinear fractional random differential equation:

$$\begin{cases} \partial_t^\alpha x(t,w) = \sum_{i=1}^n \frac{\partial^2}{\partial w_i^2} x(t,w) + f(t, x(t,w), w), \\ x(0,w) = g(x). \end{cases} \tag{4.1}$$

where ∂_t^α is a Riemann-Liouville fractional partial derivative of order $0 < \alpha < 1$. Let $([0,1], \mathcal{A}, P)$ be a complete probability measure space. We denote by

$$(E, \|.\|) = (L^2([0,\pi]), \mathcal{A}, P), \|.\|_{L^2([0,\pi])})$$

the space of square integrable maps $x = x(t,w) = x(t)(\zeta)$, with

$$\|x(t)\| = \Big(\int_{[0,1]} |x(t,w)|^2 dP(w)\Big)^{1/2}.$$

Denote by L the Laplace operator $\dfrac{\partial^2}{\partial \zeta^2}$ with the domain

$$D(A) = \{x(.) \in L^2([0,\pi]) : \frac{d^2x}{ds^2}, \frac{dx}{ds} \text{ are absolutely continuous on } ; x(0) = x(\pi) = 0\}.$$

It is well known that A is self-adjoint with compact resolvent and is the infinitesimal generator of an analytic, strongly continuous, compact and self-adjoint semigroup $U(t)_{t\geq 0}$ satisfying

$$\|U(t)\| \leq e^{-t}, \text{ for } t \geq 0.$$

Example 4.1. Let

$$\begin{cases} \alpha = \frac{2}{3} \\ p = 2 \\ f(t, x(t,w), w) = x(t,w) + \arctan(x(t,w)), \\ g(x)(t) = \int_0^t sx(s)ds, \quad \text{where } x \in L^p([0,1]), \ s \in [0,1]. \end{cases}$$

The function $f : x \mapsto x + \arctan(x)$ clearly satisfies the L^2-Carathéodorys condition. Moreover, since $\lim_{x\to+\infty} \frac{|f(t,x)|}{x} = 1$ for all $t \in [0,1]$ then for all $\epsilon > 0$, there exists $1 < C_{1,\epsilon} < 1 + \epsilon$, $C_{2,\epsilon} > 0$ such that
$|f(t,x)| \leq C_{1,\epsilon}|x| + C_{2,\epsilon}$, for all $\mathbb{R}$, $t \in [0,1]$. Next, it is easy to show that the functions g and f satisfies to

(1) $\|g(x)\|_{L^2([0,\pi])} \leq \pi \|x\|_2$
(2) $\|f(t,x(t))\|_{L^2([0,\pi])} \leq \|x(t)\|_{L^2([0,\pi])} + \frac{\pi^2}{2}$
(3) $\|f(t,x_1(t)) - f(t,x_2(t))\|_{L^2([0,\pi])} \leq 2\|x_1(t) - x_2(t)\|_{L^2([0,\pi])}$,
(4) Denote by

$$\chi(X) = \lim_{\epsilon\to 0} \sup_{x\in X} \sup\{\|x(t;\zeta+h) - x(t;\zeta)\|_2, \quad |h| \leq \epsilon\}.$$

It is known [2] that χ is a measure of noncompactness on $L^2([0,T]), T > 0$. To check conditions $(C_f)(iii)$ and $(C_g)(iii)$, it is enough to take $k_1 = \frac{\pi^3}{3}$ and $k_2(t) = 2\pi$. Applying the result obtained in Theorem 3.2, we deduce that equation (4.1) has a mild solution.

5. Conclusion

Combining the techniques of fractional calculus, measure of noncompactness, and fixed point theorem with respect to a k-set-contractive operator, we obtain a new result on the existence of mild solutions with the assumption that the nonlinear term satisfies some growth condition and noncompactness measure condition.

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